Colloquium
ITA Universität Heidelberg

Hydrostatic Planetary Models from the Point of View of Dynamical Systems

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Outline

1. Model and Equations
2. The Isothermal Case
3. The Polytropic Generalisation
4. Application: Understanding of Mass Spectra Predictions
5. Stability

1 Model and Equations

2 The Isothermal Case

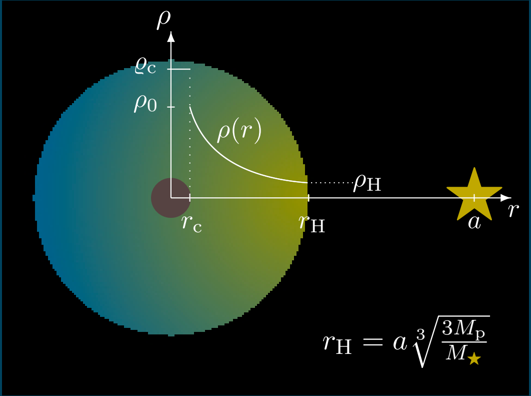
3 The Polytropic Generalisation

4 Application: Understanding of Mass Spectra Predictions

5 Stability

general assumption

- spherical symmetry



assumptions

core

- solid body
- constant density ϱ_c
- characteristic quantity: $M_c = M(r_c)$

envelope

- hydrostatic equilibrium
- ideal gas
- polytropic relation
- extends to Hill radius
- characteristic quantity: $\rho_0 = \rho(r_c)$

Model and Equations

connections between polytropic and isentropic relations

polytropic relation

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

isentropic relation

$$\left(\frac{d\ln T}{d\ln P}\right)_S = \nabla \quad \Longleftrightarrow \quad \frac{T}{T_0} = \left(\frac{P}{P_0}\right)^{\nabla}$$

equivalent descriptions in case of ideal gas

ideal gas: $$\frac{P}{P_0} = \frac{T}{T_0}\frac{\rho}{\rho_0} \quad \Longrightarrow \quad \nabla = 1 - \frac{1}{\gamma}$$

Model and Equations

hydrostatic equilibrium

$$\frac{d\rho}{dr} = -\frac{Gm\rho_0^{\gamma-1}}{kT_0\gamma}\frac{M\rho^{2-\gamma}}{r^2}$$

mass-density relation

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

transformation of variables ρ and M into homology invariants

global gravitational potential

$$\varphi := \frac{Gm}{kT}\frac{M}{r} \qquad (= V)$$

local gravitational potential

$$\psi := \frac{Gm}{kT} 4\pi r^2 \rho \qquad (= UV)$$

$$\frac{d\varphi}{ds} = \psi - \varphi + \nabla\varphi^2$$

$$s := \ln\frac{r}{r_c}$$

$$\frac{d\psi}{ds} = \psi\left[2 + \varphi(2\nabla - 1)\right]$$

Model and Equations

parameters for all examples

general

- core density $\varrho_c = 5500\,\mathrm{kg\,m^{-3}}$
- star mass $M_\star = M_\odot$
- atomic mass $m = 4.12 \cdot 10^{-27}\mathrm{kg}$
  ($X = 0.76$, $Y = 0.24$, molecular form)

Jovian region

- nebula temperature $T_\mathrm{H} = 123\,\mathrm{K}$ (cf. [Hayashi et al. 1985])
- orbital distance $a = 5.2\,\mathrm{AU}$

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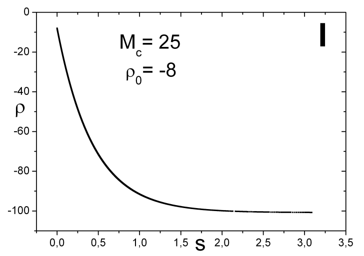
The Isothermal Case

