A new numerical method for solving the equations of hydrodynamics and of ideal magnetohydrodynamics

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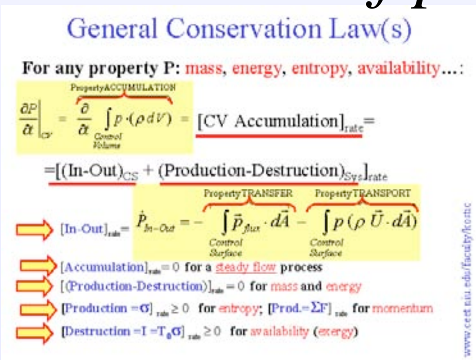
lecture ITA Heidelberg, May 2, 2007

nature

rules of physics



General Conservation Law(s)

For any property P: mass, energy, entropy, availability...:

$$\left.\frac{\partial P}{\partial t}\right|_{CV} = \overbrace{\frac{\partial}{\partial t}\int_{\substack{Control\\Volume}} p\cdot(\rho\, dV)}^{\text{Property ACCUMULATION}} = [\text{CV Accumulation}]_{rate} =$$

$$= [(\text{In-Out})_{CS} + (\text{Production-Destruction})_{Sys}]_{rate}$$

$$\Rightarrow [\text{In-Out}]_{rate} = \dot{P}_{In-Out} = -\overbrace{\int_{\substack{Control\\Surface}} \vec{p}_{flux}\cdot d\vec{A}}^{\text{Property TRANSFER}} - \overbrace{\int_{\substack{Control\\Surface}} p\,(\rho\,\vec{U}\cdot d\vec{A})}^{\text{Property TRANSPORT}}$$

⇨ $[\text{Accumulation}]_{rate} = 0$ for a steady flow process

⇨ $[(\text{Production-Destruction})]_{rate} = 0$ for mass and energy

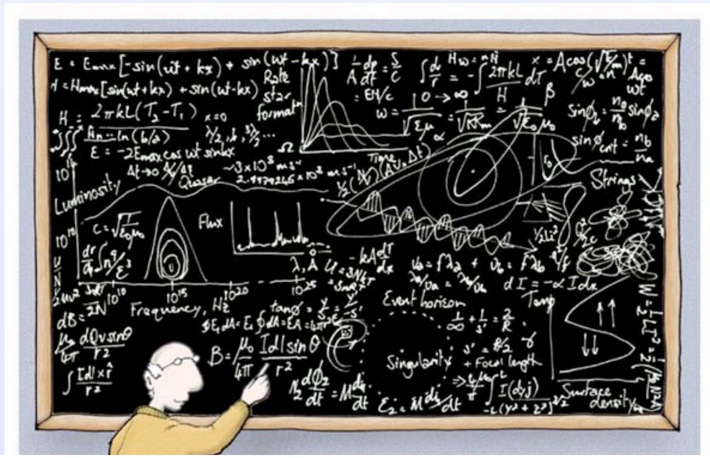
⇨ $[\text{Production} = \sigma]_{rate} \geq 0$ for entropy; $[\text{Prod.} = \Sigma F]_{rate}$ for momentum

⇨ $[\text{Destruction} = I = T_0\sigma]_{rate} \geq 0$ for availability (exergy)

www.cesr.siu.edu/faculty/kostic

computer simulations

mathematics



We model physical phenomena by conservation laws.

- conservation of mass
- conservation of momentum
- conservation of total energy

etc.

This gives rise to partial differential equations.

Their solution can only be found by approximating the solution by numerical discretisation.

Euler equations of compressible gas dynamics:

$$\rho_t + (\rho u)_x = 0$$ conservation of mass

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$ conservation of momentum

$$E_t + \big(u(E+p)\big)_x = 0$$ conservation of total energy

closure relationship - equation of state: $$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2$$ *polytropic gas*

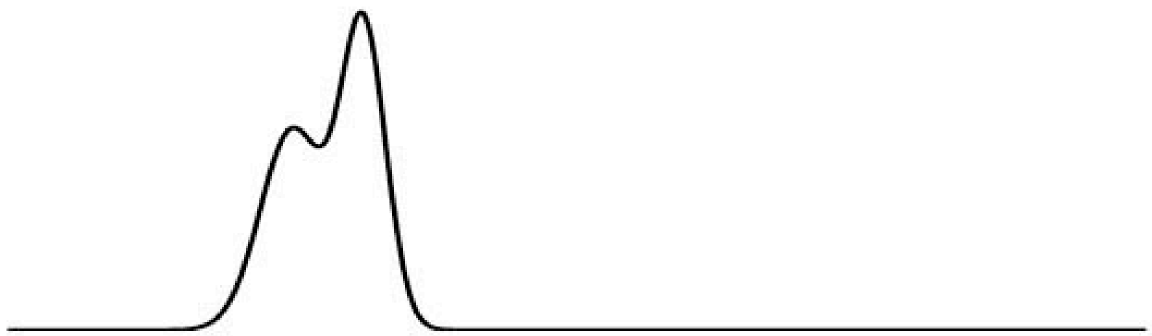
In order to study the numerical discretization of such equations we first study simpler equations.

Advection equation

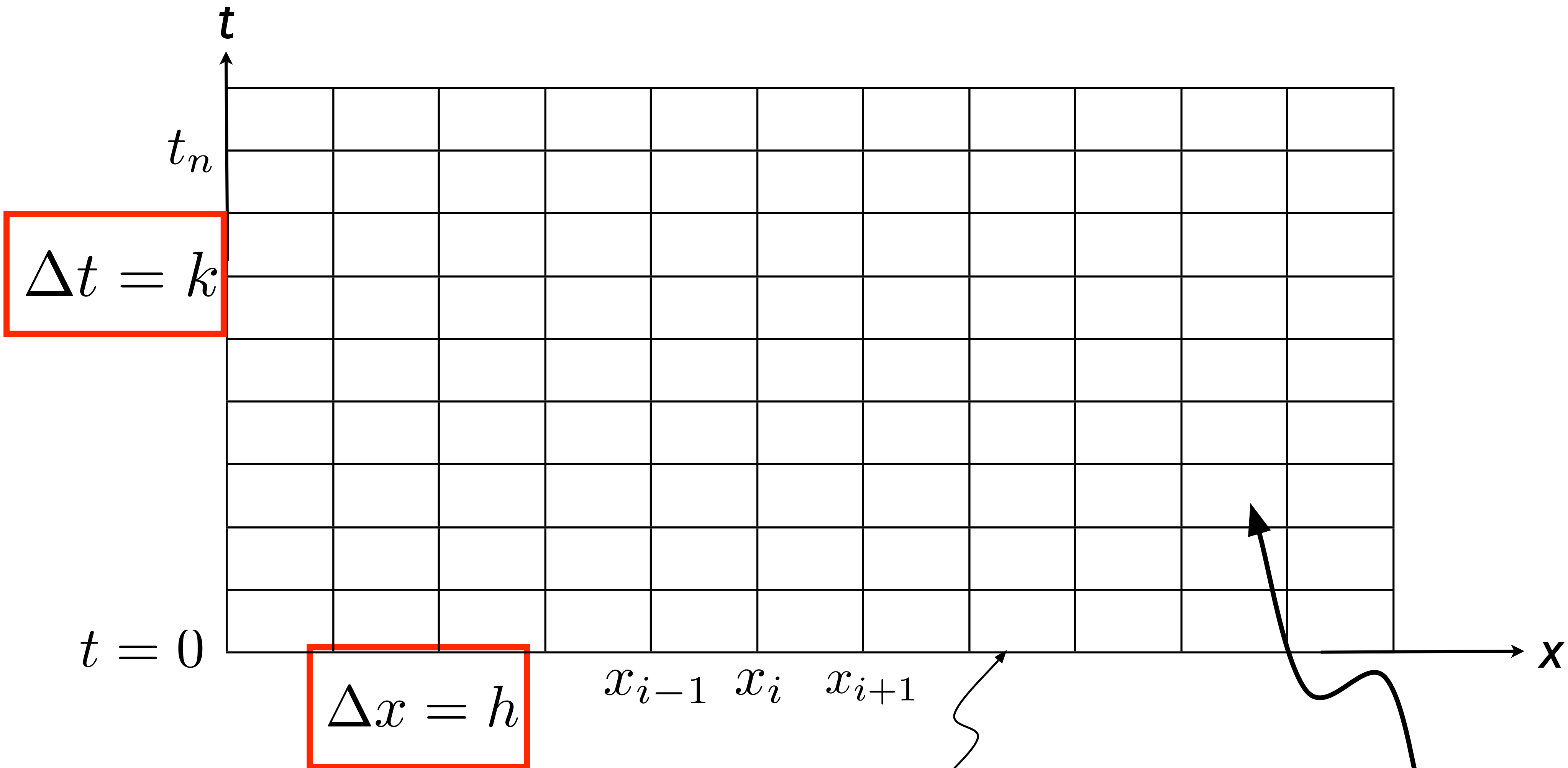
$$q_t + uq_x = 0$$

True solution: $q(x,t) = q(x - ut, 0)$

Assume $u > 0$ so flow is to the right.



Numerical methods use space- and time discretization:



the state of the gas is given initially

we then need to determine it at later times.

Finite difference method

Based on point-wise approximations:

$$Q_i^n \approx q(x_i, t_n), \quad \text{with } \; x_i = ih, \;\; t_n = nk.$$

Approximate derivatives by finite differences.

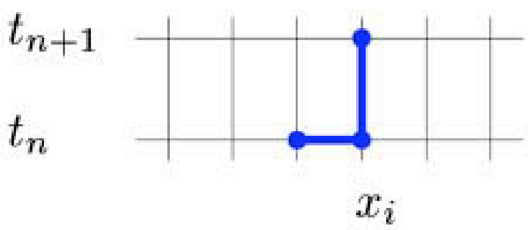
Ex: Upwind methods for advection equation $q_t + uq_x = 0$:

$$\frac{Q_i^{n+1} - Q_i^n}{k} + u\left(\frac{Q_i^n - Q_{i-1}^n}{h}\right) = 0$$

or

$$Q_i^{n+1} = Q_i^n - \frac{k}{h}u(Q_i^n - Q_{i-1}^n).$$

Stencil:



Finite volume method (linear equation)

Based on cell averages:

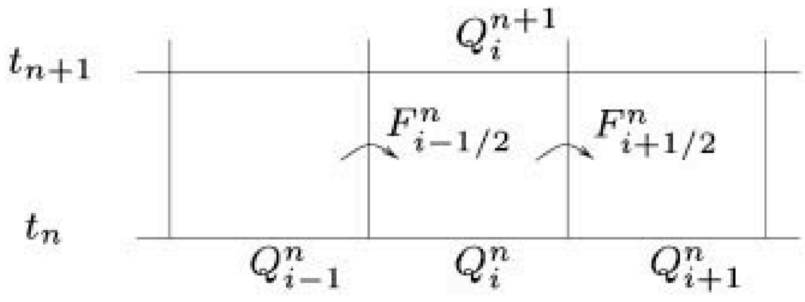
$$Q_i^n \approx \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$$

Update cell average by flux into and out of cell:

Ex: Upwind methods for advection equation $q_t + uq_x = 0$:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{k(uQ_{i-1}^n - uQ_i^n)}{h} \\ &= Q_i^n - \frac{ku}{h}(Q_i^n - Q_{i-1}^n) \end{aligned}$$

Stencil:



Finite volume method (nonlinear equation): $q_t + f(q)_x = 0$

Integral form: $$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

Integrate from t_n to t_{n+1} $\Longrightarrow$

$$\int q(x,t_{n+1})\,dx = \int q(x,t_n)\,dx + \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))\,dt$$

$$\frac{1}{h}\int q(x,t_{n+1})\,dx = \frac{1}{h}\int q(x,t_n)\,dx - \frac{k}{h}\left(\frac{1}{k}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t)) - f(q(x_{i-1/2},t))\,dt\right)$$

Numerical method: $$Q_i^{n+1} = Q_i^n - \frac{k}{h}(F_{i+1/2}^n - F_{i-1/2}^n)$$ $$Q_i^n = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x,t_n)dx$$

Numerical flux: $F_{i-1/2}^n \approx \dfrac{1}{k}\displaystyle\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt.$

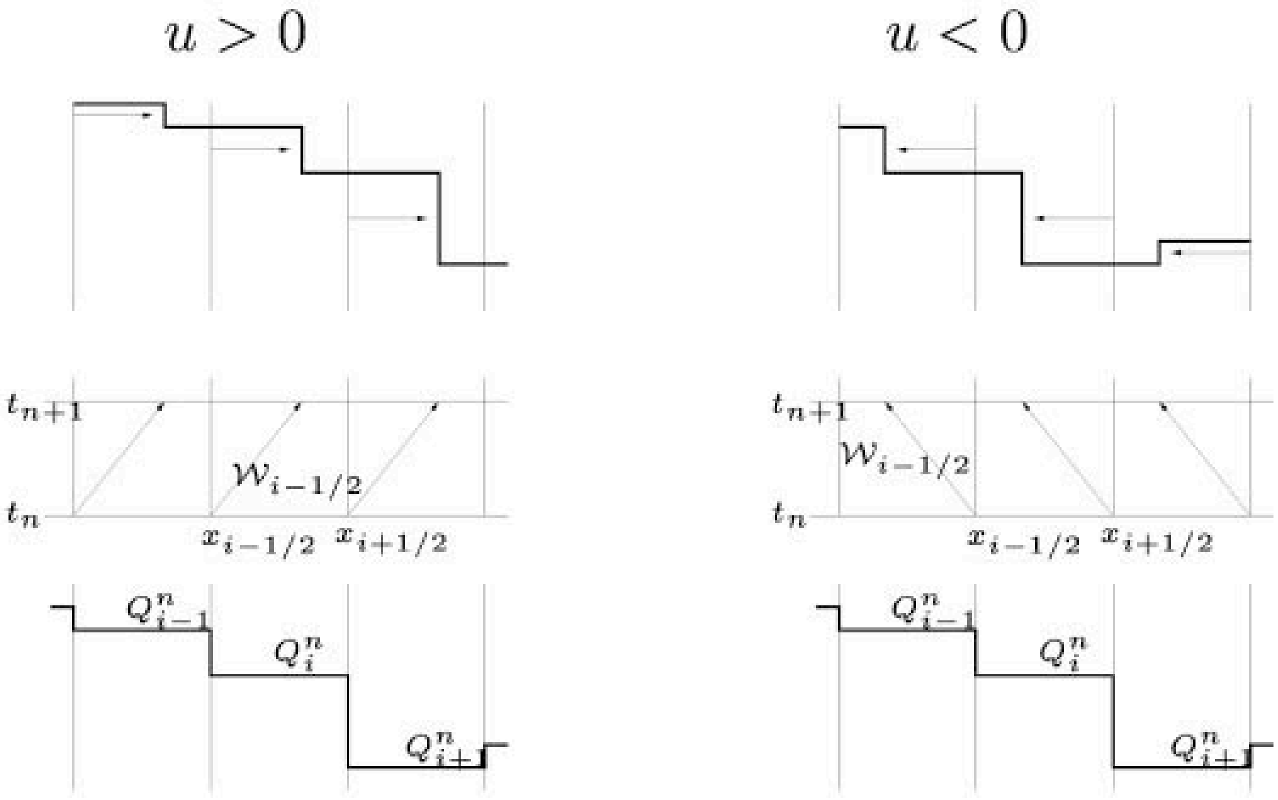
Godunov's method for advection

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \;\text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Longrightarrow$ Riemann problems.

$$q_t + uq_x = 0$$



Godunov's method

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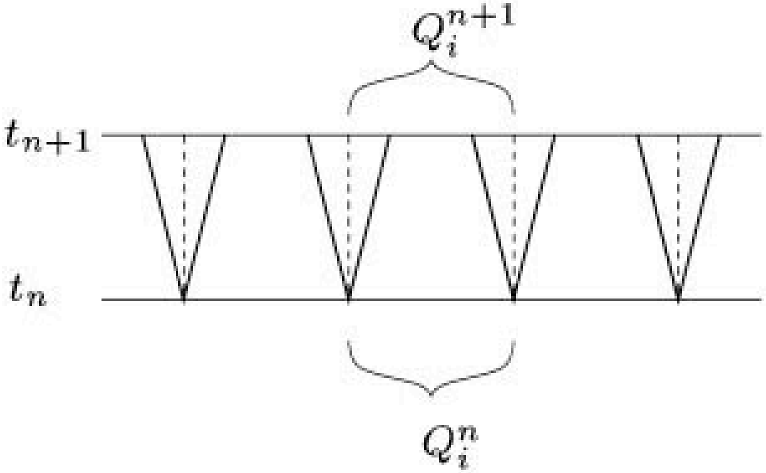
Discontinuities at cell interfaces $\implies$ Riemann problems.



