Chapter 2

A Newtonian Universe

Strictly speaking, Newtonian dynamics cannot give a description of the expansion of spacetime. Newtonian dynamics describes the motion of bodies in a *fixed* spacetime. For a *dynamic* spacetime we must go into General Relativity, which we shall do in Chapter 3. It turns out, however, that *with some tricks* one can create a Newtonian description of the expanding universe that is surprisingly accurate. Because of its relative simplicity, this description is so useful that it is used very often in studies of cosmology, and we will make ample use of it in this lecture. So in spite of the fact that a Newtonian approach is not a fully self-consistent description of the universe, it is extremely useful, and we will therefore discuss it here.

2.1 Velocity field (Hubble Flow)

In the Newtonian picture, it is not space that expands, but the galaxies that move away from each other. Let us take a coordinate system (x, y, z) such that $(0, 0, 0)$ is our location. We assume ourselves to be at zero velocity with respect to the microwave background radiation, so that $\vec{v} = (0, 0, 0)$ at our location. Far away galaxies then have a velocity

$$\vec{v}(\vec{r}) = H_0\vec{r} \tag{2.1}$$

where H_0 is the Hubble constant. We must now check whether this universe can be homogeneous: How would an observer at location $\vec{r}_1$ see the expansion of the universe? He would put $\vec{v}' = (0, 0, 0)$ at his location while for us his velocity is $\vec{v}_1 = H_0\vec{r}_1$. However, by a Galilei transformation:

$$\vec{v}'(\vec{r}) = \vec{v}(\vec{r}) - H_0\vec{r}_1 \tag{2.2}$$

we would obtain a velocity field that has $\vec{v}'(\vec{r}_1) = (0, 0, 0)$ and has an identical shape as that of ours, just that it is centered on the observer at $\vec{r}_1$:

$$\vec{v}'(\vec{r}) = H_0(\vec{r} - \vec{r}_1) \tag{2.3}$$

In other words: there is no “center of the universe”: each position is equivalent to the other, even in this Newtonian description.

2.2 Problem with Newtonian description + solution

Newtonian dynamics works well for systems with a finite size and with non-relativistic velocities. Both conditions will evidently be broken at large scales in the above Hubble Flow picture. If we, however, would focus only on a small portion of the universe, we break the condition of homogeneity. Suppose we would consider only a large-but-finite spherical volume around ourselves with some radius R and try to understand the universe with only that spherical portion of the universe, we would put ourselves at the "center of the universe" (the center of the sphere), which we know is artificial.

It turns out, however, that this is nonetheless a reasonable trick to do. Consider the universe as being built up of shells of matter centered on us. The density is in each shell the same: ρ. The mass of the shell between r and $r + dr$ is $4\pi\rho r^2 dr$. A particle at a distance R away from us, it feels the gravity force of all shells with $r < R$, but does not feel the gravity of the shells with $r > R$. This is because, as we know from classical mechanics, a spherical mass shell has a constant gravitational potential inside and hence any particle inside this shell feels no gravitational force from the shell.

One could argue: The choice of using ourselves as the central point of all these shells is arbitrary. It is. That is why the Newtonian description is somewhat artificial. But it turns out to work nonetheless!

2.3 A first simple powerlaw expansion model

Consider a sphere of homogeneously distributed *cold* (i.e. pressureless) matter. The sphere has a radius $R(t)$, where t ist time. The mass of the sphere is constant, because of mass conservation. So if the sphere expands, it means that the matter density $\rho(t)$ in the sphere declines:

$$M = \text{const} = \frac{4\pi}{3}\rho(t)R(t)^3 \tag{2.4}$$

A parcel of matter at the edge of the sphere feels a gravitational force per unit mass $f(t)$ of

$$f(r) = -\frac{GM}{R(t)^2} \tag{2.5}$$

Since this force decellerates the gas parcel, one can write

$$\frac{d^2R(t)}{dt^2} = f(r) = -\frac{GM}{R(t)^2} \tag{2.6}$$

Let's see if we can find a powerlaw solution to this equation. So let us take as *Ansatz*:

$$R(t) = R_0\left(\frac{t}{t_0}\right)^\alpha \tag{2.7}$$

Inserting this into the above equation yields:

$$\alpha(\alpha-1)\frac{R_0}{t_0}t^{\alpha-2} = -\frac{GM}{R_0^2}t_0^{2\alpha}t^{-2\alpha} \tag{2.8}$$