isothermal $\rightarrow$ $\nabla = 0\,, \gamma = 1$

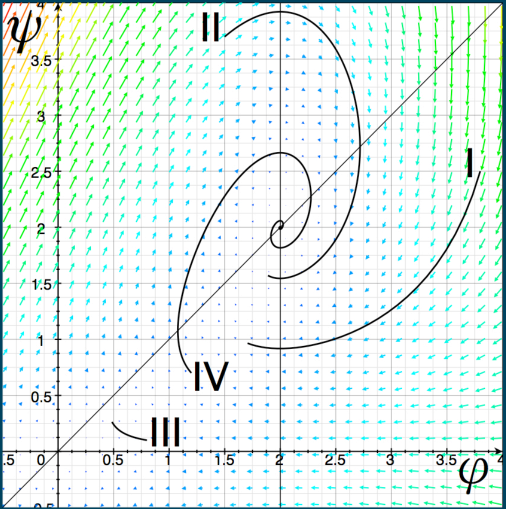
$$\frac{d\varphi}{ds} = \psi - \varphi \qquad (\varphi \sim M/r)$$

$$\frac{d\psi}{ds} = \psi(2 - \varphi) \qquad (\psi \sim r^2 \rho)$$

radial density profiles



phase portrait of the system



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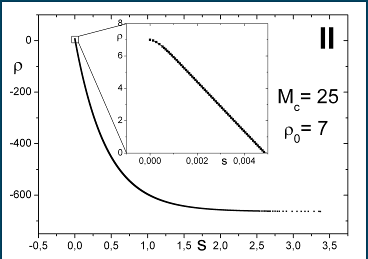

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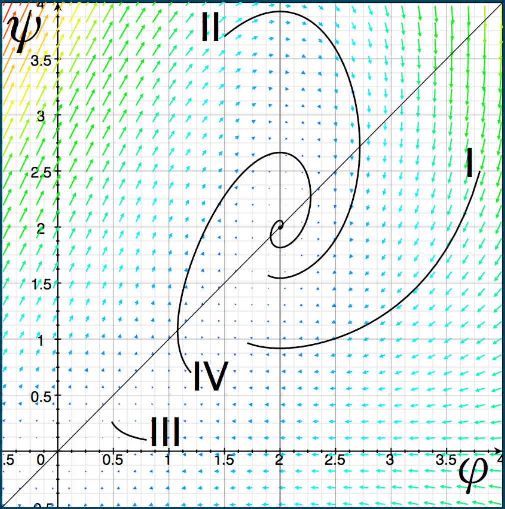

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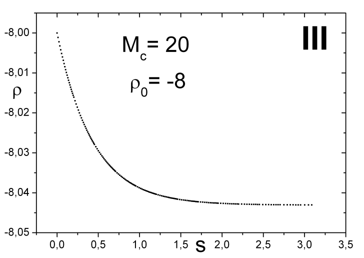

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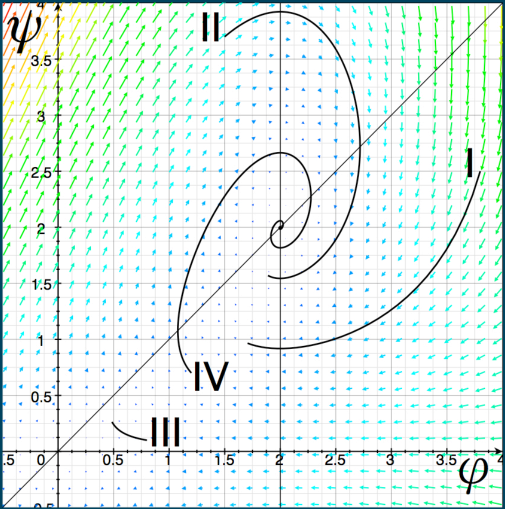

The Isothermal Case

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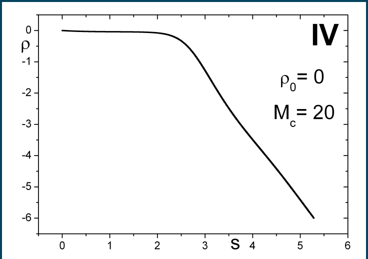

