Chapter 3

General Relativity in a Nutshell

As already mentioned before, a Newtonian description of the universe has only limited validity, especially in the early universe. This lecture, however, does not assume any prior knowledge of General Relativity (GR). It will be impossible to give a sufficiently in-depth introduction to GR in this lecture, but I also do not want to skip it entirely. So we will start this chapter with a very "quick and dirty" introduction to GR. We will, however, assume that you have knowledge of tensor calculus and that you are familiar with special relativity.

3.1 Minkowski spacetime revisited

Before we start to discuss curved space and curved spacetime, let us recapitulate the concept of Minkowski spacetime and the metric, just to get the notations right and refresh our memories.

In special and general relativity space and time are treated as a unity. We thus get a 4-D spacetime with x^μ denoting a point in this spacetime. The x^0-coordinate denotes time (with $x^0 \equiv ct$, with c the light speed and t the time in seconds) while the x^i-coordinates (with $i = 1, 2, 3$) denote the spatial coordinates. In flat spacetime with a cartesian coordinate system the metric is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu} \tag{3.1}$$

This is called the Minkowski metric. Note that the sign convention is also often chosen to be +1 for the time component and -1 for the spatial components. This is just a convention. As another convention we shall denote spacetime indices with Greek symbols $\alpha, \beta, \mu, \nu = 0, 1, 2, 3$, while latin characters denote purely spatial coordinates: $i, j, k, l = 1, 2, 3$. Finally, we always adopt the *Einstein summation convention*, i.e. we sum over all indices that appear twice in a formula. In this way the length of a 3-vector v^i is $|v|^2 = g_{ij}v^iv^j$ and the length of a 4-vector u^μ is $|u|^2 = g_{\mu\nu}u^\mu u^\nu$.

Any coordinate transformation that leaves the form of the metric of Eq. 3.1 intact is called a *Lorentz transformation* (which includes rotations). We assume that you know what these transformations are.

The spatial location x^i of some observer as a function of time t is called the *world line* of that observer, and is written as $x^i(t)$. At any time we can define the 3-velocity v^i as

$$v^i(t) = \frac{dx^i(t)}{dt} \tag{3.2}$$

This velocity is limited by the light speed, i.e. $g_{ij}v^i v^j < c^2$.

If we define the *proper time* τ as the time that the observer would measure him/herself (i.e. the time on his/her own watch), then we can define the 4-velocity u^μ as

$$u^\mu(t) = \frac{dx^\mu(t)}{d\tau} \tag{3.3}$$

and we have

$$g_{\mu\nu}u^\mu u^\nu = -c^2 \tag{3.4}$$

Its energy-momentum 4-vector is

$$p^\mu = mu^\mu \tag{3.5}$$

For an observer that is moving at speeds very much slower than the light speed with respect to our coordinate system we have $u^\mu \simeq (c + \frac{1}{2}v^2/c, v^1, v^2, v^3)^\mu$.

If we have not just one particle moving through our Minkowski spacetime, but a collection of particles that together can be regarded as a gas, then we have to introduce the *stress-energy tensor* $T^{\mu\nu}$.

Suppose that the matter is so cold that it has neglible pressure. Relativists call this "dust", though this should not be taken literally. The energy-momentum tensor for "dust" with a density ρ and 4-velocity u^ν is

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{3.6}$$

If in our current coordinate system, and at a given point P, the dust is not moving, then the stress-energy tensor of this dust is thus:

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

If, however, our gas *does* have a pressure p, then the stress energy tensor becomes

$$T^{\mu\nu} = \rho u^\mu u^\nu + p\left(g^{\mu\nu} + \tfrac{1}{c^2}u^\mu u^\nu\right) \tag{3.8}$$

If in our current coordinate system, and at a given point P, the gas is not moving, then the stress-energy tensor of this gas is thus:

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{3.9}$$

Note that the density ρ in this case includes thermal energy:

$$\rho = \rho_{\text{restmass}} + \frac{\epsilon_{\text{thermal}}}{c^2} \tag{3.10}$$