$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\vee}(Q_{i-1}, Q_i) \quad \text{for } t > t_n.$$

$$F_{i-1/2}^n = \frac{1}{k} \int_{t_n}^{t_{n+1}} f(q^{\vee}(Q_{i-1}^n, Q_i^n))\, dt = f(q^{\vee}(Q_{i-1}^n, Q_i^n)).$$

First order REA Algorithm

1. **Reconstruct** a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

2. **Evolve** the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. **Average** this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{h} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

In our

Graduiertenkolleg
"Theoretische Astrophysik und Teilchenphysik"

in Würzburg involving particle physics, astrophysics and mathematics

among other things we model the temporal evolution of compressible flow.

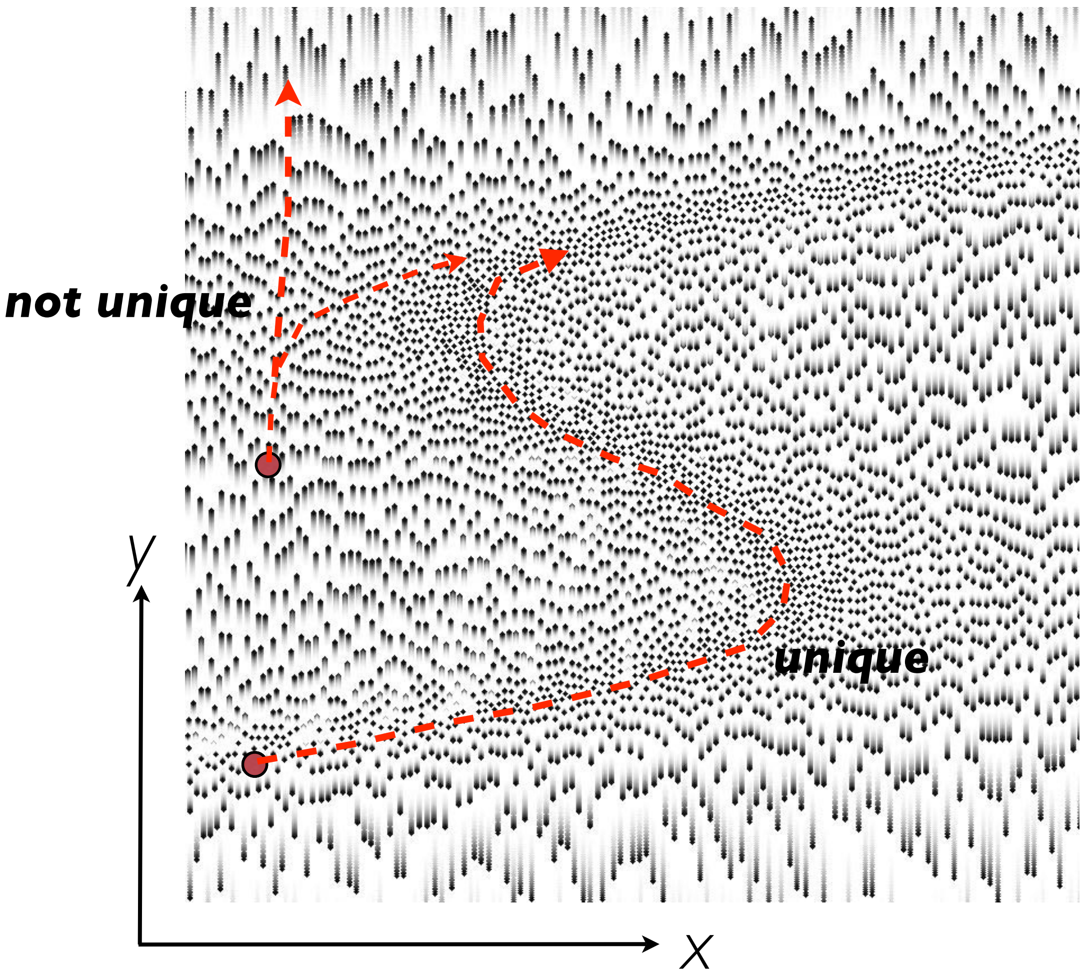
Many phenomena in continuum mechanics may be modelled as systems of hyperbolic conservation laws:

$$\frac{\partial U(x,t)}{\partial t} + \nabla F\big(U(x,t)\big) = 0$$

Their solutions need to be considered together with some *admissibility* condition,

also called *entropy* condition.

analogy: dynamical system



Candidates for admissibility:

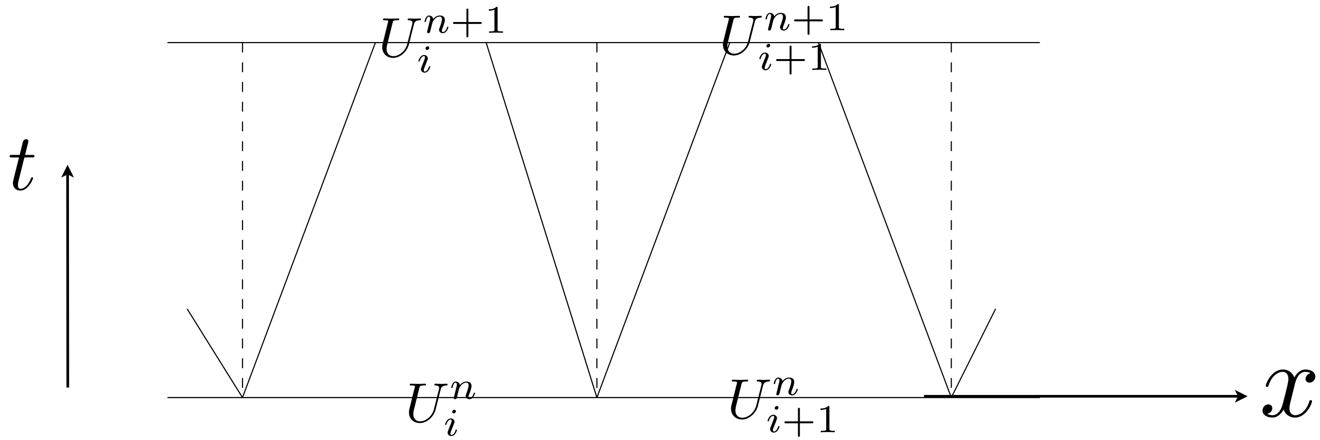
- *second law of thermodynamics*: the solution should satisfy an additional differential inequality, *entropy inequality*
- take into account *viscous effects:* take limit of vanishing viscosity

We shall use the following admissibility (or entropy) condition:

$$(\rho\phi(s))_t + \mathrm{div}(\rho\mathbf{u}\phi(s)) \leq 0$$

where ϕ, is an appropriately chosen convex functional.

Approximate this by a Godunov scheme



$$U_i^{n+1} - U_i^n + \frac{\Delta t}{h_i}[F^c(U_i^n, U_{i+1}^n) - F^c(U_{i-1}^n, U_i^n)] = 0, \quad h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

where the discrete solution satisfies

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{h_i}[G^c(U_i^n, U_{i+1}^n) - G^c(U_{i-1}^n, U_i^n)] \leq 0$$

discrete entropy inequality

Such an a priori bound ensures that we compute physically relevant shocks.

For gas dynamics we want to also have:

$$\text{if } \rho^n > 0 \text{ and } e^n > 0, \text{ then } \rho^{n+1} > 0 \text{ and } e^{n+1} > 0.$$

POSITIVITY

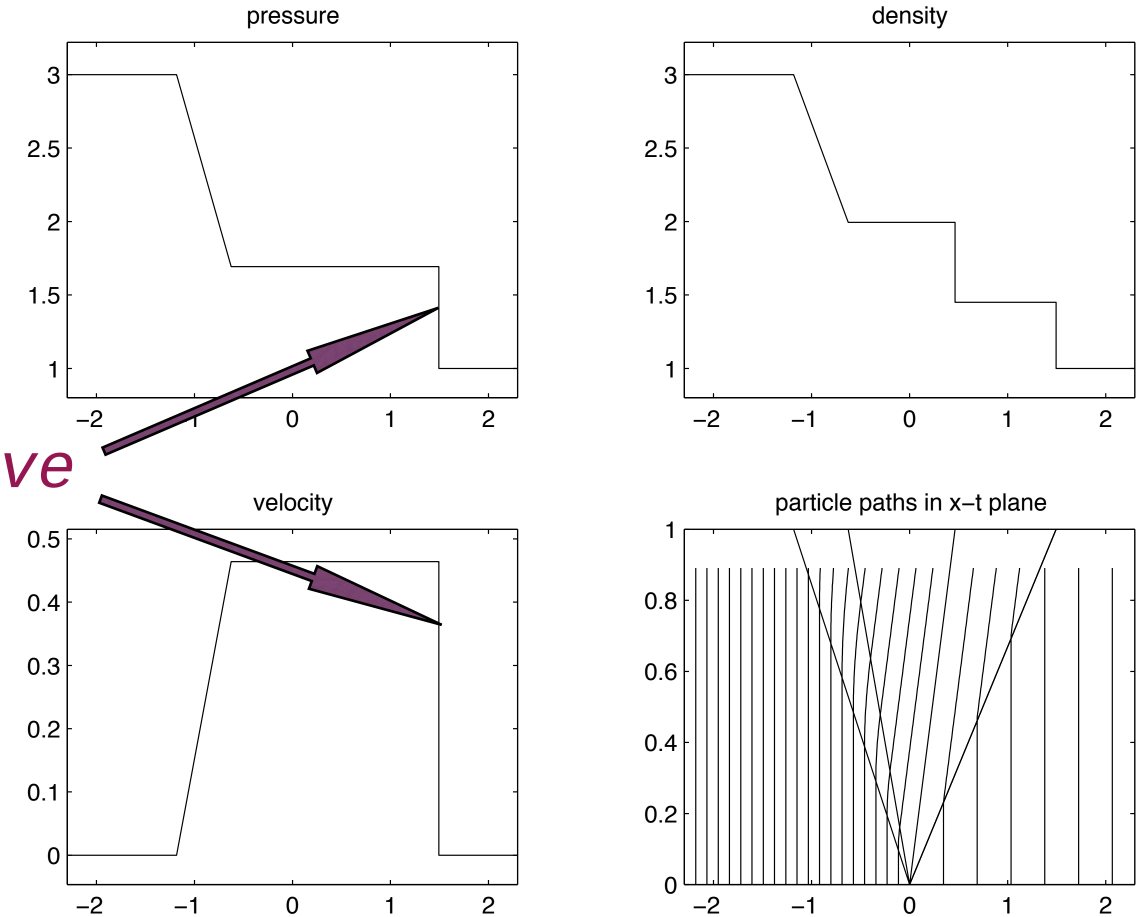
Phil Roe 1981 introduced an approximate Riemann solver by a local linerization of the flux which is consistent and conservative.

Shock tube problem

at time $t = 0$

membrane at $x = 0$

at time $t = 1$



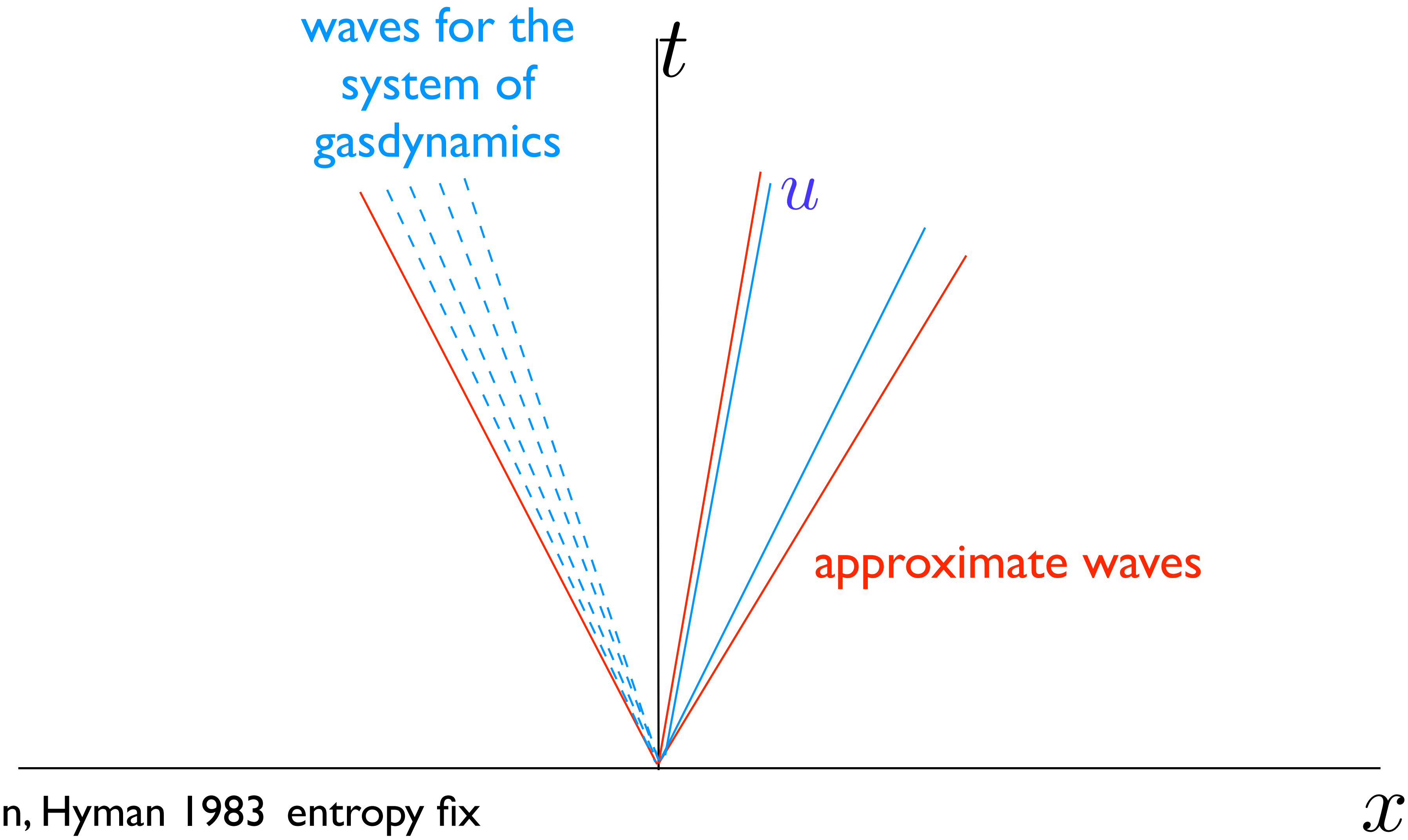
shock wave

x - axis

x - axis

This is called a Riemann problem.

For the Euler equations Roe's approximate Riemann solver consists of three constant states separated by jumps.



Harten, Hyman 1983 entropy fix

Harten, Lax, van Leer 1983 even simpler approximate Riemann solver
with only two waves, called the "HLL" solver.

Toro et. al. 1994 for gas dynamics improved this by inroducing a middle wave,
the "HLLC" solver.

Siliciu (~1996), Coquel (~1998), Coquel & Kl. (1999)
noticed that the HLLC solver could be improved by a relaxation approach.