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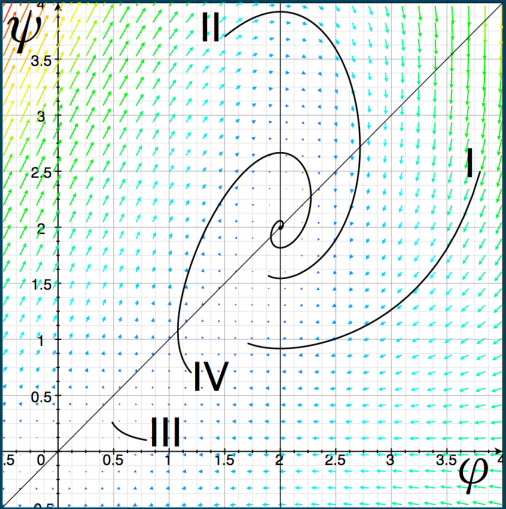


The Isothermal Case

envelope mass

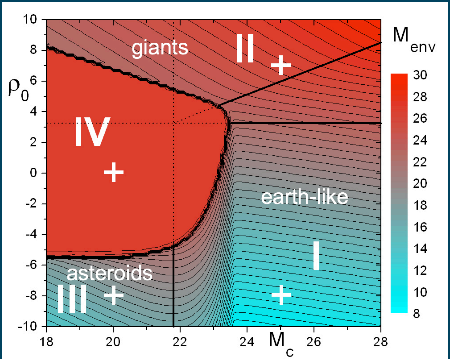
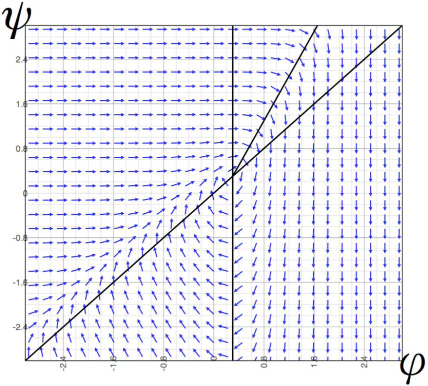


illustration cf. [Pečnik and Wuchterl 2005]

logarithmic phase portrait



equivalence between critical lines in different graphical representations

Example: $\psi = \varphi \quad \Rightarrow \quad \rho = \frac{M}{4\pi r^3} = \frac{\langle \rho \rangle}{3} \quad \Rightarrow$ core surface: $\rho_0 = \frac{\varrho_c}{3}$

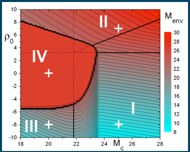
useful applications of the description with homology invariants

properties of the different regions

- formulas for radial mass and density profiles
- dependencies like $M_{\rm env}(M_{\rm c}, \rho_0)$ or $\rho_{\rm H}(M_{\rm c}, \rho_0)$
- clear physical approximations, e.g. region II: $\psi \gg \varphi$ at core surface

analytic descriptions of borders between regions

- Which core mass has a significant gravitational influence?
- When self-gravity overcomes the core potential?
- Where is the critical core mass?



$$\left[\text{e.g.:} \quad M_{\rm c}^{\rm crit} = \left(\frac{3}{4\pi\varrho_{\rm c}}\right)^{\frac{1}{2}} \left(\frac{kT}{Gm} \ln \frac{4\pi a^3 \varrho_{\rm c}}{27 M_\star}\right)^{\frac{3}{2}}\right]$$

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The Polytropic Generalisation

general remarks

polytropes

- two important temperature gradients: $\nabla_{\mathrm{rad}}(P, T, \kappa, L)$ and $\nabla_{\mathrm{ad}}(P, T)$
- thermodynamics: $\nabla_{\mathrm{ad}} = 0...2/5$
- Schwarzschild-criterion: $\nabla = \min(\nabla_{\mathrm{rad}}, \nabla_{\mathrm{ad}})$
- $\Rightarrow$ $\nabla(r)$ is "some" function, varying between $0...2/5$! $[\gamma(r) = 1...5/3]$

temperature

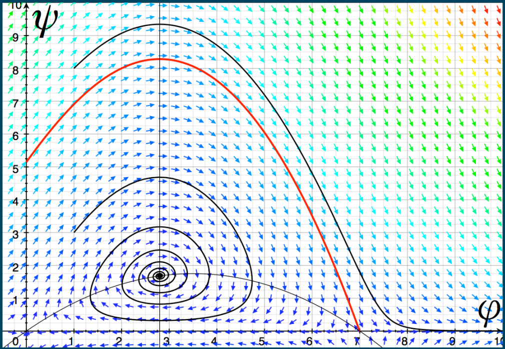
- not constant $\quad \Rightarrow \quad T(r)$
- choice of boundary condition: T_0 or T_{H} ?