For non-relativistic matter the $\epsilon_{\text{thermal}}/c^2$ term is negligibly small, so one can write $\rho \simeq \rho_{\text{restmass}}$. For isotropic radiation we have the other extreme: the $\rho_{\text{restmass}} = 0$ so that $\rho = \epsilon_{\text{thermal}}/c^2$. It turns out that for an isotropic radiation field we have

$$\rho = \frac{\epsilon_{\text{thermal}}}{c^2} = \frac{3p}{c^2} \tag{3.11}$$

Conservation of energy and momentum can be written compactly as

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} \equiv \partial_\nu T^{\mu\nu} \equiv T^{\mu\nu}{}_{,\nu} = 0 \tag{3.12}$$

Now let us place these concepts into GR. In GR we can always locally regard spacetime as a Minkowski spacetime; it is only globally that it becomes clear that spacetime is curved by the presence of matter. So let us have a look at what "curved spacetime" is by taking examples from "curved space" that we are familiar with.

3.2 Curved space: An introduction to the concept

GR consists for 90% out of the mathematical characterization of curved spacetime. The mathematical theory behind this is called Riemannian Geometry. So let us first get some understanding of what "curved space" is. The easiest example is the 2-D surface of a sphere. Take the Earth. As you know, if you make a map of the earth on a flat sheet of paper, there is no way to avoid geometric distortions. Usually the projections used will strongly distort the north- and southpolar regions, making Siberia and Greenland appear much larger than they really are. The surface of a sphere is therefore a 2-D curved space. To measure the curvature you first have to be able to measure lengths and distances. If you could see human beings on the map of the earth, you would notice that their shape (as seen on the 2-D flat projection) is strongly stretched in longitudinal direction. More mathematically: we introduce a metric, which can be visualized as a unit circle, at every point on the surface. Now you see that the circles near the polar region are (in the flat projection) ellipses that are strongly stretched in longitudinal direction. By looking how these metric circles change between different points on the map you can tell something about the spatial curvature. As we know, the metric is a second-rank covariant tensor g_{ij}. For a flat 2-D space (x, y) one would have

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{ij} \tag{3.13}$$

(if one chooses the coordinate system wisely). For the surface of the Earth, with coordinate system (θ, ϕ), we would instead have a metric that depends on where you are:

$$g_{ij} = R^2_{\text{Earth}} \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{pmatrix}_{ij} \tag{3.14}$$

Here θ is the lattitude (in radians from the equator) and ϕ is the longitude (in randians from Greenwich).

The metric is also often written as a *line element* ds. For flat 2-D space (x, y) we would have:

$$ds^2 = dx^2 + dy^2 \tag{3.15}$$

while for the above surface of the Earth example we have:

$$ds^2 = R^2_{\text{Earth}}(d\theta^2 + \cos^2\theta d\phi^2) \tag{3.16}$$

For any space (curved or not) we can always find a coordinate system where the metric becomes $g_{ij} = \delta_{ij}$ (as in Eq. 3.13) at *any single (!) point in space of your choice*. But where a curved space is different from a flat one is that for a curved space one cannot find a global coordinate system for which $g_{ij} = \delta_{ij}$ at *all points at the same time*, while for flat space we can. This is the reason why we can always make a local map for any place on Earth which looks undistorted. But as soon as we want to make a map of the entire Earth, we always get distortions. A curved space, if we look at small enough scales, looks approximately flat. Mathematically more complete: if we choose a point P in space, then we can always find a coordinate system for which $g_{ij} = \delta_{ij}$ and even $\partial_k g_{ij} = 0$, i.e. physically straight lines will also look approximately straight. However, if space is curved one can usually *not* find a coordinate system for which, at that point P, *also* the second derivatives $\partial_k\partial_l g_{ij}$ are zero. The second derivatives of the metric will, as we will see, contain information about the intrinsic curvature of space, i.e. "physical curvature", i.e. curvature that is not just a bad choice of coordinate system.

Now that we have seen some examples of 2-D curved spaces, let us from now on immediately generalize this to 4-D curved spacetime! This works in exactly the same way as we wrote above; just we replace the Latin indices with Greek indices.