This gives an algebraic equation for α:

$$\alpha - 2 = -2\alpha \qquad \text{therefore} \qquad \alpha = \frac{2}{3} \tag{2.9}$$

Now we can insert this again into the above equation to find:

$$\frac{2}{3}\left(-\frac{1}{3}\right)\frac{R_0}{t_0^{2/3}}t^{-4/3} = -\frac{GM}{R_0^2}t_0^{4/3}t^{-4/3} \tag{2.10}$$

which gives

$$\frac{2}{9}R_0^3 = GMt_0^2 \tag{2.11}$$

or in other words:

$$\frac{2}{9}R^3 = GMt^2 \tag{2.12}$$

The Hubble "constant" H is defined as

$$H = \frac{\dot{R}}{R} = \frac{d\lg R}{dt} \tag{2.13}$$

which in this case is

$$H = \frac{2}{3}\frac{1}{t} \tag{2.14}$$

So we can replace t in the above equation with $t = (2/3)/H$, so that we obtain

$$\frac{2}{9}R^3 = GM\frac{4}{9}\frac{1}{H^2} \qquad \rightarrow \qquad R^3 = \frac{2GM}{H^2} \tag{2.15}$$

Now replace M with $(4\pi/3)\rho R^3$ and we get

$$1 = \frac{2G}{H^2}\frac{4\pi}{3}\rho \tag{2.16}$$

or in other words:

$$\rho = \frac{3H^2}{8\pi G} \equiv \rho_{\rm crit} \tag{2.17}$$

This is in fact the *critical density of cosmology*, the density for which the expanding universe is a flat spacetime and for which the expansion is between a closed and an open universe.

2.4 Analogy to escape velocity from a point mass

In the above example we looked at a single particle at radius $R(t)$ which only feels the gravity of all the mass inward of $R(t)$. Since this mass does not change with time (as all that mass is also collapsing), and since gravity of a sphere is the same as gravity from a point mass, we can regard the above problem identical to the problem of a test particle being thrown away radially from a point-like body of mass M. If thrown too slowly, the test particle will fall back. If thrown too fast, it will reach an asymptotically non-zero velocity. If thrown at a velocity that is "just right", it will just about escape, but with no energy to spare. The velocity is then (at every distance) the local escape velocity $v(R) = v_{\rm esc}(R) = \sqrt{2GM/R}$. The above powerlaw solution is this critical solution. The critical density is thus the density such that for each R the mass M is such that the recession velocity $v = H_0R = v_{\rm esc} = \sqrt{2GM/R}$.

This means that we can expect three types of universe expansions:

1. Subcritically expanding universe ($\rho > \rho_{\rm crit}$), which will eventuall recollapse onto itself. We will discuss this case in Section 2.5. In the general relativistic version of cosmology this will be the *closed universe* with positive curvature.

2. Critically expanding universe ($\rho = \rho_{\rm crit}$), which will expand forever, but with ever-decreasing velocity. We have discussed this case already in Section 2.3. The Hubble constant goes as $H(t) = \frac{2}{3t}$. In the general relativistic version of cosmology this will be the *flat universe*.

3. Supercritically expanding universe ($\rho < \rho_{\rm crit}$), which will expand forever, with a non-zero asymptotic velocity for each given galaxy. Since this galaxy is then (in this limit) linearly moving away from us, the Hubble constant will then, at large times, go as $H(t) = 1/t$. In other words: at large t the universe expands ballistically as if there were no gravity. In the general relativistic version of cosmology this will be the *open universe* with negative curvature.