The Polytropic Generalisation

comments

- new topology (after bifurcation at $\nabla = 0$)
- "isothermal" part
- "compact" part
- separatrix

phase portrait for $\nabla = 1/7 \quad (\gamma = 7/6)$



general analytic form of the separatrix

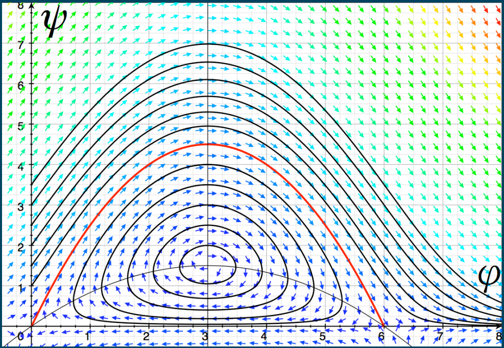
$$\psi_{\text{sep}}(\varphi) = -\frac{1}{2}\left(\varphi - \frac{2}{1-2\nabla}\right)^2 + \frac{1}{2}\left(\frac{1-4\nabla}{\nabla(1-2\nabla)}\right)^2$$

The Polytropic Generalisation

comments

- integrable, conservative system
- $\Rightarrow$ constant of "motion"
- "periodic" part (elliptic functions)
- "compact" part
- last appearance of a separatrix
- bifurcation point

phase portrait for $\nabla = 1/6 \quad (\gamma = 6/5)$



constant of "motion"

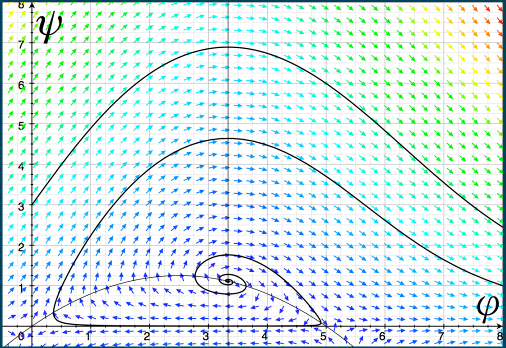
??? $$C(\psi, \varphi) = \psi^{\frac{1}{2}} \left(2\psi - 6\varphi + \varphi^2\right)$$???

The Polytropic Generalisation

comments

- again new topology (after bifurcation at $\nabla = 1/6$)
- equilibrium point becomes unstable
- all profiles become compact
- no separatrix

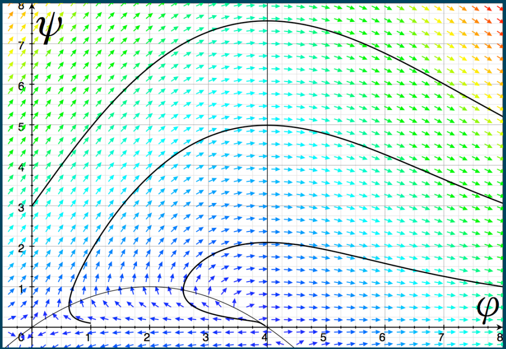
phase portrait for $\nabla = 1/5 \quad (\gamma = 5/4)$



comments

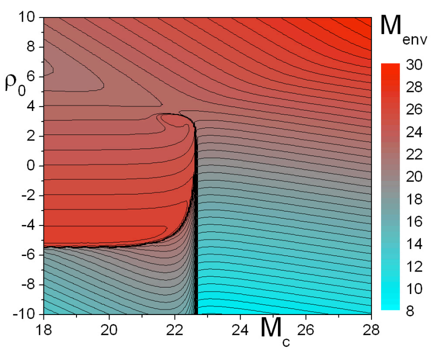
- equilibrium point is lost
- profiles become compact faster and faster
- no more qualitative changes till $\nabla = 2/5$