3.3 Defining "parallel transport"

From here until end of this chapter everything is voluntary and will not be part of the exam.

Now that we have a metric, and can thus measure distances in our space, we need to get a better understanding of how the concept of "direction" is changed in curved spaces. The way to do this is by defining the concept of "parallel transport" of vectors and tensors. The simplest example is again the Earth's surface. Suppose we make a travel from some point A on the Equator to the North Pole, turn 90 degrees to the right, travel back to the Equator, arriving there at point B, then turn again 90 degrees to the right and travel back to point A. If, on our travel, we bring along a vector pointing initially in the direction of our travel (i.e. pointing north). We keep the vector always in the same direction (but horizontal). On the second leg of our trip the vector does no longer point along the travel, but 90 degrees left. On the final leg, the arrow points again along the direction of motion, but to the back. When we arrive back at A the vector has changed direction by 90 degrees, even though we always faithfully kept the vector non-rotating during out trip. This is an effect of the curvature of the surface of the Earth. We will use this kind of "round trip" exercise to characterize this curvature in Section 3.6.

But first we have to find a mathematical description of this parallel transport. For this we need the first derivatives of the metric. Without proof, the parallel transport of a vector v^μ along an infinitesimal path in space dx^α is given by

$$v^\mu(x^\alpha + dx^\alpha) = v^\mu(x^\alpha) - \Gamma^\mu_{\rho\sigma} v^\rho(x^\alpha) dx^\sigma \tag{3.17}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the *Christoffel symbol* also called the *affine connection*. It contains information about how curved the coordinate system is, and is given (without proof) by the following expression:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\frac{\partial g_{\mu\gamma}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu} \right) \tag{3.18}$$

Note that $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$. You can always choose a choose a coordinate system that at some point P of your choice the $\Gamma^\alpha_{\beta\gamma} = 0$. However, for curved space one cannot find a coordinate system where $\Gamma^\alpha_{\beta\gamma} = 0$ everywhere. A coordinate system which has $\Gamma^\alpha_{\beta\gamma} = 0$ in some point P is, in general relativity, said to be a *local inertial coordinate system*: the elevator of Einstein's thought experiment.

3.4 Covariant derivative

In a flat space with a orthonormal coordinate system the derivative of a vector field v^α is given by

$$\frac{\partial v^\alpha}{\partial x^\beta} \tag{3.19}$$

The divergence is

$$\frac{\partial v^\alpha}{\partial x^\alpha} \tag{3.20}$$

However, if we use non-cartesian coordinates and/or if the space is curved, this simple definition of the derivative is no longer very useful, because even a vector field that is constant in space would yield non-constant derivative values because of the curved coordinates / curved space.

The problem is: How do we faithfully compare two vectors that are located on two nearby points (at point $x^\mu + dx^\mu$ and point x^μ)? The only clean way is to use parallel transport: we transport the vector v^α at point x^μ in a parallel way to point $x^\mu + dx^\mu$, and we can then compare it with the vector v^α that is located at $x^\mu + dx^\mu$. We define the *covariant derivative* as the derivative using this parallel transport method, and denote it with a capital D:

$$\begin{aligned}
\frac{Dv^\alpha}{Dx^\beta} &= \frac{v^\alpha(x^\mu + dx^\mu) - \mathrm{ParTrans}[v^\alpha(x^\mu)]}{dx^\beta} \\
&= \frac{v^\alpha(x^\mu + dx^\mu) - v^\alpha(x^\mu) + \Gamma^\alpha_{\nu\gamma} v^\gamma(x^\mu) dx^\nu}{dx^\beta} \\
&= \frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} v^\gamma
\end{aligned} \tag{3.21}$$

where we have used, that $dx^\nu/dx^\beta = \delta^\nu_\beta$.