2.5 A collapsing Universe

Let's return to the basic equation (Eq. 2.6):

$$\ddot{R} = -\frac{GM}{R^2} \tag{2.18}$$

The first integration of this equation can be written in the form

$$\dot{R}^2 = \frac{2GM}{R} + C \tag{2.19}$$

where C is an integration constant. You can verify this by taking the d/dt of Eq. (2.19):

$$2\dot{R}\ddot{R} = -\frac{2GM}{R^2}\dot{R} \tag{2.20}$$

which gives Eq. (2.18) back. The integration constant C decides which of the three solutions you'll get. Eq. (2.19) in fact says that the kinetic enegy equals the potential energy plus some constant.

If we seek a collapsing solution, there will be a time t_{ta} called the "turn around time" where the maximum $R = R_{\text{ta}}$ is reached and after which the solution collapses. The solution will be symmetric about this point in time. At this turn around time we will have $\dot{R} = 0$.

Let us rewrite Eq. (2.19) as

$$\dot{R} = \pm\sqrt{\frac{2GM}{R} + C} \tag{2.21}$$

Since, for the recollapsing solution, we know that $C < 0$ and that there will be a R_{ta}, for which

$$R_{\text{ta}} = -\frac{2GM}{C} \tag{2.22}$$

we can introduce

$$y := \frac{R}{R_{\text{ta}}} \tag{2.23}$$

for which holds that $0 \leq y \leq 1$. Eq. (2.21) then becomes

$$\dot{y} = \pm\left(\frac{2GM}{R_{\text{ta}}^3}\right)^{1/2}\sqrt{\frac{1}{y} - 1} \tag{2.24}$$

If we now also introduce a dimensionless time θ such that

$$\theta = \left(\frac{2GM}{R_{\text{ta}}^3}\right)^{1/2} t \tag{2.25}$$

we obtain the fully dimensionless equation

$$\frac{dy}{d\theta} = \pm\sqrt{\frac{1}{y} - 1} \tag{2.26}$$

To solve this, it is easiest to write this as

$$\frac{d\theta(y)}{dy} = \pm\sqrt{\frac{y}{1-y}} \tag{2.27}$$

where we regard θ as a function of y instead of vice-versa. You can verify that

$$\theta = \frac{1}{2}\arcsin(2y-1) - \sqrt{y - y^2} + D \tag{2.28}$$

(where D is an integration constant) is a solution of Eq. (2.27), where one uses $d\arcsin(x)/dx = 1/\sqrt{1-x^2}$. Now, we wish to choose D such that the start of the expansion ($y = 0$) is at $\theta = 0$. Putting $y = 0$ into Eq. (2.28) gives $\theta(y = 0) = -\pi/4 + D$, so this means that we choose $D = \pi/4$:

$$\theta = \frac{1}{2}\arcsin(2y - 1) - \sqrt{y - y^2} + \frac{\pi}{4} \tag{2.29}$$

The maximum size ($y = 1$) is then reached at $\theta_{\rm ta} = \pi/2$, and the collapse ends in a "big crunch" at $\theta_{\rm bc} = \pi$. In real time these are:

$$t_{\rm ta} = \frac{\pi}{2}\left(\frac{R_{\rm ta}^3}{2GM}\right)^{1/2} \qquad t_{\rm bc} = \pi\left(\frac{R_{\rm ta}^3}{2GM}\right)^{1/2} \tag{2.30}$$

Now suppose we are currently at a time $t_0 < t_{\rm ta}$, and at this time we know the Hubble constant H_0 and the density of the universe ρ_0, for which $\rho_0 > \rho_{\rm crit}$. Can we now calculate when the universe reaches its maximal size and when it crunches? To do this we go back to Eq. (2.21), and insert Eq. (2.22) to eliminate the integration constant C:

$$\dot{R} = \pm\sqrt{\frac{2GM}{R} - \frac{2GM}{R_{\rm ta}}} \tag{2.31}$$

With $H_0 = \dot{R}/R$ at $t = t_0$ and squaring we get

$$H_0^2 = \frac{2GM}{R_0^3}\left(1 - \frac{R_0}{R_{\rm ta}}\right) \tag{2.32}$$

Replacing M with $(4\pi/3)\rho_0 R_0^3$ we get

$$H_0^2 = \frac{8\pi G}{3}\rho_0\left(1 - \frac{R_0}{R_{\rm ta}}\right) \tag{2.33}$$