phase portrait for $\nabla = 1/4 \quad (\gamma = 4/3)$

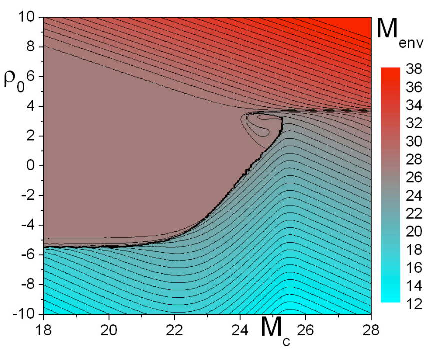


The Polytropic Generalisation

envelope mass $\nabla = 1/6$, fixed T_0



envelope mass $\nabla = 1/6$, fixed T_{H}



Analytic descriptions of occurring structures can be given in analogy to the isothermal case

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Application: Understanding of Mass Spectra Predictions

excursion: a more realistic model [Broeg 2006]

- realistic, tabulated eos
- core luminosity due to accretion of planetesimals
- radiative transfer (diffusion approximation) with tabulated opacities
- convection (Schwarzschild criterion) with tabulated ∇_{ad}

▼

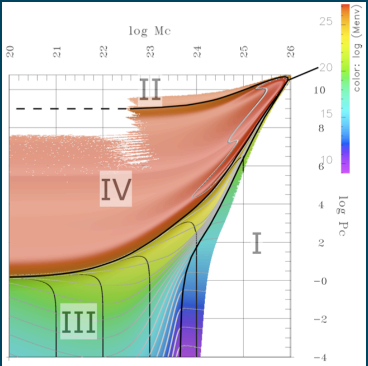
computing $M_{\mathrm{tot}}(M_{\mathrm{c}}, \rho_0)$ $\Rightarrow$ log-equidistant (scale free) grid $\Rightarrow$ count

$\Rightarrow$ mass spectrum, valid with the following assumptions:

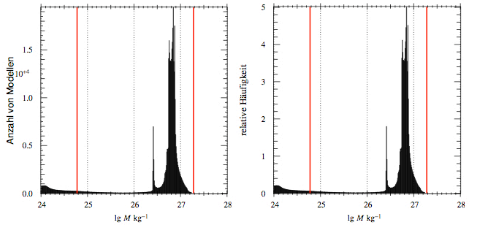
- all states equiprobable (stability?! $\rightarrow$ cf. last section)
- all states form quasi-static (otherwise: valid for pre-dynamic phase)

Application: Understanding of Mass Spectra Predictions

$M_{\mathrm{env/tot}}(M_{\mathrm{c}}, P_0)$ [Broeg2006]



numerical mass spectrum [Broeg 2006]



(a) unnormiert (b) normierte Verteilung

Abbildung 4.1: **Das Massenspektrum für Jupiters Position.** Es zeigt die Anzahl von Modellen bzw. die relative Häufigkeit aller Gleichgewichtszustände pro Masseninterval $d\lg M_{\mathrm{tot}}$ als Funktion der Gesamtmasse $\lg M_{\mathrm{tot}}$. Die beiden vertikalen roten Linien markieren die Position einer Erdmasse, M♁, bzw. einer Jupitermasse, M♃.

Das Massenspektrum von Jupiter hat offenbar einen deutlichen Peak bei $\lg M \sim 26{,}7$. Um die Positionen der unter Umständen mehreren Maxima quantifizieren zu können, werde ich stets zwei Größen bestimmen: den Modus und den Median der Anzahlhäufigkeiten.[1]

Im dem hier vorgestellten Beispiel gibt es nur ein Maximum, welches stark auf Null abfällt: Modus: 26,878 (0,001), Median: 26,814 (26,52–28).[2] Dies entspricht 0,40 bzw. 0,35 M♃. Insge-

analytical comprehension of the characteristic mass peak

peak corresponds to characteristic mass in region IV (the “island”)