For convenience of notation we often write

$$\frac{\partial v^\alpha}{\partial x^\beta} \quad =: \quad \partial_\beta v^\alpha =: v^\alpha{}_{,\beta} \tag{3.22}$$

$$\frac{Dv^\alpha}{Dx^\beta} \quad =: \quad \nabla_\beta v^\alpha =: v^\alpha{}_{;\beta} \tag{3.23}$$

We can now rewrite the conservations laws of energy and momentum in a curved spacetime as

$$\nabla_\beta T^{\alpha\beta} \equiv T^{\alpha\beta}{}_{;\beta} = 0 \tag{3.24}$$

and the conservation of mass as

$$\nabla_\mu(\rho v^\mu) \equiv (\rho v^\mu)_{;\mu} = 0 \tag{3.25}$$

3.5 A geodesic path through space(-time)

A geodesic path is a path through space(time) that is *locally straight* at every point. Consider again the example of the Earth's surface. If you walk in a manner that you consider straight, your path will be a circle around the world: after about 40,000 km you will arrive back at where you started from. A locally straight path $x^\mu(\tau)$ is defined such that the direction vector $v^\mu(\tau) \equiv dx^\mu(\tau)/d\tau$ has zero covariant derivative along the path:

$$\frac{Dv^\alpha(\tau)}{D\tau} := v^\mu(\tau)\nabla_\mu v^\alpha(\tau) = 0 \tag{3.26}$$

Working this equation out:

$$\begin{aligned}
v^\mu(\tau)\nabla_\mu v^\alpha(\tau) &= v^\mu(\tau)\partial_\mu v^\alpha(\tau) + v^\mu(\tau)\Gamma^\alpha_{\mu\nu} v^\nu(\tau) \\
&= \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}
\end{aligned} \tag{3.27}$$

so we get as our final equation the so called *geodesic equation*:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{3.28}$$

While we derived Equation (3.28) for a geodesic through 3-D space, *it is also valid in 4-D spacetime, and according to Einstein's relativity principle, it describes the path through spacetime of a free-falling particle*. This equation is therefore one of the cornerstones of general relativity. In this formalism τ is the time as it is measured by a clock moving along this geodesic through spacetime. The normalization is then such that

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2 \tag{3.29}$$

3.6 Riemann-Christoffel curvature tensor

Now that we know how a free-falling body moves in 4-D spacetime, we must find the equations for the structure of spacetime itself, or in other words for the metric $g_{\mu\nu}$ as a function of spatial coordinates x^α. Einstein postulated that the curvature of spacetime must be locally related to the presence or absense of matter. We must therefore find a mathematical description of curvature of spacetime. The Christoffel symbol describes this in principle, but in itself it is not a measure of curvature of spacetime only: it also measures the curvature of the coordinate system. In other words, $\Gamma^\alpha_{\mu\nu}$ can be non-zero even for flat space, if the coordinates are curved. Since gravity should not depend on the choice of coordinates, it cannot be correct to merely use $\Gamma^\alpha_{\mu\nu}$ as a description of gravity. Moreover, one can always choose a coordinate system such that at some point P in spacetime one has $\Gamma^\alpha_{\mu\nu} = 0$ even if the curvature of spacetime at point P is decidedly non-zero.

To find a better measure of curvature of space(time) we must think of an experiment that we can carry out in such space(time) that gives us an objective measure of curvature of space(time). Remember the experiment with the vector that we parallelly transport along a closed path we carried out in Section 3.3. Let us repeat this in a more formal way.

Let us start with a vector v^α located at point P given by coordinates x^μ. Let us do an infinitesimal step $dx^\mu_{(1)}$ in space(time), so that we arrive at $x^\mu + dx^\mu_{(1)}$. We take the vector v^α along with us in a parallel transport manner. Now we take another infinitesimal step, this time in direction $dx^\mu_{(2)}$, so we arrive at $x^\mu + dx^\mu_{(1)} + dx^\mu_{(2)}$, again taking v^α along with us in a parallel transport manner. Let us call the vector v^α that we have obtained in this way v^α_I. According to the formulae of parallel transport (Section 3.3) this vector is:

$$\begin{aligned}
v^\alpha_I &= \left[v^\alpha(x^\mu) - \Gamma^\alpha_{\beta\gamma}(x^\mu)v^\gamma(x^\mu)dx^\beta_{(1)}\right] - \Gamma^\alpha_{\rho\sigma}(x^\mu + dx^\mu_{(1)})\left[v^\sigma(x^\mu) - \Gamma^\sigma_{\beta\gamma}(x^\mu)v^\gamma(x^\mu)dx^\beta_{(1)}\right]dx^\rho_{(2)} \\
&= v^\alpha(x^\mu) - \Gamma^\alpha_{\beta\gamma}(x^\mu)v^\gamma(x^\mu)dx^\beta_{(1)} - \Gamma^\alpha_{\rho\sigma}(x^\mu + dx^\mu_{(1)})v^\sigma(x^\mu)dx^\rho_{(2)} \\
&\quad + \Gamma^\alpha_{\rho\sigma}(x^\mu + dx^\mu_{(1)})\Gamma^\sigma_{\beta\gamma}(x^\mu)v^\gamma(x^\mu)dx^\beta_{(1)}dx^\rho_{(2)}
\end{aligned} \tag{3.30}$$

Now let us repeat this, but in opposite order: First move along $dx^\mu_{(2)}$ and *then* along $dx^\mu_{(1)}$. We obtain a vector v^α_{II} in this way, which, in curved space(time) is *not* necessarily the same as v^α_I:

$$\begin{aligned}
v^\alpha_{II} &= \left[v^\alpha(x^\mu) - \Gamma^\alpha_{\rho\sigma}(x^\mu)v^\sigma(x^\mu)dx^\rho_{(2)}\right] - \Gamma^\alpha_{\beta\gamma}(x^\mu + dx^\mu_{(2)})\left[v^\gamma(x^\mu) - \Gamma^\gamma_{\rho\sigma}(x^\mu)v^\sigma(x^\mu)dx^\rho_{(2)}\right]dx^\beta_{(1)} \\
&= v^\alpha(x^\mu) - \Gamma^\alpha_{\rho\sigma}(x^\mu)v^\sigma(x^\mu)dx^\rho_{(2)} - \Gamma^\alpha_{\beta\gamma}(x^\mu + dx^\mu_{(2)})v^\gamma(x^\mu)dx^\beta_{(1)} \\
&\quad + \Gamma^\alpha_{\beta\gamma}(x^\mu + dx^\mu_{(2)})\Gamma^\gamma_{\rho\sigma}(x^\mu)v^\sigma(x^\mu)dx^\rho_{(2)}dx^\beta_{(1)}
\end{aligned} \tag{3.31}$$

The difference between these two vectors, to first order, is therefore

$$\begin{aligned}
\Delta v^\alpha &= v^\alpha_{II} - v^\alpha_I \\
&= \left[\Gamma^\alpha_{\rho\sigma,\beta}v^\sigma - \Gamma^\alpha_{\beta\gamma,\rho}v^\gamma + \Gamma^\alpha_{\beta\gamma}\Gamma^\gamma_{\rho\sigma}v^\sigma - \Gamma^\alpha_{\rho\sigma}\Gamma^\sigma_{\beta\gamma}v^\gamma\right]dx^\beta_{(1)}dx^\rho_{(2)} \\
&= \left[\Gamma^\alpha_{\rho\sigma,\beta} - \Gamma^\alpha_{\beta\sigma,\rho} + \Gamma^\alpha_{\beta\gamma}\Gamma^\gamma_{\rho\sigma} - \Gamma^\alpha_{\rho\gamma}\Gamma^\gamma_{\beta\sigma}\right]v^\sigma dx^\beta_{(1)}dx^\rho_{(2)} \\
&\equiv R^\alpha{}_{\sigma\beta\rho}v^\sigma dx^\beta_{(1)}dx^\rho_{(2)}
\end{aligned} \tag{3.32}$$

where we defined the *Riemann-Christoffel curvature tensor*:

$$R^\alpha{}_{\sigma\beta\rho} \equiv \Gamma^\alpha_{\rho\sigma,\beta} - \Gamma^\alpha_{\beta\sigma,\rho} + \Gamma^\alpha_{\beta\gamma}\Gamma^\gamma_{\rho\sigma} - \Gamma^\alpha_{\rho\gamma}\Gamma^\gamma_{\beta\sigma} \tag{3.33}$$