The resulting approximate Riemann solver was

- more accurate
- entropy consistent
- positivity preserving

outline of what follows:

- 1. we have developed new Riemann solvers
- 2. we tested them in an astrophysics code

literature:

to 1.: *Bouchut, Klingenberg, Waagan: "A multiwave Riemann solver for MHD", part 1, part 2, Numerische Mathematik, 2007*

to 2.: *Klingenberg, Waagan, Schmidt, "Numerical comparisons of Riemann solvers", Journal Computational Physics, 2007*

Boltzmann equation

Boltzmann (1844 – 1906)

interacting particles are modelled at a "microscopic" level

distinguish between particles with different velocities v

density distribution $f(t, x, v)$

evolution equation is given by the so called Boltzmann equation:

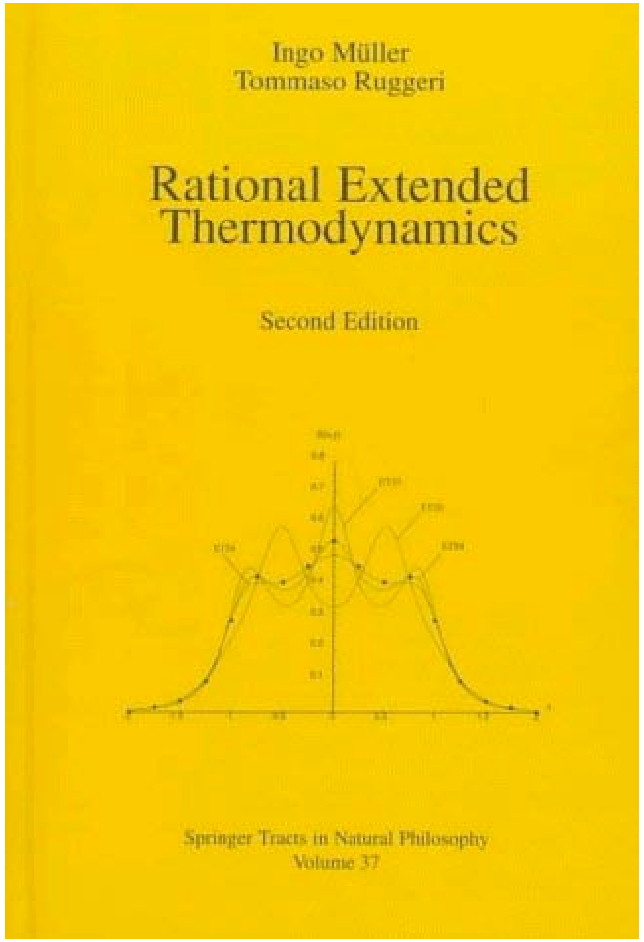
$$f_t + v.\nabla_x f = Q(f)$$

collision term

use this to obtain a PDE description

description by physicaly measurable quantities, like $\rho, \quad v, \quad T$

these can be found by taking moments of Boltzmann



get the evolution equations of the moments:

$$
\begin{array}{lcccc}
\partial_t F & + & \partial_k F_k & = & 0 \\
\partial_t F_i & + & \partial_k F_{ik} & = & 0 \\
\partial_t F_{ij} & + & \partial_k F_{ijk} & = & P_{\langle ij \rangle} \\
 & \vdots & & \vdots & \\
\partial_t F_{i_1 \cdots i_N} & + & \partial_k F_{i_1 \cdots i_N k} & = & P_{i_1 \cdots i_N}
\end{array}
$$

for example Grad's 13 moment expansion:

Euler

$$\partial_t \rho + \partial_x \rho v = 0$$

$$\partial_t \rho v + \partial_x (\rho v^2 + p + \sigma) = 0$$

$$\partial_t (\rho v^2 + 3p) + \partial_x (\rho v^3 + 5pv + 2\sigma v + 2q) = 0$$

$$\partial_t (\tfrac{2}{3}\rho v^2 + \sigma) + \partial_x (\tfrac{2}{3}\rho v^3 + \tfrac{4}{3} pv + \tfrac{7}{3}\sigma v + \tfrac{8}{15} q) = -\tfrac{4}{5} B\rho\sigma$$

$$\partial_t (\rho v^3 + 5pv + 2\sigma v + 2q) + \partial_x (\rho v^4 + 8pv^2 + 5\sigma v^2 + \tfrac{32}{5} qv + \tfrac{p}{\rho}(5p + 7\sigma))$$
$$= -\tfrac{8}{5} B\rho(\tfrac{2}{3} q + \sigma v)$$

can identify small parameter such that this is of the form

$$\partial_t U + \mathrm{div} F(U) = \frac{1}{Kn} P(U)$$

Knudsen number (small)

we mimic this procedure as follows:

embed your system of conservation laws into a more complete model

this is reminscent of extended thermodynamics

the enlargarged system has a small parameter $\epsilon > 0$ s.th.

$\epsilon > 0$ enlarged system

$\epsilon = 0$ original system

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + \pi)_x = 0$$
$$E_t + [(E + \pi)u]_x = 0$$
$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p - \pi}{\epsilon}$$

For smooth solutions of the Euler equations

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$E_t + \big(u(E+p)\big)_x = 0$$

we can write an evolution equation for the pressure:

$$(\rho p)_t + (\rho u p)_x + \rho^2 p'(\rho) u_x = 0$$

Replace p by a new dependant variable π and let c replace the soundspeed $\rho\sqrt{p'(\rho)}$

$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p-\pi}{\epsilon}$$

Siliciu (1995), Coquel, Kl. (1999)

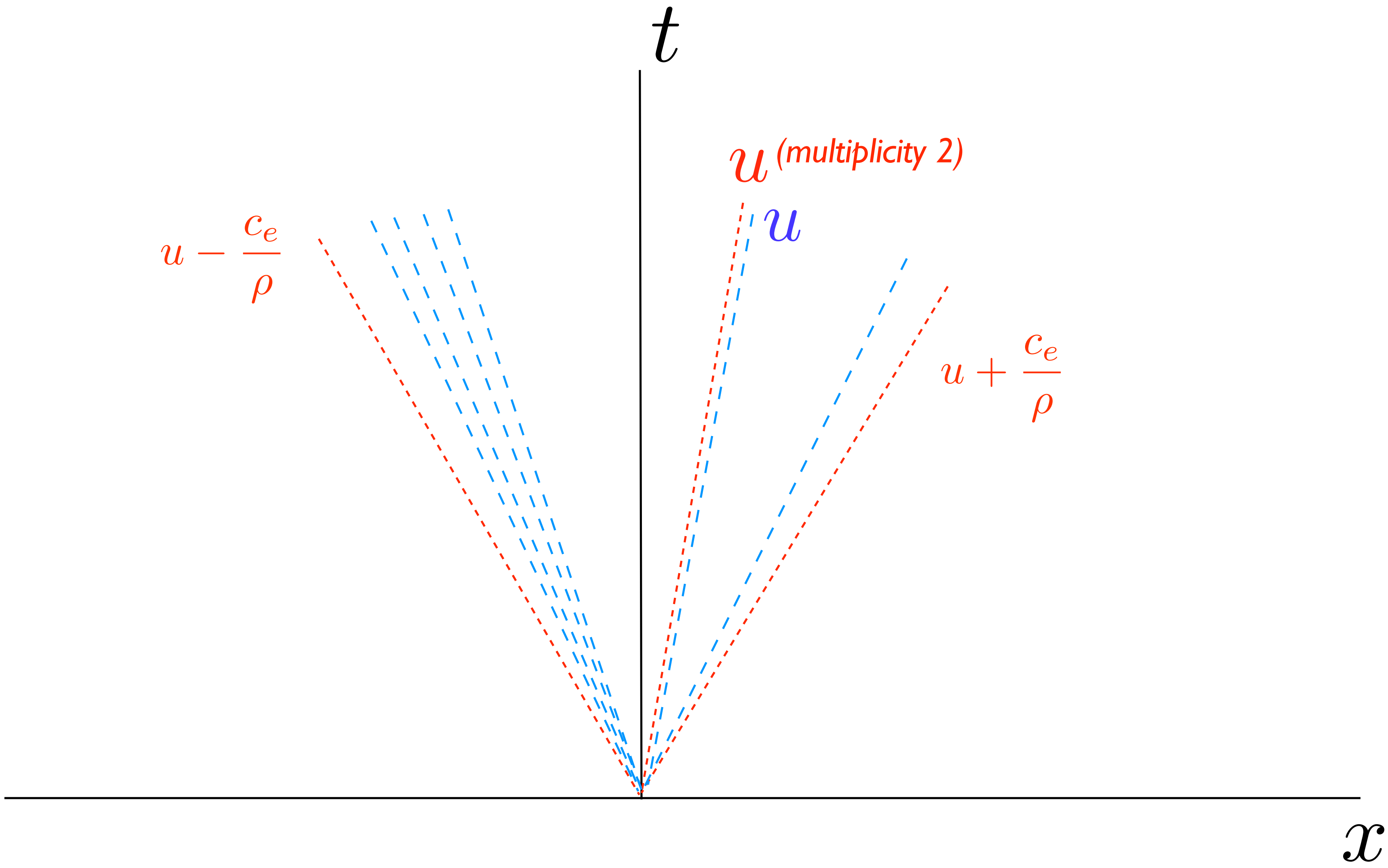
One advantage of the extended system is that by making the pressure a new dependent variable it easy to solve the Riemann problem for the homogeneous part of the extended system.

Also the constant c replaces the soundspeed, which is a nonlinear function.

The choice of c determines the "stability' of this relaxation:

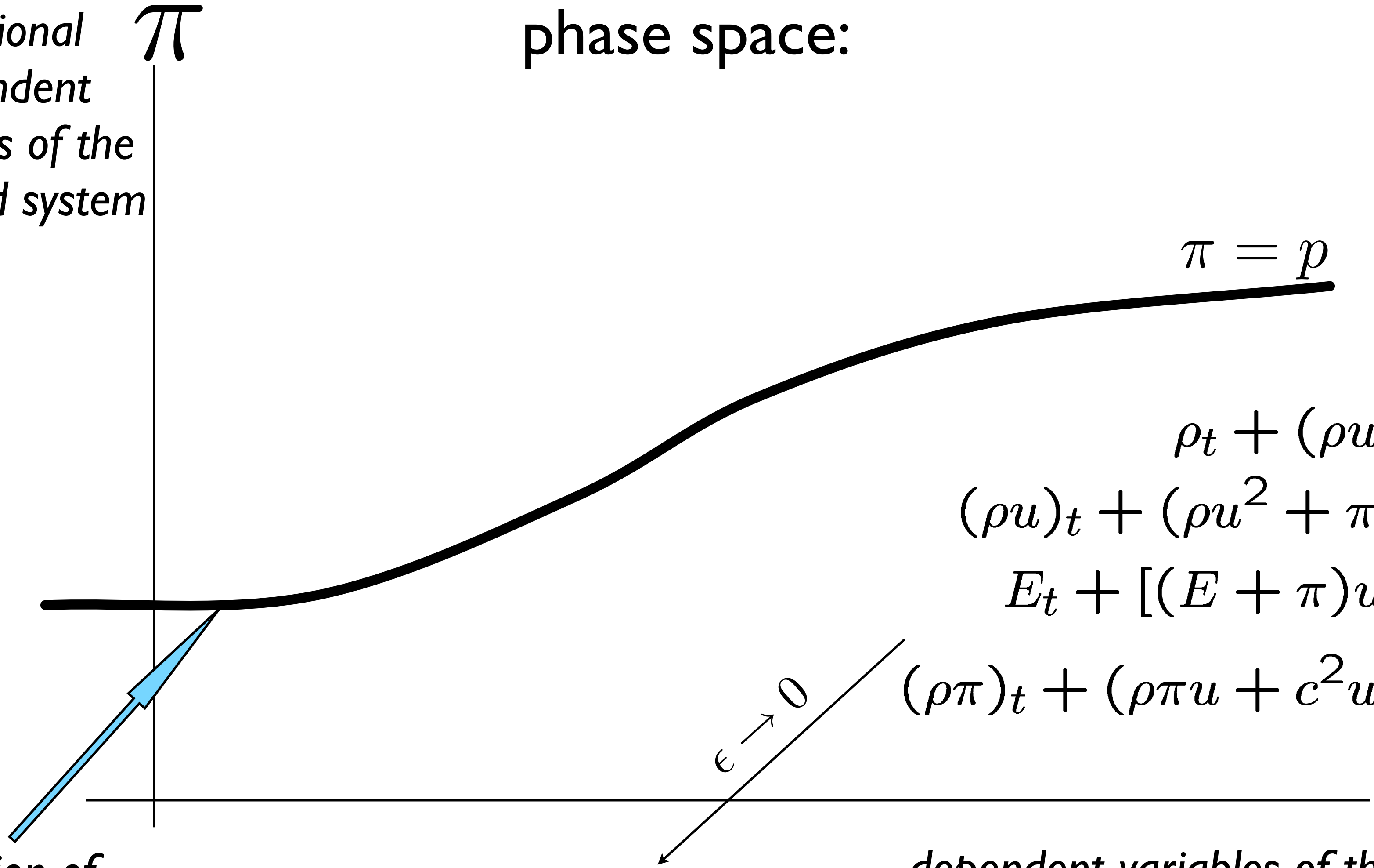
"subcharacteristic condition" $c > \rho\sqrt{p'(\rho)}$

wave speeds for the system of extended gasdynamics:



waves for the original system of gasdynamics:

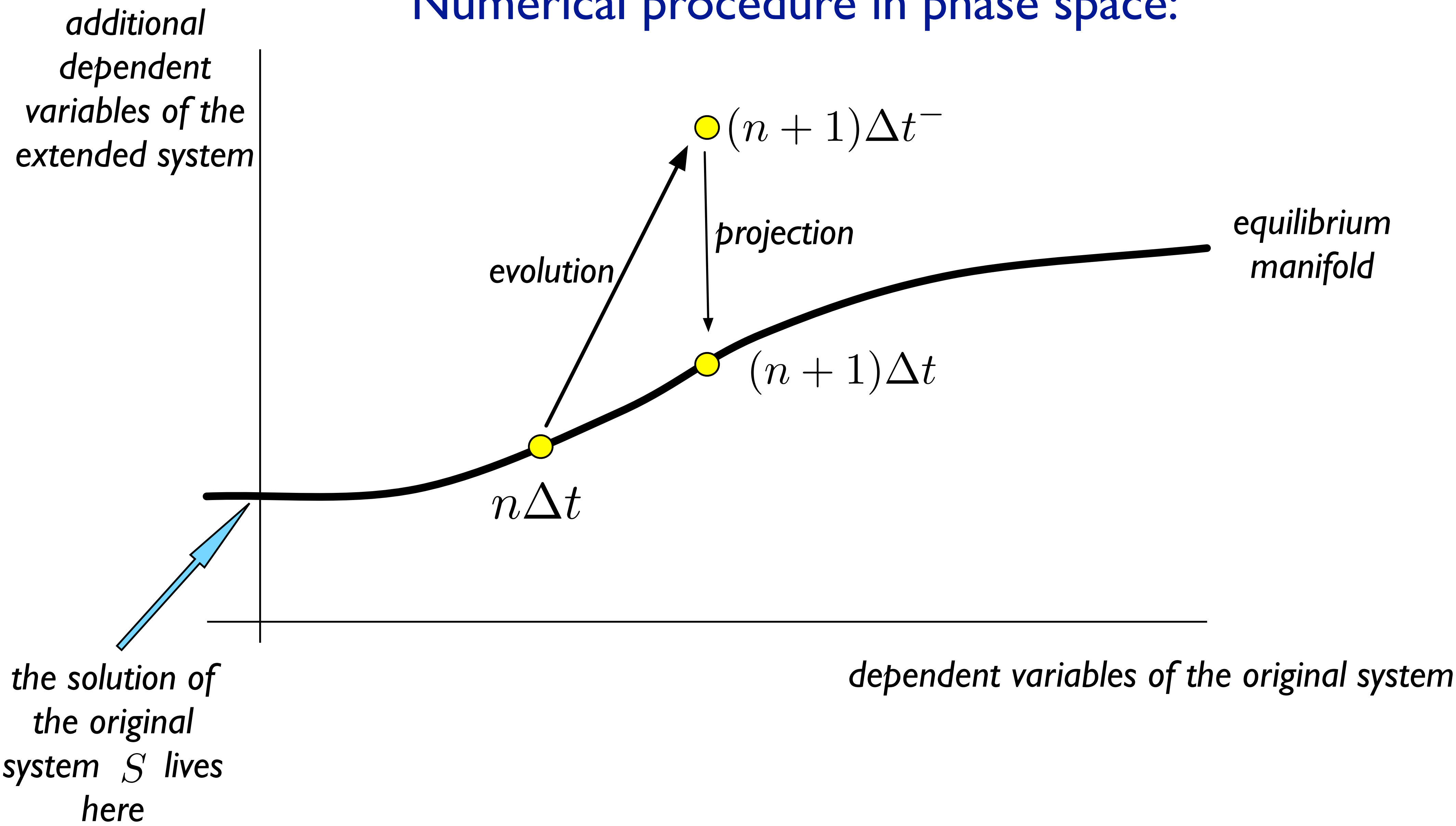
phase space:



$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + \pi)_x = 0$$
$$E_t + [(E + \pi)u]_x = 0$$
$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p - \pi}{\epsilon}$$

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$E_t + \big(u(E + p)\big)_x = 0$$

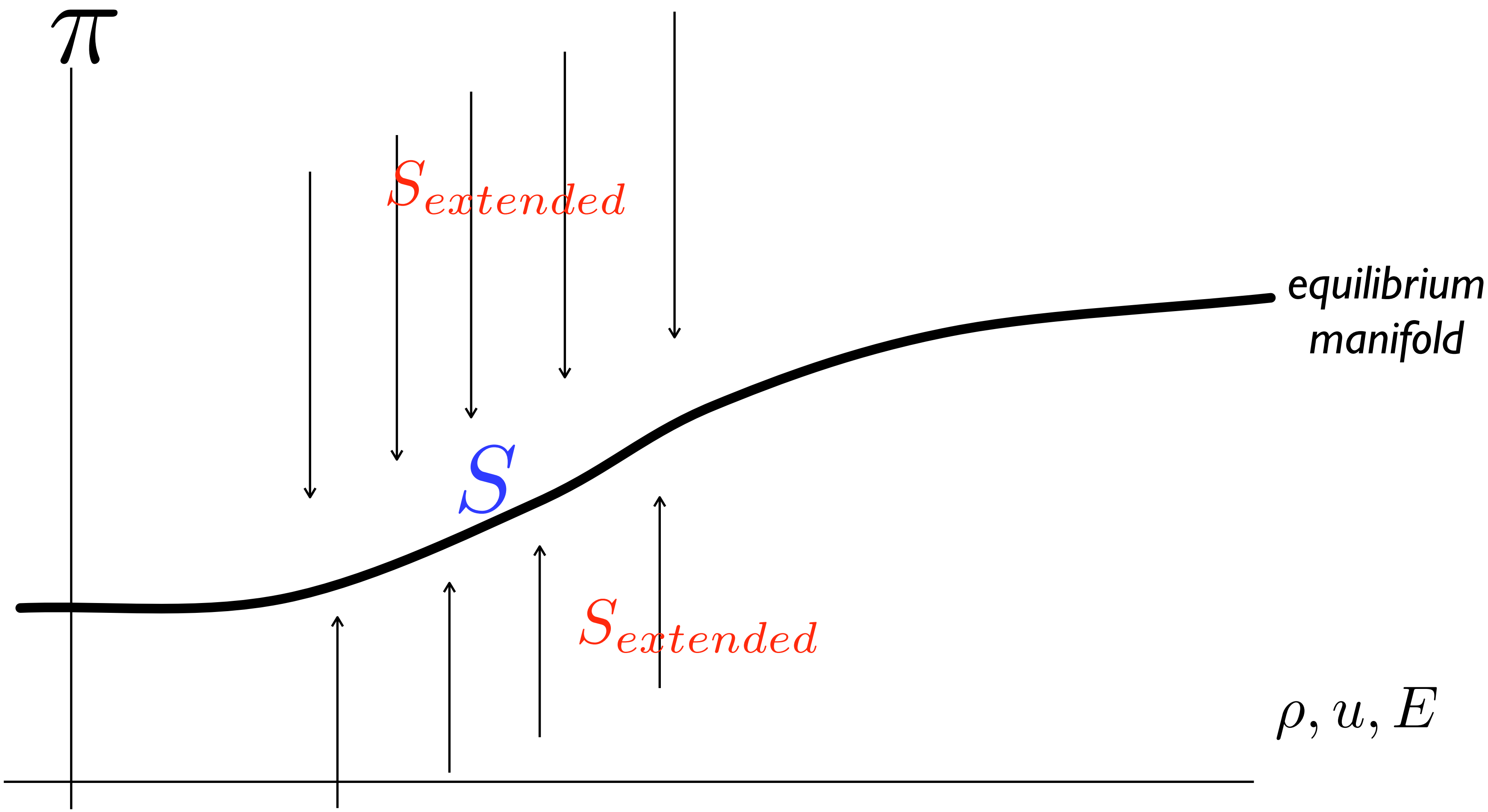
Numerical procedure in phase space:



This results in a numerical method for the original system.

It is possible to extend the entropy S of the original system of gas dynamics to an entropy $S_{extended}$ of the system of extended gas dynamics

such that for $\epsilon \longrightarrow 0$ the extended entropy converges to the original entropy.



this procedure translates Riemann solvers for the extended system to Riemann solvers for the original system

- preserves $\rho \geq 0$
- can handle vacuum
- this ensures that the "second law of thermodynamics" is staisfied by the numerical solution of our original system

more generally:

$U = (\rho, \rho u, E)$

Given a system of conservation laws $U_t + f(U)_x = 0$

$\psi = (\rho, \rho u, E, \pi)$

we associate with it an extended system of balance laws $\psi_t + A(\psi)_x = r(\psi)$

and an equilibrium mapping: $\psi = M(U)$ and a linear operator L

$M(U) = (\rho, \rho u, E, p)$

such that $LM(U) = U$.

The fluxes of the two systems are connected by the relation $LA(M(U)) = f(U)$

This defines approximate Riemann solvers for the original system.

Given an entropy pair for the equilibrium equation (η, G)

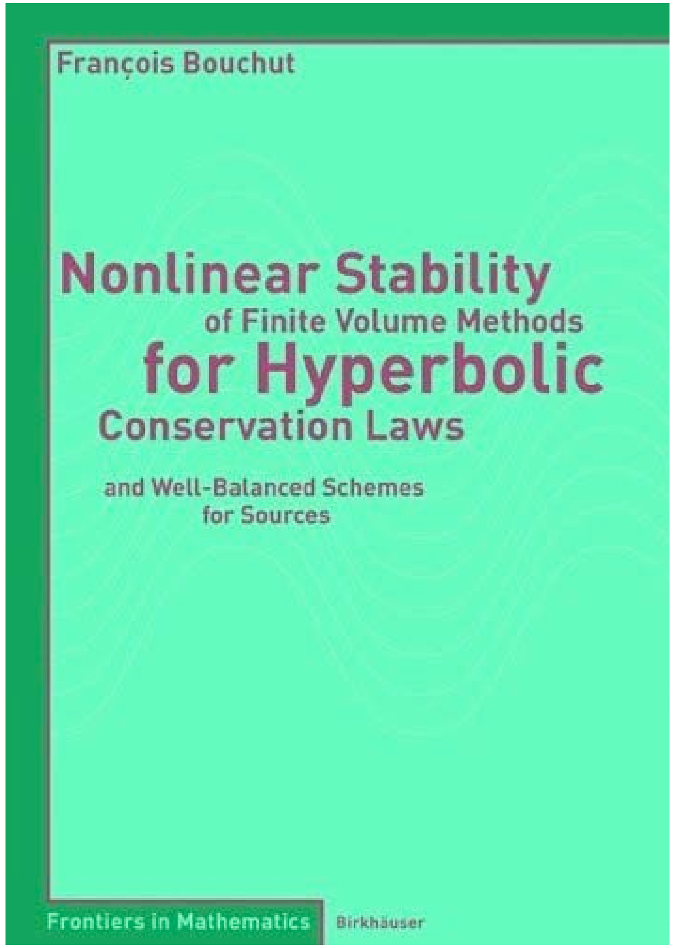
Let the extended system have an entropy pair $(\mathcal{H}, \mathcal{G})$ such that

$$\mathcal{H}(M(U)) = \eta(U) \qquad \mathcal{G}(M(U)) = G(U)$$

and the inequality holds $\mathcal{H}(M(L\psi)) \leq \mathcal{H}(\psi)$ for any ψ

Then this entropy extension will ensure that the approximate Riemann solver deduced for the equilibrium equation will be entropy consistent with respect to η .

2005



We will apply these ideas to the Magnetohydrodynamics (MHD) Equations

Bouchut, Klingenberg, Waagan: *A multiwave approximate Riemann solver for ideal MHD based on relaxation I - theoretical framework*, Numerische Mathematik (2007)

ionized compressible gas subject to magnetic fields

couple the Euler equations of compressible gas dynamics to equations for magnetic fields

Ideal MHD: Ignore resistivity ("viscous effect") $\Longrightarrow$ hyperbolic system.

New issues:

- Coupled with elliptic constraint $\nabla \cdot \vec{B} = 0$.
- Nonstrictly hyperbolic
- Nonconvex (not strictly hyperbolic) $\Longrightarrow$ compound waves

Conservation laws of MHD

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\vec{u}\\ \vec{B}\\ E\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\vec{u}\\ \rho\vec{u}\vec{u}+I\left(\left(p+\frac{1}{2}B^2\right)-\vec{B}\vec{B}\right)\\ \vec{u}\vec{B}-\vec{B}\vec{u}\\ \left(E+p+\frac{1}{2}B^2\right)\vec{u}-\vec{B}(\vec{u}\cdot\vec{B})\end{bmatrix}=0.$$

In components:

$$q=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ B^{(x)}\\ B^{(y)}\\ B^{(z)}\\ E\end{bmatrix},\qquad f(q)=\begin{bmatrix}\rho u\\ \rho u^2+p+\frac{1}{2}B^2-(B^{(x)})^2\\ \rho uv-B^{(x)}B^{(y)}\\ \rho uw-B^{(x)}B^{(z)}\\ 0\\ vB^{(x)}-B^{(y)}u\\ wB^{(x)}-B^{(z)}u\\ u\left(E+p+\frac{1}{2}B^2\right)-B^{(x)}(uB^{(x)}+vB^{(y)}+wB^{(z)})\end{bmatrix}$$

One-dimensional MHD

$$q_t + f(q)_x = 0$$

Note that

$$\frac{\partial}{\partial t} B^{(x)} = 0$$

In 1-D, $\nabla \cdot \vec{B} = 0$ means $B^{(x)} =$ constant.

Variations in $B^{(x)}$ remain stationary.

1-D equations reduce to **7-wave system** for

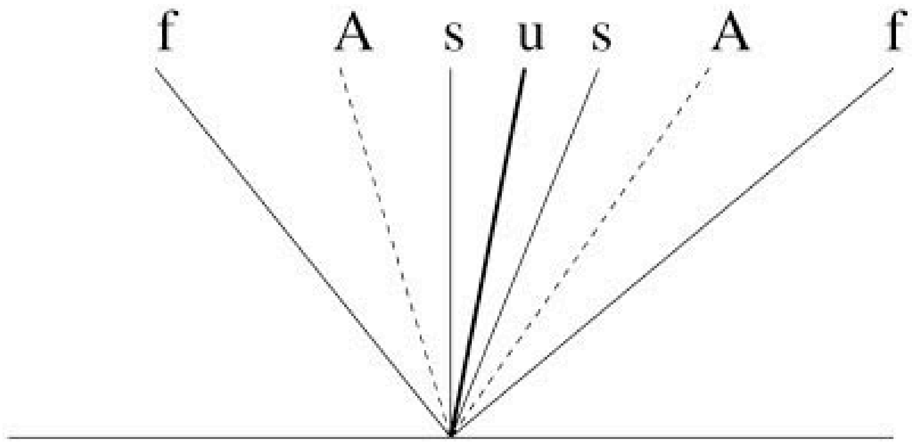
$$\tilde{q} = (\rho, \rho u, \rho v, \rho w, B^{(y)}, B^{(z)}, E).$$

Jacobian matrix has 7 eigenvalues (wave speeds)

$$u, \quad u \pm c_s, \quad u \pm c_A, \quad u \pm c_f$$

Waves in one-dimensional MHD

u	entropy waves — contact discontinuities
$u \pm c_s$	slow magnetosonic waves
$u \pm c_A$	Alfvén waves
$u \pm c_f$	fast magnetosonic waves



Magnetosonic waves are genuinely nonlinear

The divergence of B

In theory $\nabla \cdot \vec{B} \equiv 0$.

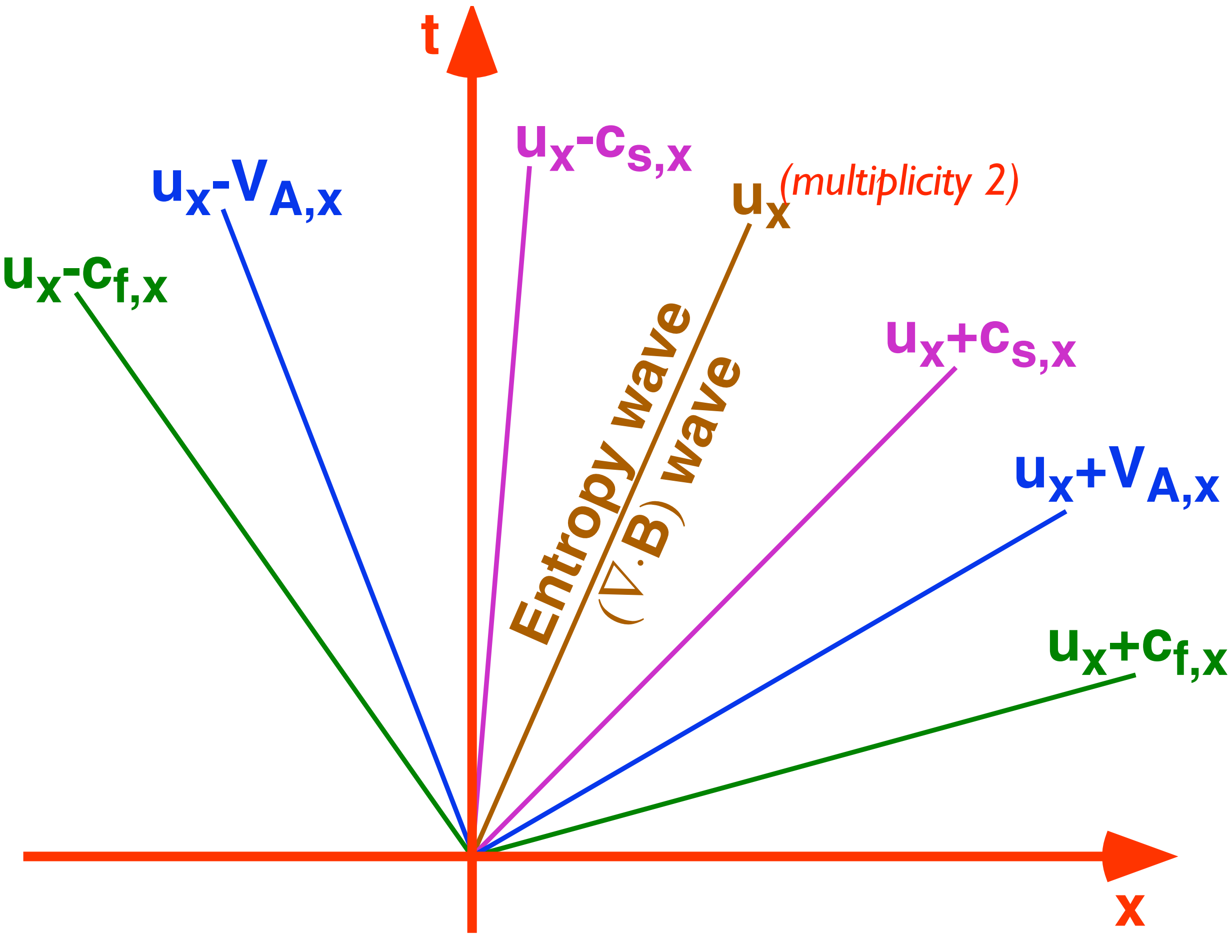
True at $t = 0 \implies$ true for all time.