$\in$ island $\iff$ $\in$ attractor $\iff$ outer envelope parts have $\nabla < 1/6$

$\Rightarrow$ mean total mass in region IV is determined by the attractor

$$\langle M_{\text{tot}}^{\text{IV}} \rangle = M_{\text{peak}} = \left(\frac{3\,k\,T_{\text{H}}\,a}{Gm\,(3M_{\star})^{\frac{1}{3}}} \right)^{\frac{3}{2}} =$$

numerical results [Broeg 2006]:

$M_{\text{peak}}^{\text{Broeg}} = 0.35\,M_{♃}$ (median) or $0.40\,M_{♃}$ (modus)

Application: Understanding of Mass Spectra Predictions

analytical comprehension of the characteristic mass peak

peak corresponds to characteristic mass in region IV (the "island")

$\in$ island $\iff$ $\in$ attractor $\iff$ outer envelope parts have $\nabla < 1/6$

$\Rightarrow$ mean total mass in region IV is determined by the attractor

$$\langle M_{\rm tot}^{\rm IV} \rangle = M_{\rm peak} = \left(\frac{3\,k\,T_{\rm H}\,a}{Gm\,(3M_{\star})^{\frac{1}{3}}} \right)^{\frac{3}{2}} = 0.375\,M_{♃}$$

numerical results [Broeg 2006]:

$M_{\rm peak}^{\rm Broeg} = 0.35\,M_{♃}$ (median) or $0.40\,M_{♃}$ (modus)

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Stability

linear stability analysis

introducing perturbations of the form

$$\rho(r,t) = \rho_{\text{static}}(r) + \delta\rho(r)\, e^{i\omega t}$$

$$M(r,t) = M_{\text{static}}(r) + \delta M(r)\, e^{i\omega t}$$

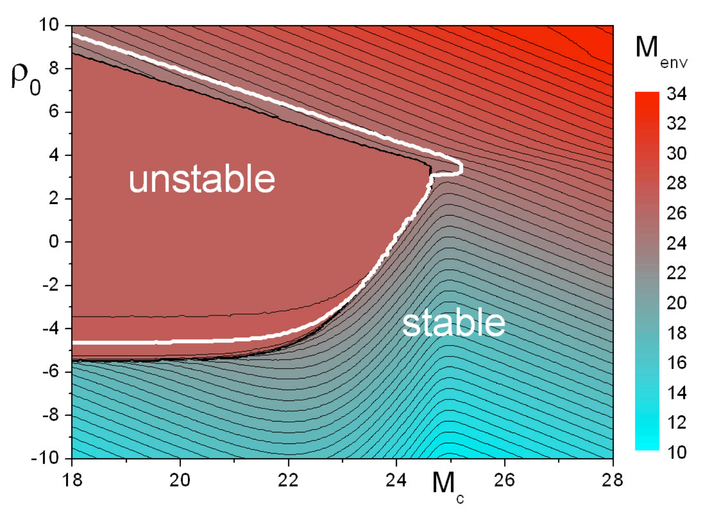
$$\vdots$$

$\Rightarrow$ Eigenvalue problem for δM
with Eigenvalues ω_n^2 and Eigenfunctions $\delta M_n(r)$

$\Rightarrow$ sign of the smallest Eigenvalue ω_0^2 decides about stability

Stability

envelope mass and border of stability for $\nabla = 1/8$ ($\gamma = 8/7$) , fixed T_{H}



Conclusions

- A clear description of core-envelope structures is possible with the help of the two homology invariants φ and ψ.
- There are two basic behaviours of trajectories in the φ - ψ - space:
  1. approach of certain fixed values (attractor)
  2. $\psi \rightarrow 0$ (compact).
- Fundamental changes occur for $\nabla = 0$ (isothermal) and $\nabla = 1/6$.
- Characteristic physical quantities, e.g. the critical core mass, can be given analytically.
- Main peak in mass spectra can be understood.
- Structures near the attractor appear to be unstable.

Thanks for your attention!

(japanese cat paying attention)