Exercises for
Introduction to Cosmology (WS2011/12)

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Exercise sheet 1

1. **Newtonian Friedmann Equations**
In the lecture we have derived the expansion of the Universe by analogy of an expanding self-gravitating sphere of homogeneous density. The not-so-elegant aspect of this derivation is that the size R and mass M of the sphere appear explicitly in the equations. Let us eliminate these, and thus derive a more elegant equation. Let us write the Hubble flow as
$$\vec{r}(t) = a(t)\vec{x} \tag{1}$$
where $\vec{x}$ is a comoving position vector: A given galaxy moving with the Hubble flow has constant $\vec{x}$. The factor $a(t)$ is the scale factor, which by definition is $a(t_{\mathrm{now}}) = 1$ today.

   (a) Show that with $\dot{\vec{r}} = \dot{a}\vec{x}$ the Hubble "constant" is
$$H = \frac{\dot{a}}{a} \tag{2}$$

   (b) Show that
$$\rho(t) = \frac{\rho_0}{a(t)^3} \tag{3}$$
where ρ_0 is the average density in the Universe today.

   (c) Show that the equation for $a(t)$ is
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho \tag{4}$$
This is the Newtonian version of the second Friedmann equation.

   (d) Show that this can be integrated to
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{B}{a^2} \tag{5}$$
where B is a constant. This is the Newtonian version of the first Friedmann equation.

   (e) Show that this can be written as
$$H^2(t) = H_0^2\left(\frac{\rho}{\rho_{\mathrm{crit},0}}\right) + \frac{B}{a^2} \tag{6}$$
where H_0 is the current Hubble constant and $\rho_{\mathrm{crit},0}$ is the current critical density.

   (f) Show that, if $\rho = \rho_{\mathrm{crit}}$ at one point in time, it holds that $\rho = \rho_{\mathrm{crit}}$ at any other point in time.

2. **Present-day galaxy count**

(a) Currently the best estimate of the Hubble constant is $H_0 = 70.4\,\text{km/s/Mpc}$. Compute the critical density at the present time.

(b) There is strong evidence that the density of luminous + dark matter today is about 25% of the critical density. If we assume an average mass of large galaxies (luminous + dark matter) of $10^{12} M_\odot = 2 \times 10^{45}$ gram, and assume all luminous and dark matter of the Universe to be in the form of such galaxies, how many such galaxies does one expect per Mpc^3?

(c) If we estimate the visible Universe to be a sphere with a Hubble distance in radius, give a very rough estimate of the number of observable large galaxies (ignore complications of expansion etc, the idea here is to get a rough estimate).