Numerical methods may not preserve this.

Various approaches:

- Don't worry about it
  (ok for smooth solutions to order of method)
- Divergence-cleaning — projection onto $\nabla \cdot \vec{B} = 0$
- Constrained transport:
  Staggered grids and updating formula that preserves
  $\nabla \cdot \vec{B} = 0$
- 8-wave solver — advect $\nabla \cdot \vec{B}$ away

wave speeds for the original system of MHD:

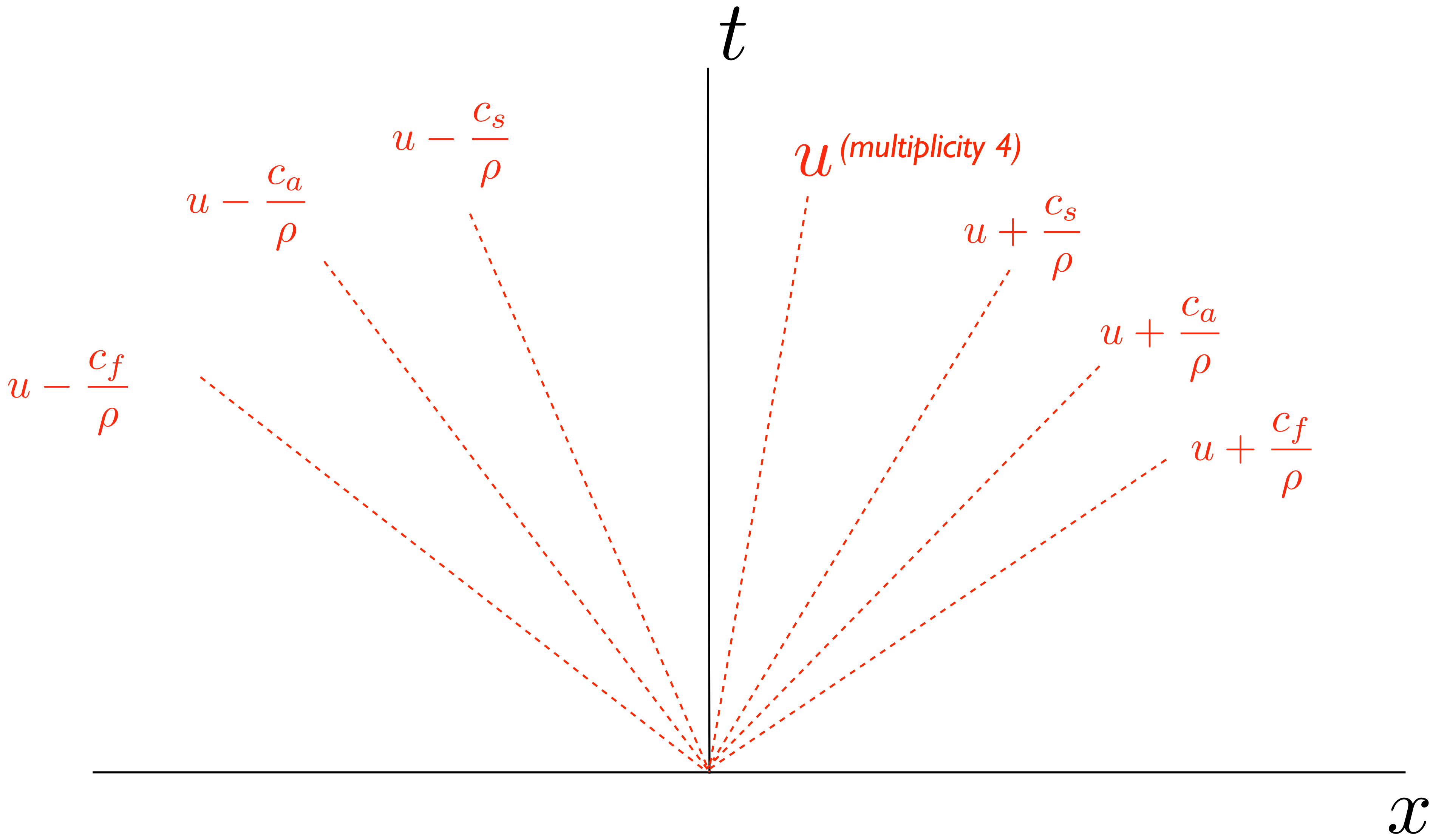


the Powell 8-wave structure

The extended system for MHD:

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + \pi)_x &= 0 \\
(\rho u_\perp)_t + (\rho u v + \pi_\perp)_x &= 0 \\
E_t + [(E + \pi)u + \pi_\perp \cdot u_\perp]_x &= 0 \\
(B_\perp)_t + (B_\perp u - B_x u_\perp)_x &= 0 \\
(\rho\pi)_t + [\rho\pi u + (c_s^2 + c_f^2 - c_a^2)u - c_a b \cdot u_\perp]_x &= \rho \frac{p + \frac{1}{2}B_\perp^2 - \frac{1}{2}B_x^2 - \pi}{\epsilon} \\
(\rho\pi_\perp)_t + (\rho\pi_\perp u + c_a^2 u - c_a b u)_x &= \rho \frac{-B_x B_\perp - \pi_\perp}{\epsilon}
\end{aligned}
$$

wave speeds for the system of extended magnetohydrodynamics:



A three wave approximate Riemann solver is obtained by:

$$\text{Set } c_s = c_a = c_f$$

Theorem

The approximate Riemann solver defined by this 3-wave relaxation is positive and defines a discrete entropy inequality if for all intermediate states we have:

$$\frac{1}{\rho_2} - \frac{B_x^2}{c_a^2} \geq 0$$

$$\left| \frac{B_\perp^1 + B_\perp^2}{2} - \frac{B_x b}{c_a} \right|^2 \leq \left(\frac{c_s^2 c_f^2}{c_a^2} - (\rho^2 p')_{1,2} \right) \left(\frac{1}{\rho_2} - \frac{B_x^2}{c_a^2} \right)$$

The proof of the discrete entropy inequality

$$\rho_i^{n+1}\phi(s(\rho_i^{n+1}, e_i^{n+1})) - \rho_i^n\phi(s(\rho_i^n, e_i^n)) + \frac{\Delta t}{h}\left(G^s_{i+\frac{1}{2}} - G^s_{i-\frac{1}{2}}\right) \leq 0$$

is given in Bouchut, Kl., Waagan (2006).

A formal derivation of this for smooth solutions is available by a Chapman-Enscog expansion.

Write $\pi = p + \frac{1}{2}B_\perp^2 - \frac{1}{2}B_x^2 + g(\epsilon) + O(\epsilon^2)$ $\quad \pi_\perp = -B_x B_x + +g_\perp \epsilon + O(\epsilon^2)$

Insert this into the extended system

$$\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + \pi)_x &= 0 \\
(\rho u_\perp)_t + (\rho u v + \pi_\perp)_x &= 0 \\
E_t + [(E + \pi)u + \pi_\perp \cdot u_\perp]_x &= 0 \\
(B_\perp)_t + (B_\perp u - B_x u_\perp)_x &= 0 \\
(\rho\pi)_t + [\rho\pi u + (c_s^2 + c_f^2 - c_a^2)u - c_a b \cdot u_\perp]_x &= \rho\frac{p + \frac{1}{2}B_\perp^2 - \frac{1}{2}B_x^2 - \pi}{\epsilon} \\
(\rho\pi_\perp)_t + (\rho\pi_\perp u + c_a^2 u - c_a b u)_x &= \rho\frac{-B_x B_\perp - \pi_\perp}{\epsilon}
\end{aligned}$$

This gives

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + \pi)_x = \epsilon \left[\left(\frac{c_s^2 + c_f^2 - c_a^2}{\rho} - (\rho p' + B_\perp^2) \right) u_x + (B_x B_\perp - \frac{B_x b}{c_a})(u_\perp)_x \right]_x + O(\epsilon^2)$$

$$(\rho u_\perp)_t + (\rho u v + \pi_\perp)_x = \epsilon \left[(B_x B_\perp - \frac{B_x b}{c_a}) u_x + (\frac{c_a^2}{\rho} - B_x^2)(u_\perp)_x \right]_x + O(\epsilon^2)$$

$$E_t + [(E+\pi)u + \pi_\perp \cdot u_\perp]_x = \epsilon \Bigg[u \left(\frac{c_s^2 + c_f^2 - c_a^2}{\rho} - (\rho p' + B_\perp^2) \right) u_x + u(B_x B_\perp - \frac{B_x b}{c_a}) \cdot (u_n)_x$$

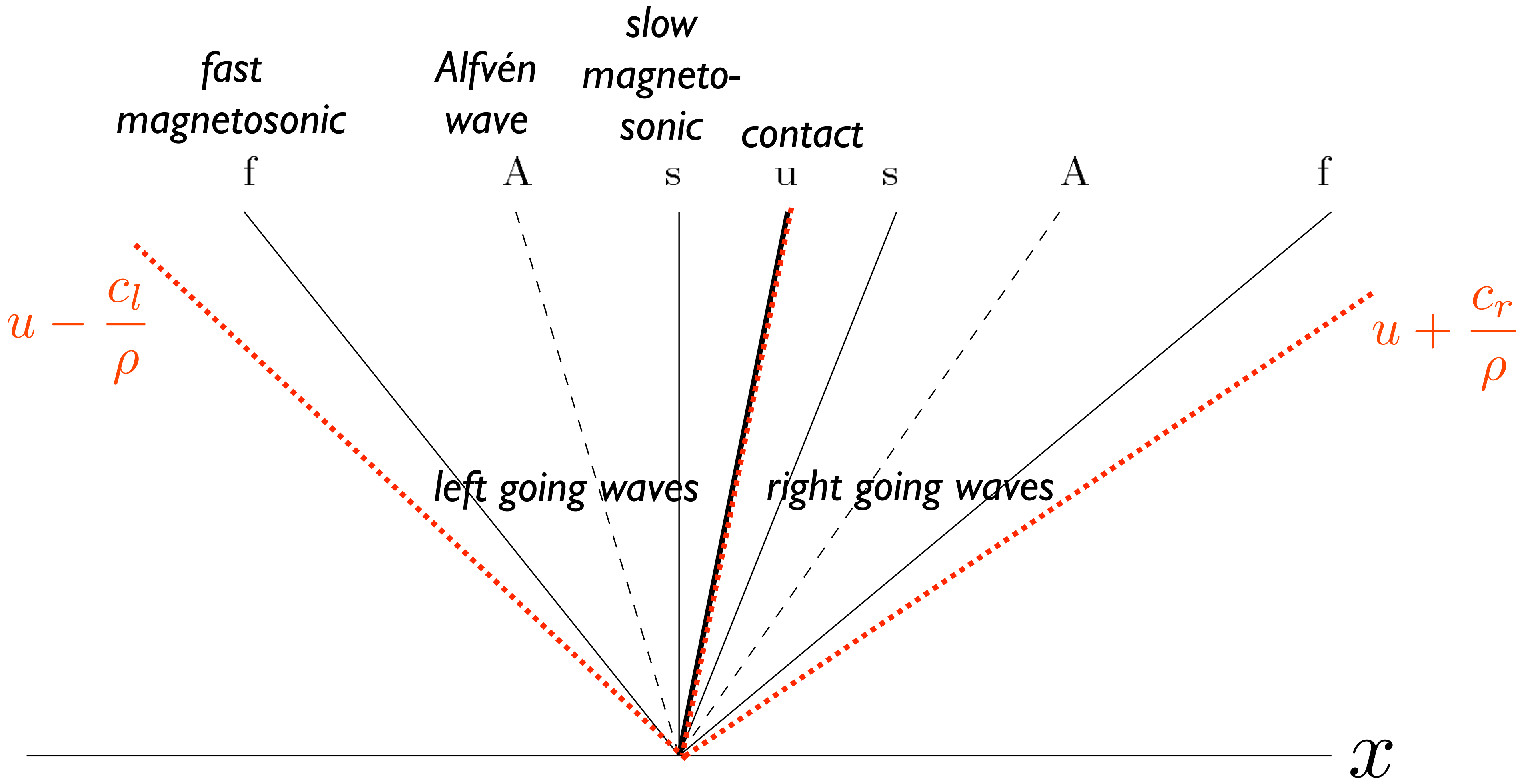
$$+ u_\perp \cdot (B_x B_\perp - \frac{B_x b}{c_a}) u_x + u_\perp \cdot (\frac{c_a^2}{\rho} - B_x^2)(u_\perp)_x \Bigg]_x + O(\epsilon^2)$$

$$(B_\perp)_t + (B_\perp u - B_x u_\perp)_x = 0$$

The entropy is evolved by an equation of the type

$$\eta(U)_t + G(U)_x - \epsilon[\eta'(U) D(U) U_x]_x = -\epsilon D(U)^t \eta''(U) U_x \cdot U_x$$

The conditions of the theorem then ensure entropy dissipation.



the three wave solver superimposed onto the exact 8-wave solution

When devising a numerical scheme we need to get concrete speeds of the waves out of the inequality in the theorem.

Bouchut, Klingenberg, Waagan: *A multiwave approximate Riemann solver for ideal MHD based on relaxation II - numerical aspects*, manuscript (2006)

Theorem:

For the three wave solver the following relaxation speeds are sufficient to guarantee positivity and entropy stability:

$$c_l = \rho_l a_l^0 + \alpha \rho_l \left((u_l - u_r)_+ + \frac{(\pi_r - \pi_l)_+}{\rho_l \sqrt{p_l'} + \rho_r a_{qr}} \right)$$

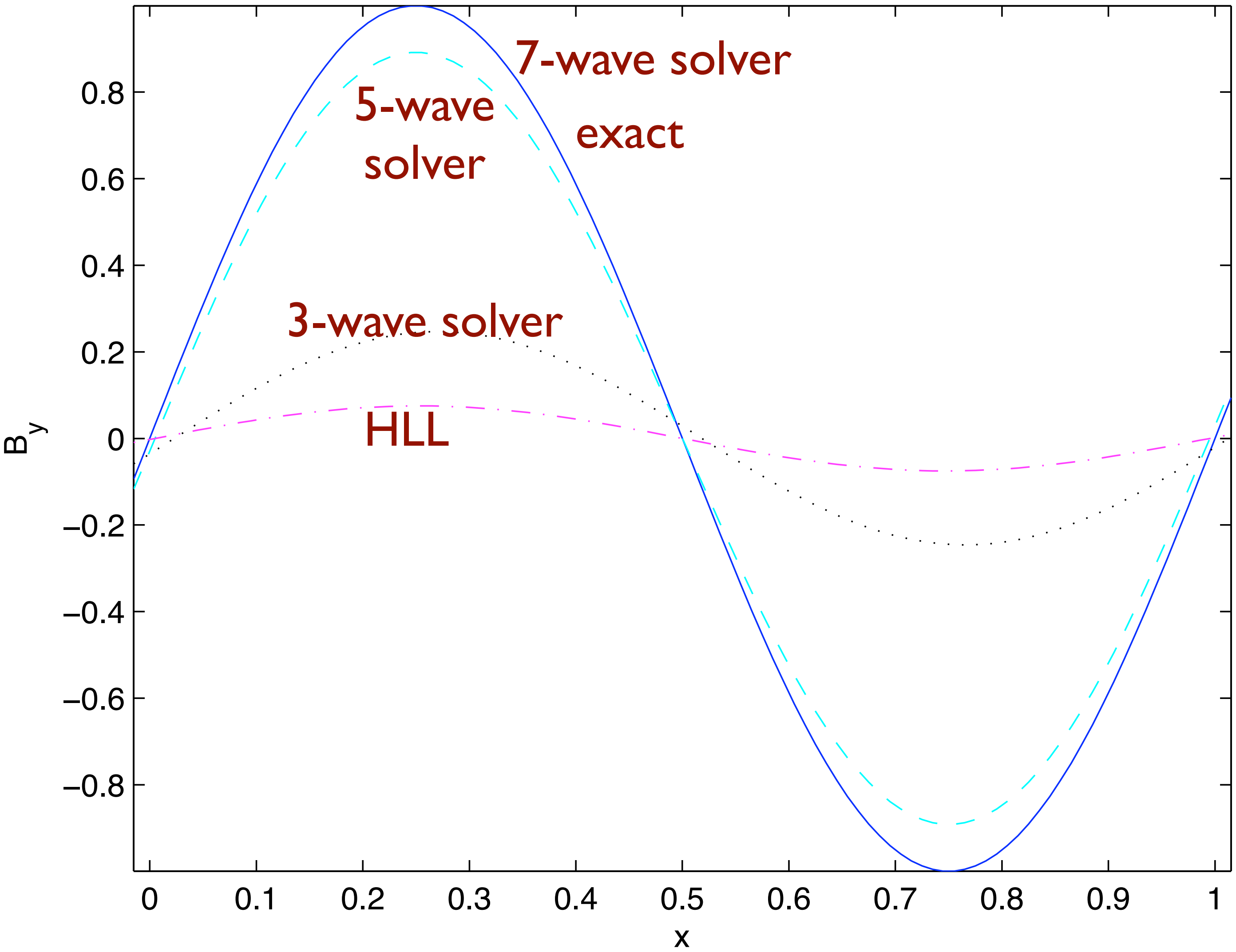
$$c_r = \rho_r a_r^0 + \alpha \rho_r \left((u_l - u_r)_+ + \frac{(\pi_l - \pi_r)_+}{\rho_r \sqrt{p_r'} + \rho_l a_{ql}} \right)$$

where $\alpha = \frac{\gamma + 1}{2}$ and $\alpha_l^0 \quad \alpha_r^0$ are given by a complicated formula.

