Exercises for
Introduction to Cosmology (WS2011/12)

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Exercise sheet 11

1. **Calculating the redshift of the release of the CMB radiation**
   In this exercise we are going to derive the numerical value of $z_{\rm cmb}$. From the lecture we already know the value: $z_{\rm cmb} \simeq 1100$, but let us do the calculation ourselves.

   (a) Let us approximate the expansion of the Universe to be according to Einstein-de-Sitter by the time of the CMB release. Show then that this leads to the following approximate relation between age of the Universe at that time $t_{\rm age}$ and scale factor a:
   $$t_{\rm age} \simeq 6 \times 10^{17}\, a^{3/2} \text{ seconds} \tag{41}$$

   (b) Show that the number density of baryons as a function of scale factor is approximately
   $$n_{\rm bary} \simeq 2.5 \times 10^{-7}\, \frac{1}{a^3}\, \frac{\text{baryons}}{\text{cm}^3} \tag{42}$$

   To express the electron number density in terms of the baryon number density, we need an expression of the ionization degree x. According to Saha's equation it obeys
   $$\frac{x^2}{1-x} = \frac{0.26}{\eta} \left(\frac{m_e c^2}{kT}\right)^{3/2} e^{-\chi/kT} \tag{43}$$
   with $\eta = 6 \times 10^{-10}$ the baryon-to-photon ratio of the Universe, m_e the electron mass, c the light speed, k the Boltzmann constant, and χ the ionization potential of a hydrogen atom.

   (c) Let us now estimate at which temperature T recombination sets in. We do so by setting $x = 0.5$ (being half-way between fully ionized material and fully neutral material). We then solve Eq. (43) using an iterative procedure, starting with a first guess T_0, and then computing T_1, T_2 etc, until convergence:

      i. First estimate T_0 by demanding the argument of the exponent in Eq. (43) to be -1. Show that you obtain $T_0 = 1.58 \times 10^5\,$K.

      ii. Now iterate the procedure:
      $$\frac{x^2}{1-x} = \frac{0.26}{\eta} \left(\frac{m_e c^2}{kT_{i-1}}\right)^{3/2} e^{-\chi/kT_i} \tag{44}$$
      with $x = 0.5$, starting with $i = 1$ (solving for T_1), then inserting the resulting T_1 into Eq. (44) for $i = 2$ etc.

      iii. What is the final resulting temperature?

      iv. How many iterations did you need to achieve convergence to within 1 %?

v. Explain why this procedure converges so extremely fast.

(d) Show that this corresponds to redshift $z_{\mathrm{recom}} \simeq 1383$.

We now know at which redshift the recombination sets in. However, this is not yet exactly the redshift at which the CMB is released (though it is close to that value). For that, we have to calculate when the value of x is so low (it will be much lower than 0.5) that photons essentially no longer scatter off electrons. To be more precise: We have to calculate when the average time τ_{scatter} between two scattering events of a single photon exceeds the age of the Universe.

(e) Argue why $\tau_{\mathrm{scatter}} \simeq 1/(cn_e\sigma_T)$, with $\sigma_T = 6.65 \times 10^{-25}$ cm^2 is the Thompson cross section of electrons, c is the light speed and n_e is the electron number density in units of cm^{-3}.

(f) Show that this implies

$$\tau_{\mathrm{scatter}} \simeq 2 \times 10^{20}\, \frac{a^3}{x} \text{ seconds} \tag{45}$$

(g) Argue that, at the redshift of last scattering, to good approximation x is given by

$$x \simeq \sqrt{\frac{0.26}{\eta}} \left(\frac{m_e c^2}{kT}\right)^{3/4} e^{-\chi/2kT} \tag{46}$$

(h) Show that this can be written in terms of a as follows:

$$x \simeq 2.1 \times 10^{11}\, a^{3/4}\, e^{-2.9\times 10^4\, a} \tag{47}$$

(i) Now set $\tau_{\mathrm{scatter}} = t_{\mathrm{age}}$. What is the equation you obtain for a?

(j) Solve this equation for a using, again, an iterative procedure very similar to what we did above. Take as starting value of a the value of onset of ionization: $a_0 = 1/(1 + z_{\mathrm{recom}})$.

(k) How close to the canonical $z_{\mathrm{cmb}} = 1100$ do you get?

(l) At which value of x is the CMB released? Show that this justifies the approximate formula Eq. (46).

This point in time is called the *last scattering surface* because it corresponds to a spherical surface around us.

(m) Estimate the thickness Δz of the last scattering surface by redoing the calculation for $\tau_{\mathrm{scatter}} = t_{\mathrm{age}}/e \simeq 0.37\, t_{\mathrm{age}}$.