Exercises for Introduction to Cosmology (WS2011/12)

Cornelis Dullemond

Exercise sheet 10

Since it is almost Christmas, the obligatory part of this exercise sheet is kept short. But if you like to learn about non-Gaussianity, bispectra and three-point correlation functions, you can also do the voluntary exercise. You can also get extra points if you do this exercise, in case you need to beef-up your average score.

1. **Linear growth in the late Universe**
   For the Einstein-de-Sitter Universe ($\Omega_m = 1$, $\Omega_\Lambda = \Omega_K = \Omega_r = 0$) we know that the growth function is linear: $D_+(a) = a$. However, our Universe at present has $\Omega_{\Lambda,0} = 0.75$, $\Omega_{m,0} = 0.25$, $\Omega_{r,0} \simeq \Omega_{K,0} \simeq 0$. In the script an approximative function for $D_+(a)$ under these conditions was given.

   (a) Show that this function is consistent with linear growth that is *linear* in a (i.e. $\delta \propto a$) in the early Universe after the CMB release ($0 \ll z \lesssim 1100$).

   (b) Once Ω_Λ is no longer negligible, the linear growth is no longer linear in a. Show this by making a plot of $D_+(a)$ (linear in $0 \le a \le 1$ and linear in $0 \le D_+ \le 1$) by calculating $D_+(a)$ for the following values and interpolating between them: $a = 0.1, 0.25, 0.5, 0.75, 1$.

2. **Bispectrum, three-point-correlation and non-linearity [VOLUNTARY]**
   In the early Universe the density perturbations $\delta(\vec{x})$ are, as far as we can currently tell, a Gaussian random noise. Any Gaussian random noise is fully described by its power spectrum, or its Fourier-equivalent: the two-point correlation function. The purpose of this exercise is to learn about higher-order statistical quantities such as the *bispectrum* and its Fourier-equivalent: the *three-point correlation function*. Signals that have non-zero bispectrum contain more information than just the power spectrum; they are therefore non-Gaussian. Linear evolution equations preserve Gaussianity. Non-linear evolution equations induce a non-zero bispectrum. This is very general: it is not only relevant to cosmology. We will therefore explore this with a very trivial example of a real function $f(\vec{x}, t)$ obeying

$$\frac{\partial f(\vec{x}, t)}{\partial t} = C f^n(\vec{x}, t) \qquad (36)$$

   where C is some arbitrary constant and n is either 1 (making the equation linear) or 2 (making it quadratic = non-linear).

   (a) Argue *in words* why, if $f(\vec{x}, 0)$ is a Gaussian random signal with $\langle f(\vec{x}, 0)\rangle = 0$ to start with, it will remain Gaussian for $t > 0$ if $n = 1$.

   (b) Argue *in words* why, if $f(\vec{x}, 0)$ is a Gaussian random signal with $\langle f(\vec{x}, 0)\rangle = 0$ to start with, it will become non-Gaussian for $t > 0$ if $n = 2$.

Now let $\hat{f}(\vec{k},t)$ be the Fourier transformed version of $f(\vec{x},t)$:

$$\hat{f}(\vec{k},t) = \int f(\vec{x},t)e^{i\vec{k}\cdot\vec{x}}d^3x \qquad , \qquad f(x,t) = \frac{1}{(2\pi)^3}\int f(k,t)e^{-i\vec{k}\cdot\vec{x}}d^3k \tag{37}$$

(c) Show that for $n = 1$ the equation for $\hat{f}(\vec{k},t)$ can be written in the form

$$\frac{\partial \hat{f}(\vec{k},t)}{\partial t} = C\int \hat{f}(\vec{k}_1,t)\delta_D(\vec{k}-\vec{k}_1)d^3k_1 \tag{38}$$

where δ_D is the Dirac-delta function.

(d) Show that for $n = 2$ the equation for $\hat{f}(\vec{k},t)$ can be written in the form

$$\frac{\partial \hat{f}(\vec{k},t)}{\partial t} = \frac{C}{(2\pi)^3}\int\int \hat{f}(\vec{k}_1,t)\hat{f}(\vec{k}_2,t)\delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)d^3k_1d^3k_2 \tag{39}$$

(e) Argue *in words* why Eq. (38) implies that modes of different $\vec{k}$ do not couple to each other.

(f) Argue *in words* why Eq. (39) implies that modes of different $\vec{k}$ *do* couple to each other. Which two modes can couple to mode $\vec{k}$?

These results show that non-linear terms induce *mode coupling*. Each individual mode $\vec{k}$ is no longer independent of the others. If we make use of the symmetry $\hat{f}(\vec{k}) = \hat{f}^*(-\vec{k})$ this suggests that we should be able, for $t > 0$, to find a correlation between $f(\vec{k})$, $f(\vec{k}_1)$ and $f(\vec{k}_2)$ for each combination for which $\vec{k}+\vec{k}_1+\vec{k}_2 = 0$:

$$\langle \hat{f}(\vec{k})\hat{f}(\vec{k}_1)\hat{f}(\vec{k}_2)\rangle = (2\pi)^3\delta_D(\vec{k}+\vec{k}_1+\vec{k}_2)B_f(\vec{k}_1,\vec{k}_2) \tag{40}$$

where $B_f(\vec{k}_1,\vec{k}_2)$ is called the *bispectrum* of the function f.

(g) We know that the power spectrum $P_f(\vec{k})$ is related to the two-point correlation function in space $\langle f(\vec{x})f(\vec{x}+\vec{y})\rangle$ (see derivation in the script). Derive, in a similar way, how the bispectrum $B_f(\vec{k}_1,\vec{k}_2)$ is related to the *three-point* correlation function in space $\langle f(\vec{x})f(\vec{x}+\vec{y}_1)f(\vec{x}+\vec{y}_2)\rangle$.