We have also found a seven wave approximate solver.

again we can prove entropy consistency under some complicated "subcharacteristic" condition

We have explicit formulas for the speeds.



$$\rho = 1.0, \quad p = 1.0$$

$$B_y = -\sin(2\pi x) \quad B_z = -\cos(2\pi x)$$

$$v = \sin(2\pi x) \quad w = \cos(2\pi x)$$

$$B_x = 1.0, \quad \gamma = 5/3.$$

stationary left-going Alfven wave

We tested such a new approximate Riemann solver in an astrophysics code:

PROMETHEUS

developed in Garching since 1989 (Müller) ported to FLASH (in Chicago) and still used today.

This code solves the hydrodynamic equations and has additional physical effects implemented.

Klingenberg, Schmidt, Waagan: *Numerical comparison of Riemann solvers for astrophysical hydrodynamics*, Journal of Computational Physics (2007)

PROMETHEUS	***PROMETHEUS - modified (preliminary)***
PPM *(piecewise parabolic method)*	PPM with our Riemann solver
This uses an "exact" Riemann solver. It is higher order accurate.	This uses our approximate Riemann solver.

Our approximate Riemann solver satisfies the entropy condition

and it also ensures that density will not become negative.

The PPM method in PROMETHEUS can not guarantee this.

Thus PPM with our Riemann solver can not guarantee this.

Hence we have also changed the numerical method in PROMETHEUS which makes the method higher order accurate.

PROMETHEUS - modified:

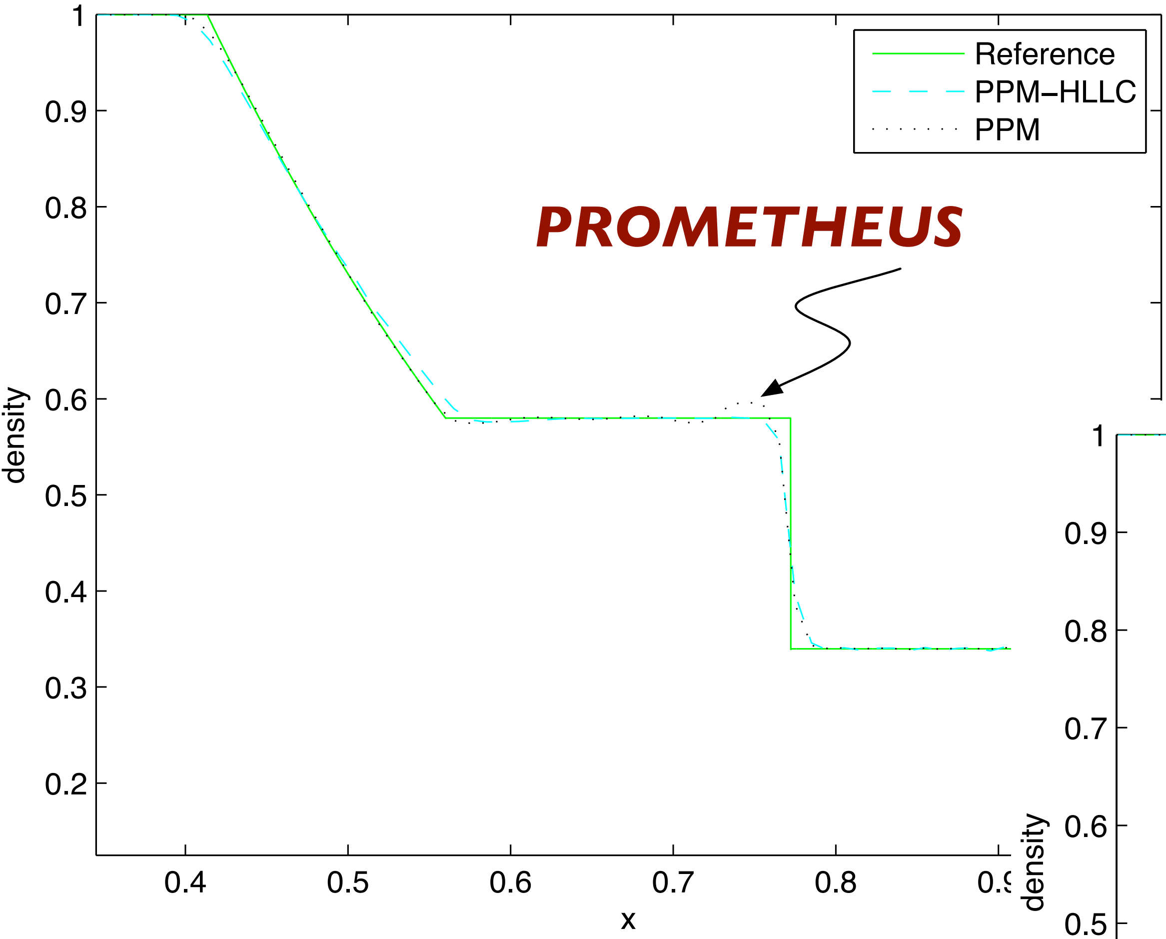
- our Riemann solver, made higher order such that positivity is preserved
- a new time integration was implemented (Runge-Kutta)

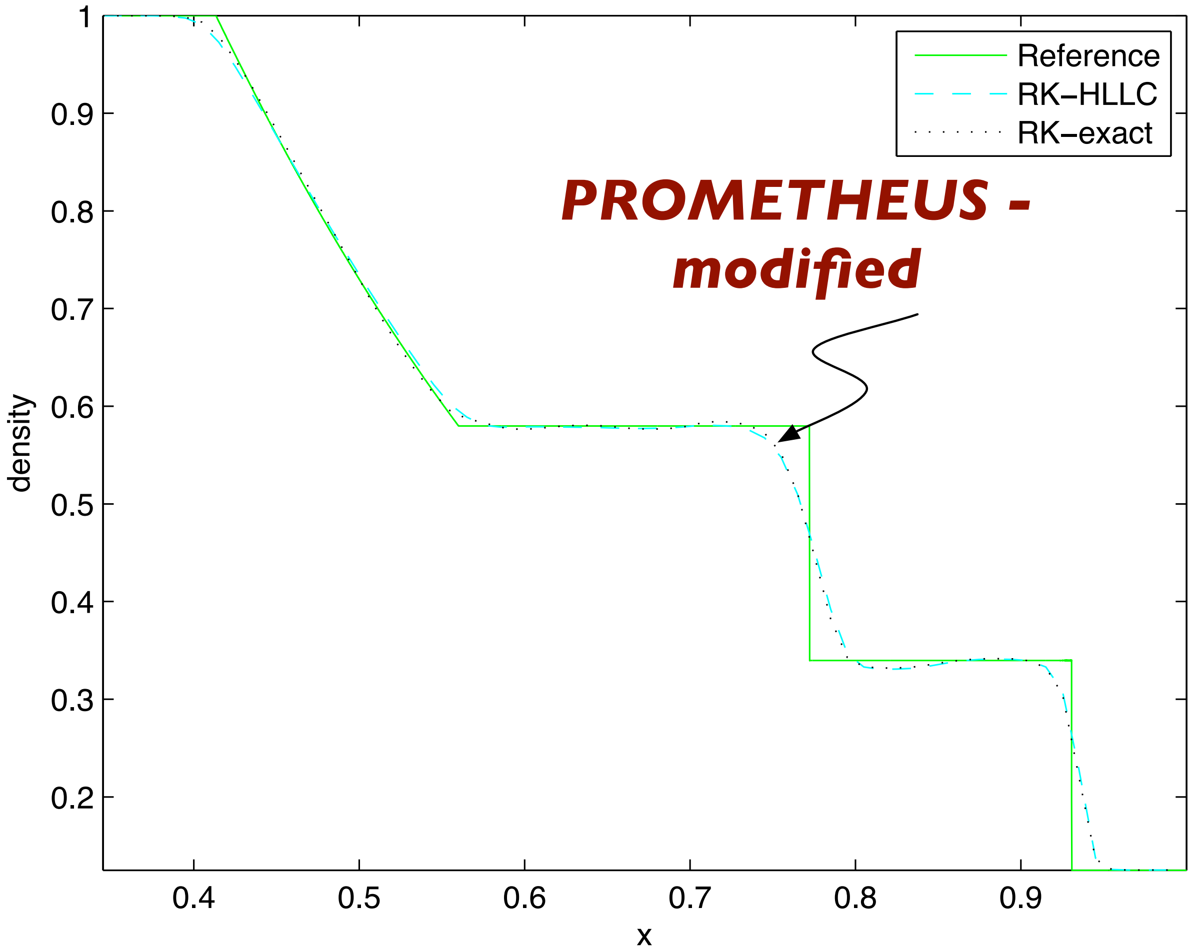
we compared these two codes:

- in one space dimension: particular Riemann problems
- in two space dimensons: mixing layers
- in three space dimensions: driven fully developed turbulence

one space dimension:

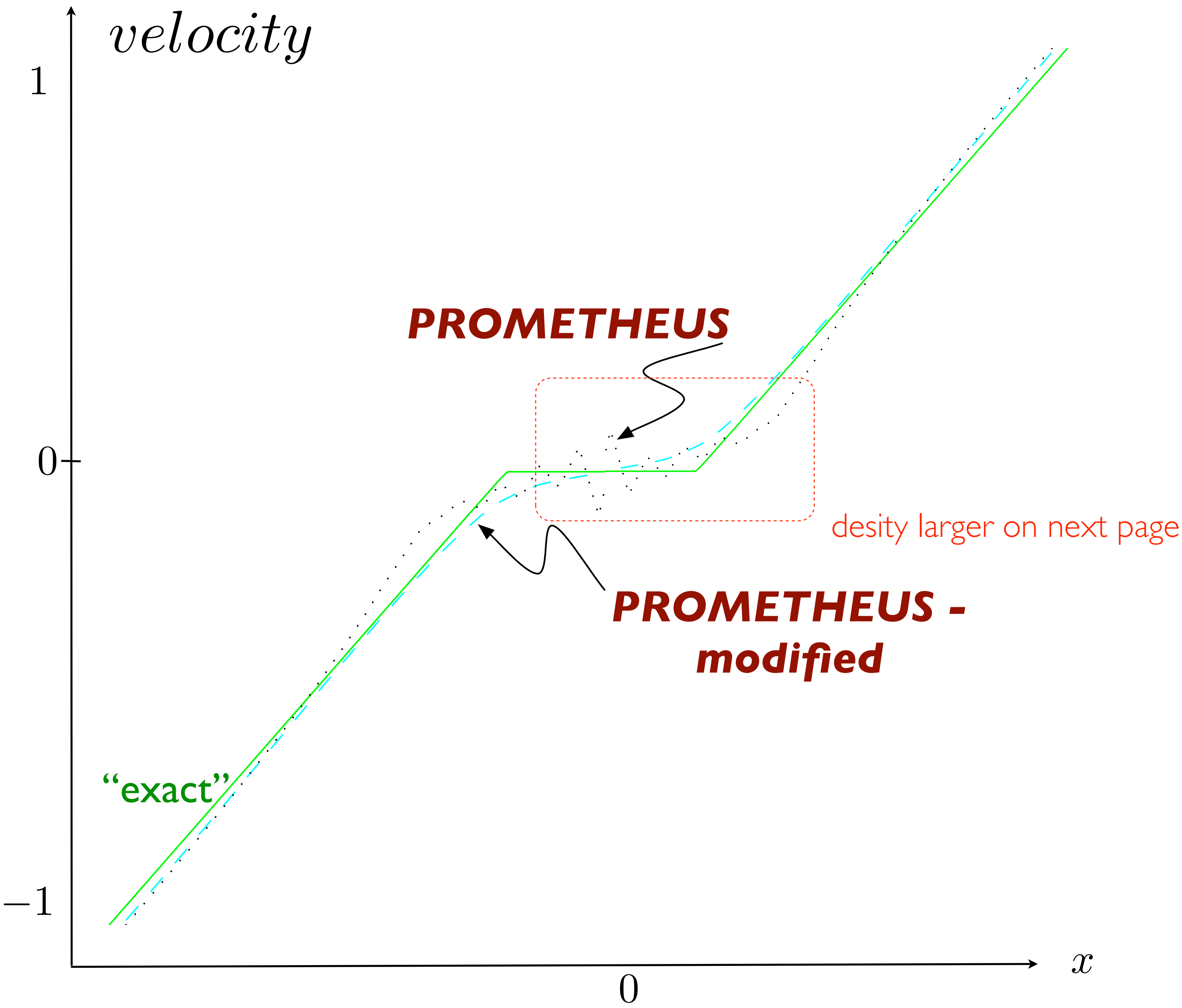
Riemann problems

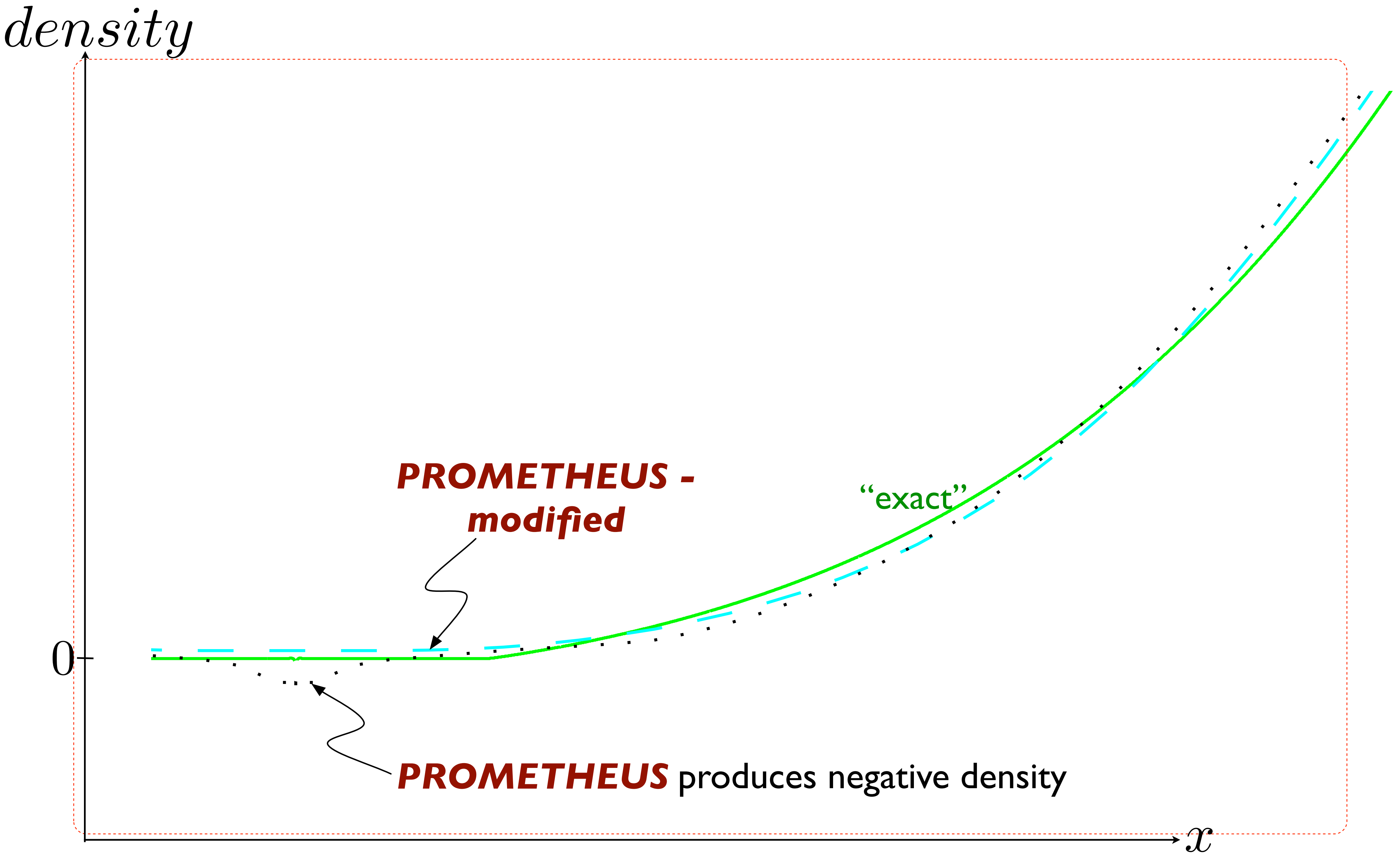




the modified code smears out a little more

This Riemann problem has two strong rarefaction waves going apart creating a low density region.