This tensor is not just built out of the Christoffel symbols, but *also out of their derivatives*. It turns out that this tensor contains all there is to know about the local curvature of space(time). If $R^{\alpha}{}_{\sigma\beta\rho} = 0$ at all locations, then the space(time) is flat. Conversely, if $R^{\alpha}{}_{\sigma\beta\rho} \neq 0$ at some points, then the space(time) is curved. You cannot transform $R^{\alpha}{}_{\sigma\beta\rho}$ to 0 as you could with $\Gamma^{\alpha}_{\mu\nu}$. Therefore the Riemann-Christoffel curvature tensor tells something about the curvature of space(time), but does *not* tell something about the curvature of the local coordinate system. It therefore has a physical meaning independent of the coordinate system you choose, and we can therefore use it to define the Einstein equations.

3.7 The Einstein equations

We now have a tensorial quantity that describes curvature of space(time) in a meaningful way. To fulfill Einstein's idea of relating this somehow to the matter content, we must find a mathematically consistent way to relate $R^{\alpha}{}_{\sigma\beta\rho}$ to the energy-momentum stress tensor $T^{\mu\nu}$. First of all, we must therefore reduce the rank-4 tensor $R^{\alpha}{}_{\sigma\beta\rho}$ into a rank-2 tensor. We define the *Ricci tensor $R_{\alpha\beta}$* as

$$R_{\alpha\beta} \equiv R^{\sigma}{}_{\alpha\sigma\beta} \tag{3.34}$$

which is a symmetric tensor ($R_{\alpha\beta} = R_{\beta\alpha}$), and the *curvature R* as

$$R \equiv g^{\mu\nu} R_{\mu\nu} \tag{3.35}$$

Finally we define the *Einstein tensor* as

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \tag{3.36}$$

(note that $R^{\alpha\beta}$ and $R_{\alpha\beta}$ are trivially related to each other, see syllabus on tensor calculus).

Without further proof it turns out that

$$G^{\alpha\beta}{}_{;\beta} = 0 \tag{3.37}$$

This is the same property as that of the energy-momentum stress tensor $T^{\mu\nu}$, which also obeys $T^{\mu\nu}{}_{;\nu} = 0$. This allows us to equate the two with some proportionality constant in between. This brings us to the *Einstein equation*

$$G^{\mu\nu} = \frac{8\pi G}{c^2} T^{\mu\nu} \tag{3.38}$$

This is a postulation, i.e., it cannot be derived in a rigorous way, but it turns out to be able to describe gravity very well in the Newtonian limit. This is the central equation of general relativity. Solving this equation for spacetimes with matter inside yields a curved spacetime. With Eq. (3.28) one can then find out how particles move in this spacetime, i.e. how they are affected by gravity.

When one is prepared to add new constants or fields to the system, then there can be various alternative versions of the Einstein equations, the most well-known is the one with a cosmological constant Λ:

$$G^{\mu\nu} = \frac{8\pi G}{c^2} T^{\mu\nu} - \Lambda g^{\mu\nu} \tag{3.39}$$

One can regard Λ as some form of energy with negative pressure, because if we write the above equation as

$$G^{\mu\nu} = \frac{8\pi G}{c^2} \left[T^{\mu\nu} - \frac{c^2}{8\pi G} \Lambda g^{\mu\nu} \right] \equiv \frac{8\pi G}{c^2} \left[T^{\mu\nu} - \tilde{\Lambda} g^{\mu\nu} \right] \equiv \frac{8\pi G}{c^2} \bar{T}^{\mu\nu} \tag{3.40}$$

then, in a local interial frame, we have

$$\bar{T}^{\mu\nu} = \begin{pmatrix} \rho c^2 + \tilde{\Lambda} & 0 & 0 & 0 \\ 0 & p - \tilde{\Lambda} & 0 & 0 \\ 0 & 0 & p - \tilde{\Lambda} & 0 \\ 0 & 0 & 0 & p - \tilde{\Lambda} \end{pmatrix} \tag{3.41}$$