With Eq. (2.17) and the definition of the scaling parameter $a(t) \equiv R(t)/R_0$ we get turn-around at a scaling parameter value of:

$$a_{\rm ta} \equiv \frac{R_{\rm ta}}{R_0} = \frac{\rho_0}{\rho_0 - \rho_{\rm crit,0}} \tag{2.34}$$

which, with Eq. (2.30), occurs at time

$$t_{\rm ta} = \frac{\pi}{2}\left(\frac{3R_{\rm ta}^3}{8\pi G\rho_0 R_0^3}\right)^{1/2} = \frac{\pi}{2H_0}\left(\frac{\rho_{\rm crit,0}R_{\rm ta}^3}{\rho_0 R_0^3}\right)^{1/2} = \frac{\pi}{2H_0}\frac{\rho_0/\rho_{\rm crit,0}}{(\rho_0/\rho_{\rm crit,0} - 1)^{3/2}} \tag{2.35}$$

and the "big crunch" happens at twice that time. Note that time is measured from the big-bang, not from today.

2.6 Adding a cosmological constant

Within the Newtonian dynamics approach it is relatively easy to add a cosmological constant. The idea is that you can add an additional body force in the system of equations, as long as this force $\vec{f}$ is proportional to $\vec{r}$. The reason for that condition is that the universe must remain isotropic and homogeneous. Isotropy means that $\vec{f}$ must be parallel to $\vec{r}$. And homogeneity means that $\dot{v}/v$ cannot depend on distance. Also, the proportionality constant must be constant with time, because otherwise one would have to introduce a new dynamically evolving quantity. By convention this constant is written as $\Lambda/3$:

$$\vec{f} = \frac{\Lambda}{3}\vec{r} \tag{2.36}$$

where $\vec{f}$ is the force per gram (i.e. an acceleration). The dimension of Λ is therefore $1/\text{second}^2$. Equation (2.6) then becomes

$$\frac{d^2R(t)}{dt^2} = -\frac{GM}{R(t)^2} + \frac{\Lambda}{3}R(t) \tag{2.37}$$

Note that this "ghost force" scales in the same way as the gravitational force. One could therefore in principle generate a static universe if you choose Λ such that

$$\Lambda = 4\pi G\rho \quad \rightarrow \quad \text{Static universe} \tag{2.38}$$

that is, if you start with $\dot{R} = 0$. However, this would be an unstable universe. A small deviation from this equilibrium would cause the universe to move exponentially away from this equilibrium. You can see this by writing Eq. (2.37) with Eq. (2.38):

$$\frac{d^2R(t)}{dt^2} = -\frac{GM}{R(t)^2} + \frac{GM}{R_0^2} \tag{2.39}$$

and linearizing via

$$R(t) = R_0(1 + \epsilon(t)) \tag{2.40}$$

You get to first order in ϵ

$$R_0\frac{d^2\epsilon(t)}{dt^2} = -\frac{GM}{R_0^2}(1 - 2\epsilon) + \frac{GM}{R_0^2} + O(\epsilon^2) \tag{2.41}$$

which reduces to

$$\frac{d^2\epsilon(t)}{dt^2} = \frac{2GM}{R_0^3}\epsilon(t) + O(\epsilon^2) = \frac{8\pi G\rho}{3}\epsilon(t) + O(\epsilon^2) \tag{2.42}$$

which has solutions

$$\epsilon(t) = \epsilon_0 \exp\left(\pm\sqrt{\frac{8\pi G\rho}{3}}t\right) \tag{2.43}$$

meaning that there are growing modes with an e-folding growth time of

$$t_{\text{growth}} = \sqrt{\frac{3}{8\pi G\rho}} \tag{2.44}$$