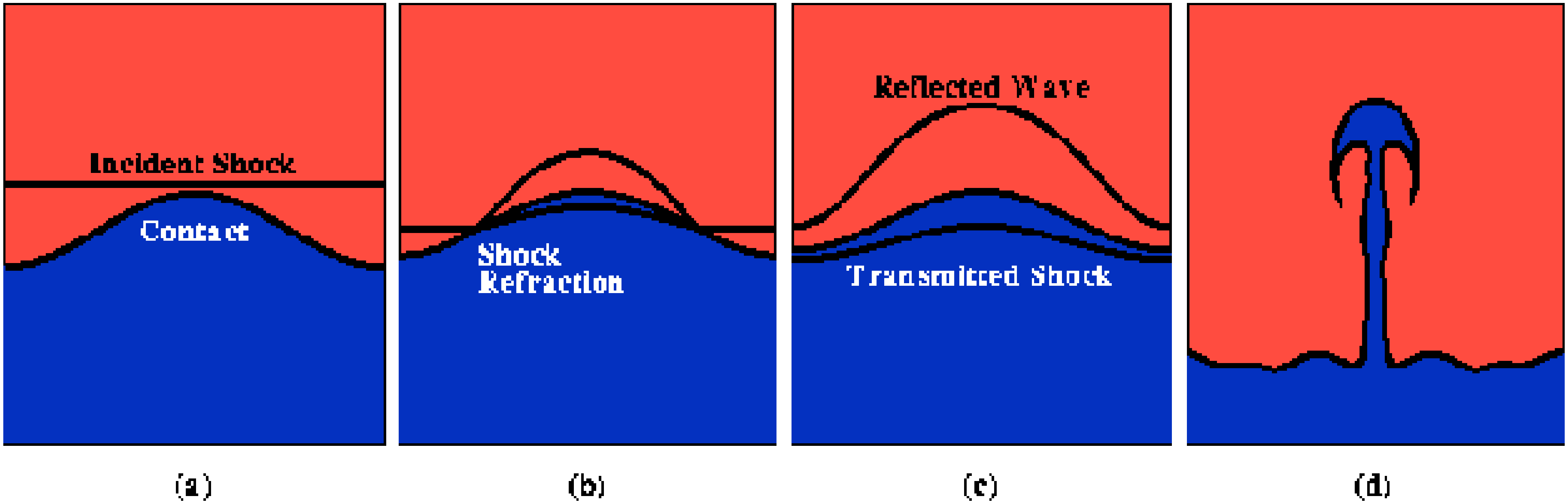


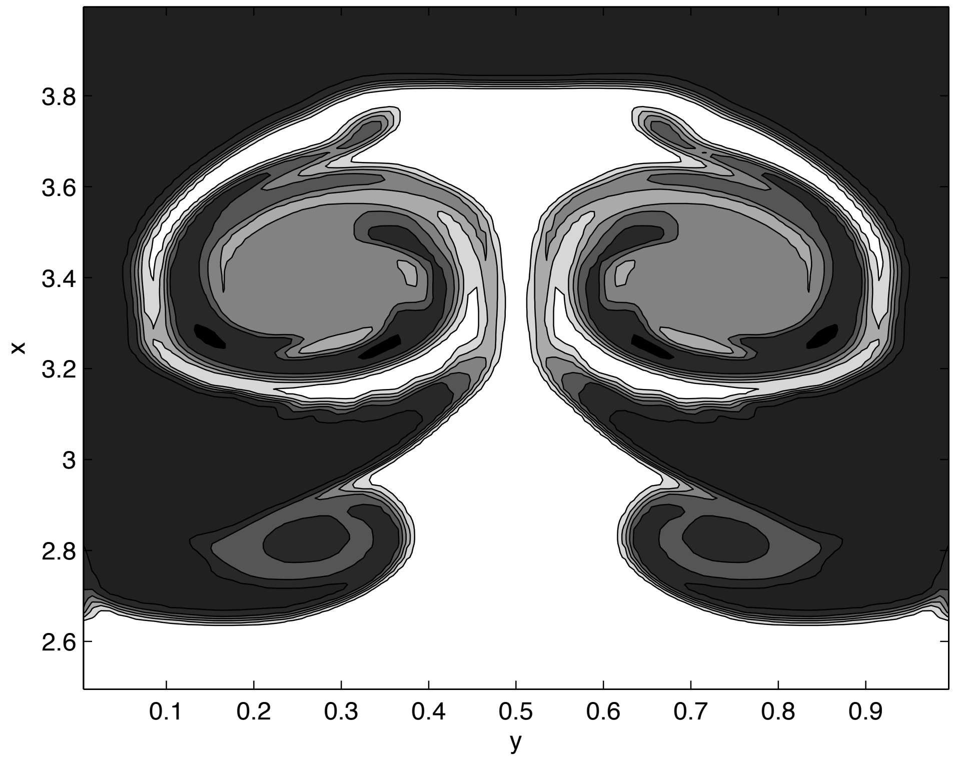


density
0
x
PROMETHEUS - modified
"exact"
PROMETHEUS produces negative density

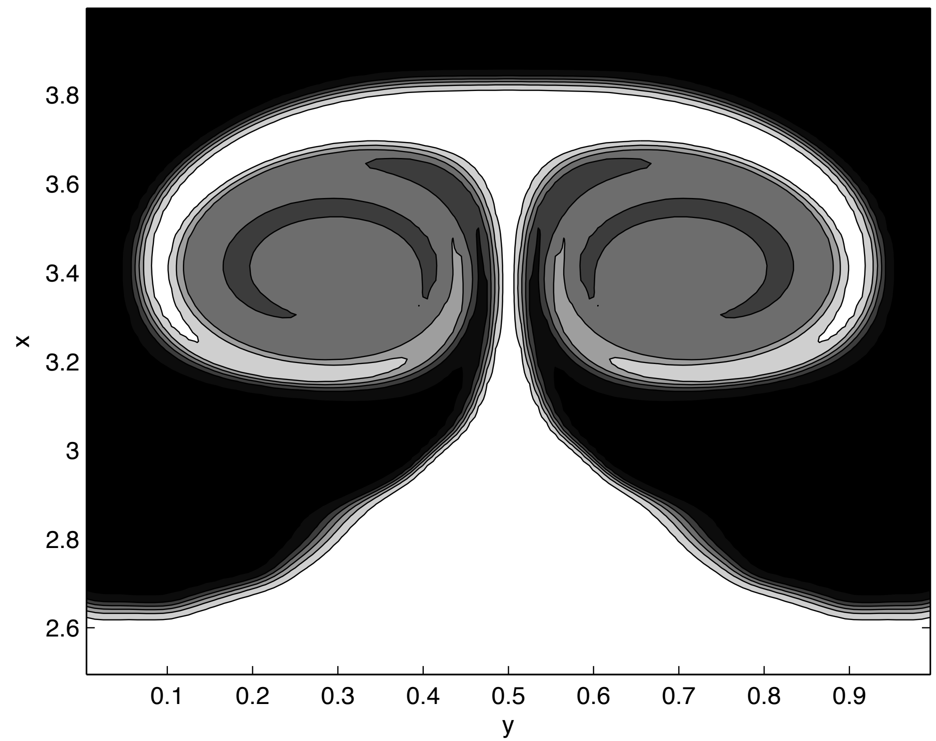
two space dimensions:

Richtmeyer-Meshkov instability

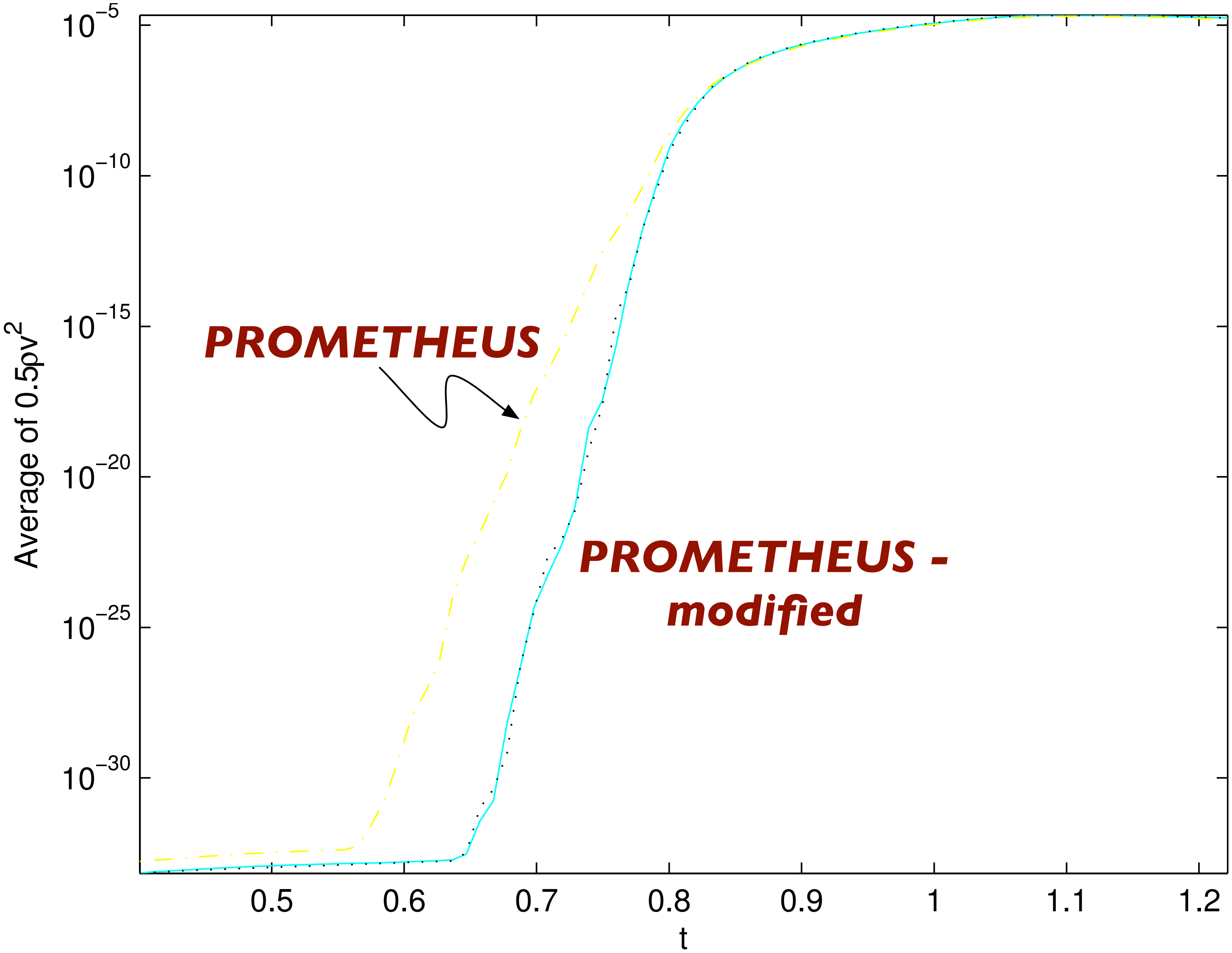




PROMETHEUS

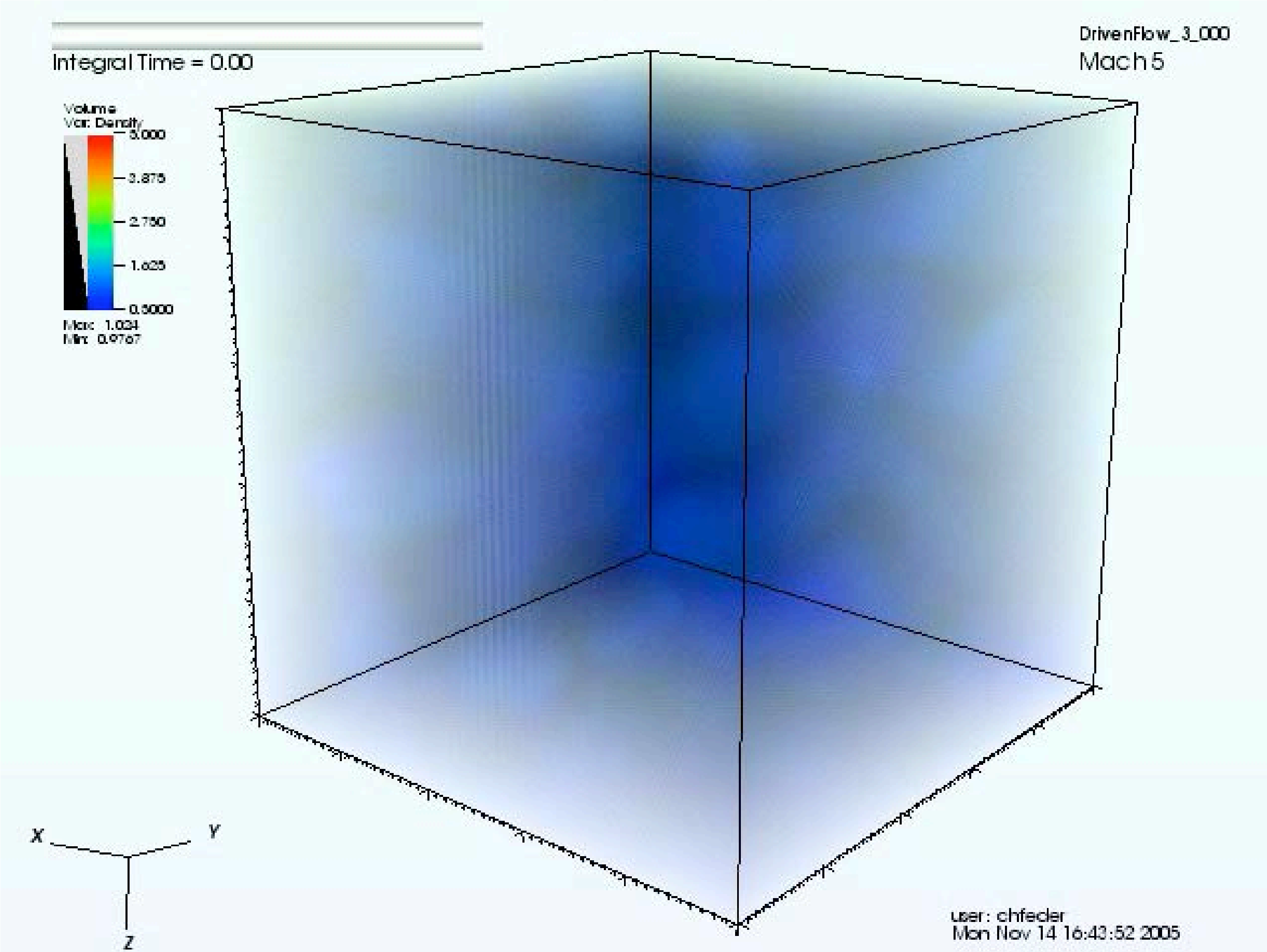


PROMETHEUS - modified:

