Exercises for Introduction to Cosmology (WS2011/12)

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Exercise sheet 9

1. **Window function in 1-D**
Consider a 1-D function $f(x)$. We want to convolve (i.e. smooth) with a window function $W_R(x-x')$ given by

$$W_R(x-x') = \frac{1}{\sqrt{2\pi}R}\exp\left(-\frac{(x-x')^2}{2R^2}\right) \tag{28}$$

The purpose is to take out short wavelength noise (noise on scales $\lambda \ll R$), leaving the longer wavelength modes ($\lambda \gg R$) untouched. The convolution is defined as

$$\tilde{f}(x) = \int_{-\infty}^{\infty} f(x')W(x-x')dx' \tag{29}$$

   (a) Argue why the normalization of Eq. (28) is $1/(\sqrt{2\pi}R)$.

   (b) Show that, in Fourier space, this convolution is simply a multiplication of $\hat{f}(k)$ and $\hat{W}(k)$.

   (c) Show that, in Fourier space, we have

$$\lim_{k\rightarrow 0}\hat{W}(k) = 1 \qquad \text{and} \qquad \lim_{k\rightarrow\infty}\hat{W}(k) = 0 \tag{30}$$

   (d) Explain why this means that indeed the window function "takes out short wavelength noise (noise on scales $\lambda \ll R$), leaving the longer wavelength modes ($\lambda \gg R$) untouched.".

2. **Non-linear mass**
As we saw in the lecture, we can at any point in time relate distance scales R with mass scales M through

$$M = \frac{4\pi}{3}R^3\rho_0 \tag{31}$$

where ρ_0 is the background density at the present time, so that R can be regarded as a comoving distance (a distance in $\vec{x}$-space, instead of $\vec{r} = a(t)\vec{x}$-space). The non-linear mass M_* at some time in the past is the mass for which the corresponding distance scale R_* is the scale at which the variance becomes δ_c^2:

$$\sigma_{R_*}^2 = 4\pi\int_0^\infty \frac{k^2dk}{(2\pi)^3}P(k)\hat{W}_{R_*}^2(k) = \delta_c^2 \tag{32}$$

   (a) Argue why one can also approximately write this as

$$\sigma_{R_*}^2 \simeq 4\pi\int_0^{k_*} \frac{k^2dk}{(2\pi)^3}P(k) \simeq \delta_c^2 \tag{33}$$

   with a suitable k_*.

(b) Give an *approximate* expression for k_* in terms of R_*.

Let us approximate the power spectrum as

$$P(k) = \begin{cases} A\,k^{-3} & \text{for} \quad k > k_0 \\ 0 & \text{for} \quad k < k_0 \end{cases} \tag{34}$$

where k_0 is the length scale of the sound horizon at t_{eq}; we therefore simply ignore the $P(k) \propto k$ part and only focus on the $P(k) \propto 1/k^3$ part. A is a function of the scale factor a: i.e. $A(a)$ which, in the matter-dominated phase obeys $A(a) \propto a$.

(c) Show that this implies

$$k_* = k_0 \exp\left(2\pi^2 \frac{\delta_c^2}{A(a)}\right) \tag{35}$$

(d) Use this expression to argue that small-mass halos form first, and larger-mass halos form later.