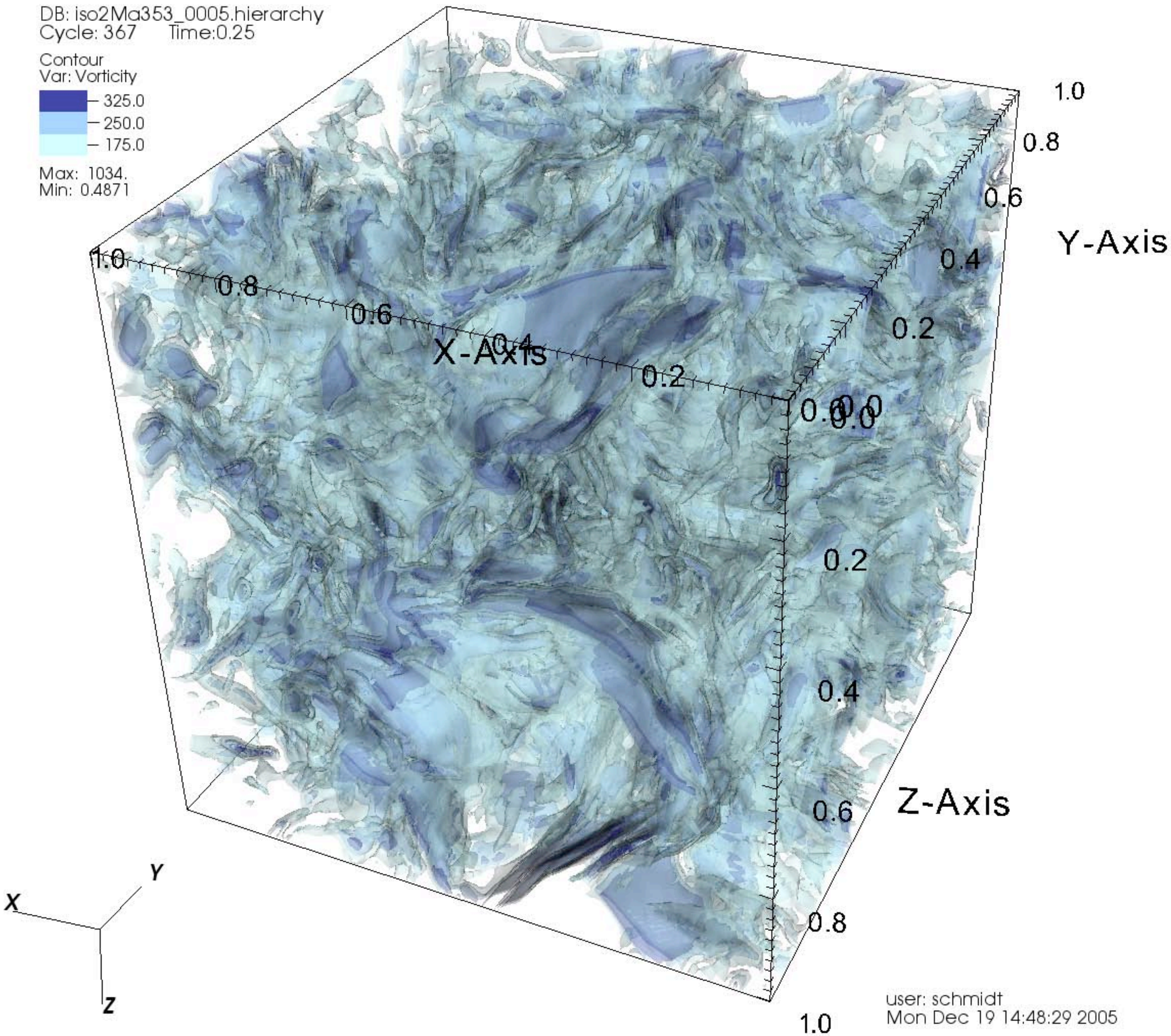


The growth of instability is similar for both codes

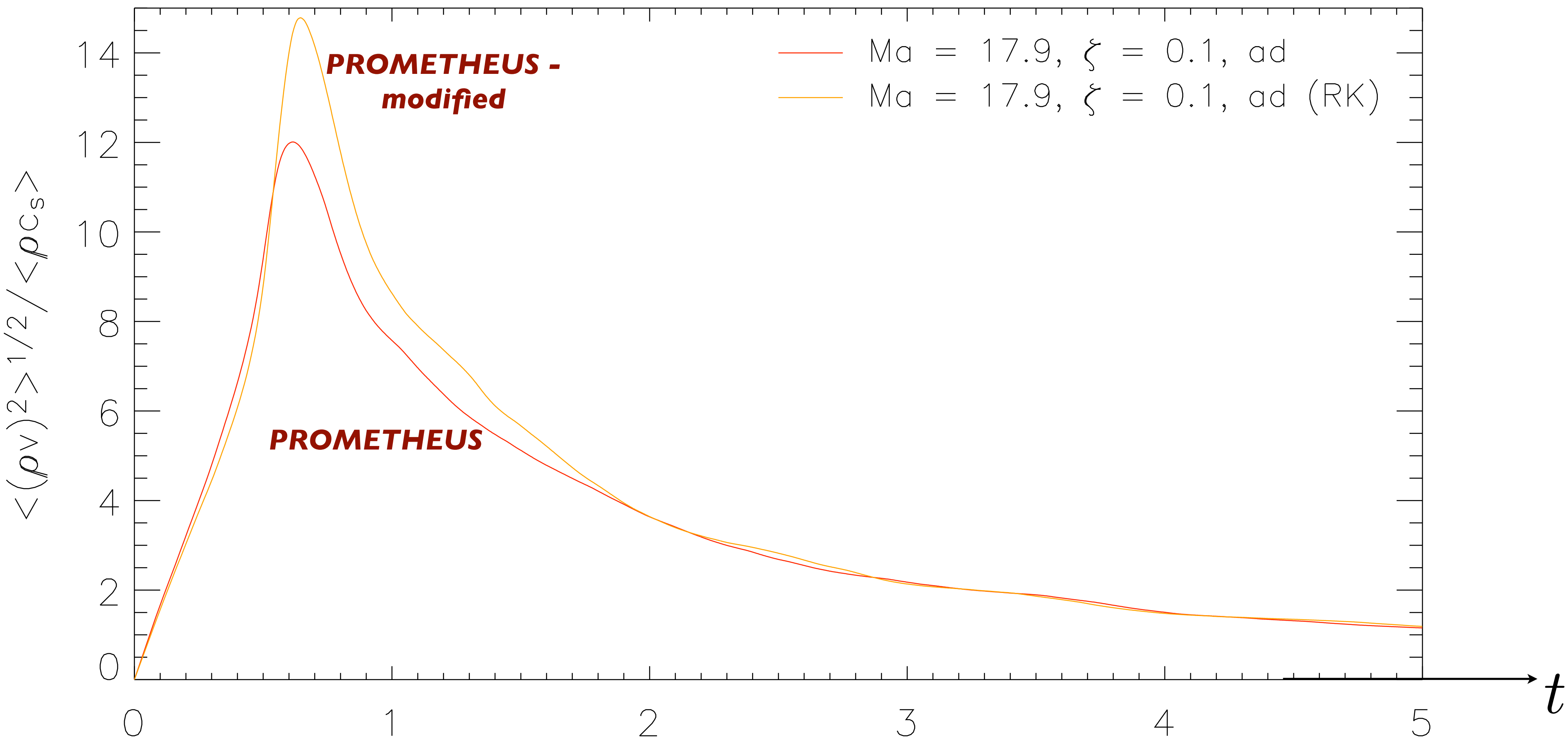
as seen here by transversal component of kinetic energy three space dimensions fully developed turbulence



Wolfram Schmidt, J. Niemeyer, Federrath (2006)

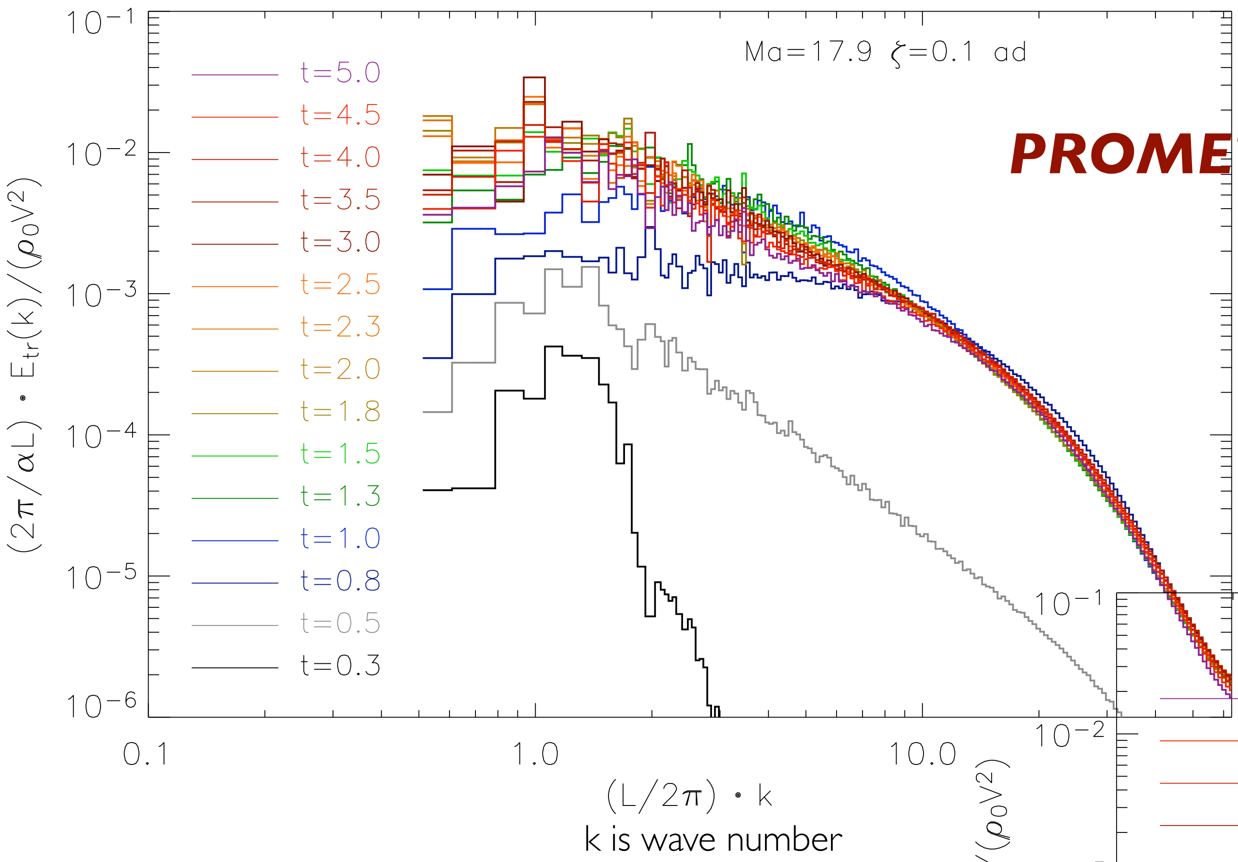


Wolfram Schmidt, J. Niemeyer (2006)

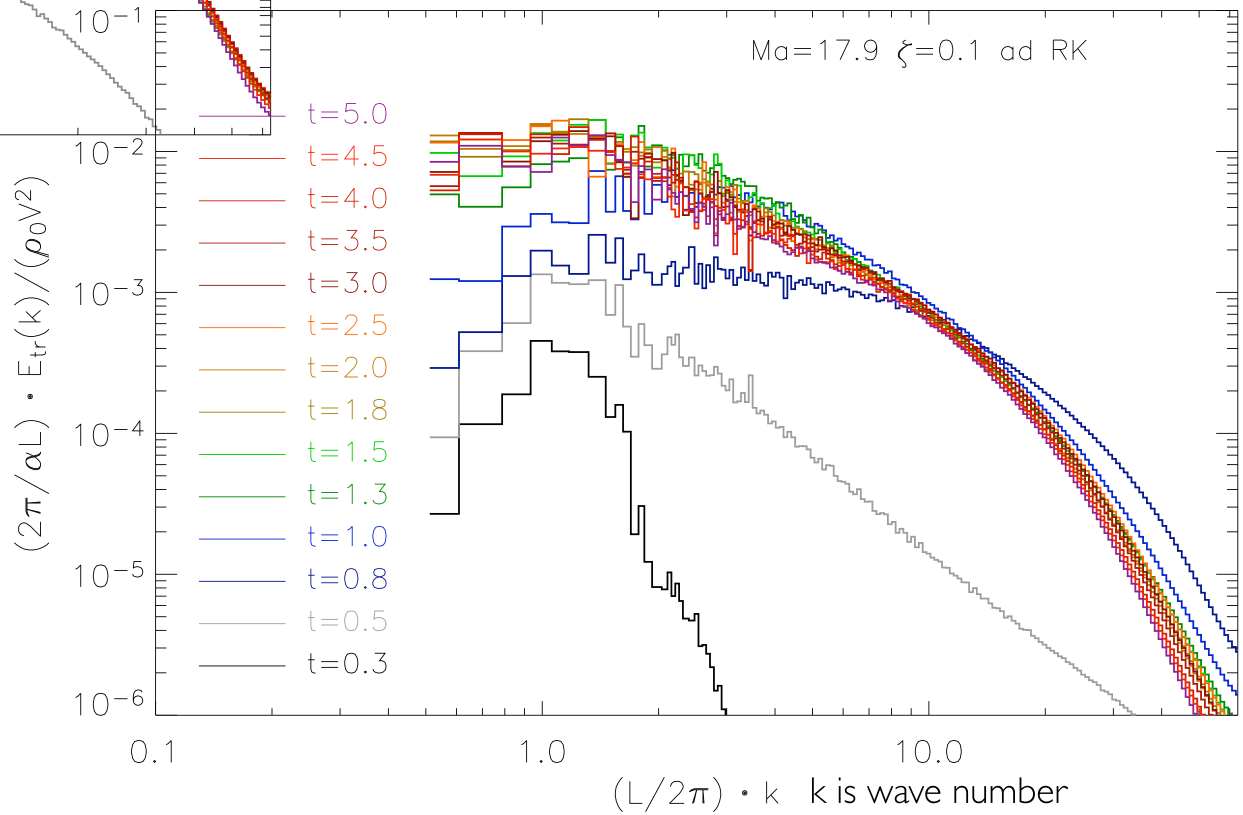


time evolution of root mean squared Mach number

PROMETHEUS



PROMETHEUS - modified



time histories of transversal energy spectra

Christoph Federrath

conclusion:

dissipativity of ***PROMETHEUS*** is independent of Mach number

dissipativity of ***PROMETHEUS-modified*** is less for higher than for lower Mach numbers

We conclude that PPM is accurate with respect to the Riemann solver.

Our approximate Riemann solver is at least 20% more efficient, though.