

Chapter 1

Equations of hydrodynamics

In this chapter we will be concerned with compressible gas flows. This forms the main focus of this lecture.

1.1 Basic quantities

The basic quantities that describe the gas are:

Name	Symbol	Unit (CGS)	Unit (SI)
Gas density	ρ	g/cm ³	kg/m ³
Particle number density	N	1/cm ³	1/m ³
Velocity	\vec{u}	cm/s	m/s
Temperature	T	K	K
Sound speed	C_s	cm/s	m/s
Isothermal sound speed	c_s	cm/s	m/s
Pressure	P	dyne/cm ²	N/m ²
Internal energy density	E	erg/cm ³	J/m ³
Internal specific energy	e	erg/g	J/kg
Internal specific enthalpy	h	erg/g	J/kg
Total specific energy	e_{tot}	erg/g	J/kg
Total specific enthalpy	h_{tot}	erg/g	J/kg

This is a large set of variables. But only 3 of these are independent:

$$\rho, \quad \vec{u}, \quad e \quad (1.1)$$

Since the velocity \vec{u} consists of 3 components, this means that there are in fact 5 independent variables:

$$\rho, \quad u_x, \quad u_y, \quad u_z, \quad e \quad (1.2)$$

These are the 5 quantities that we will model in the computer programs we will discuss in this lecture. Usually it is unpleasant to write u_x, u_y and u_z , so we will, from now on, write:

$$\vec{u} \equiv (u, v, w) \quad (1.3)$$

where $u = u_x, v = u_y$ and $w = u_z$.

All other quantities are linked to the above 5 basic quantities. Most of the relations are fundamental. The density and number density are related by:

$$\rho = N\mu \quad (1.4)$$

The μ is the mean weight of the gas particles (in gram). For atomic hydrogen, for example, $\mu = 1.67 \times 10^{-24}$ g, and for a typical cosmic mixture of molecular hydrogen and atomic helium one has roughly $\mu = 3.8 \times 10^{-24}$. The internal energy density is linked to the specific internal energy:

$$E \equiv \rho e \quad (1.5)$$

The specific enthalpy is linked to the specific energy, the density and the pressure as:

$$h \equiv e + \frac{P}{\rho} \quad (1.6)$$

The total specific energy is the internal (heat) energy plus the kinetic energy:

$$e_{\text{tot}} \equiv e + \frac{1}{2}|\vec{u}|^2 \quad (1.7)$$

and the same for the enthalpy:

$$h_{\text{tot}} \equiv h + \frac{1}{2}|\vec{u}|^2 \quad (1.8)$$

How the internal specific energy T , the adiabatic sound speed C_s , the isothermal sound speed c_s , the pressure P and the internal energy E depend on each other is dependent on the properties of the gas. This is described by the so-called *equations of state* for the gas.

1.2 Ideal gases

In this lecture we will be mainly concerned with an *ideal gas*. Often this is also called a *perfect fluid*. This is the simplest kind of gas, and of course this is an idealization of real gases. No gas is perfect. But if densities are low enough and temperatures are high enough, a gas typically starts to behave more and more as an ideal gas. For astrophysical purposes the validity of this simple equation of state is mixed: For modeling the interiors of stars or gaseous planets this equation of state is often inadequate. But when modeling gas flows around stars or in interstellar space, the ideal gas equation of state is very accurate! Therefore in astrophysics (with the exception of stellar and planetary interiors) the ideal gas law is almost always used.

In ideal gases the elementary particles (atoms or molecules) are considered to be freely moving particles, moving in straight lines and once in a while colliding with another particle and thereby changing direction. The collision events are always two-body processes, perfectly elastic, and the particles are so small that they are many orders of magnitude smaller than the mean-free path between two collisions. Also the particles are assumed to have only interactions with each other through localized collisions, so with the exception of these collisions they move along straight lines.

For the ideal gas law to work, and indeed for any *fluid-description* of a gas to work, the mean free path between consecutive collisions must be orders of magnitude smaller than the typical scales at which we study our flows. For all of the examples we shall see this condition is guaranteed.

The ideal equation of state links the temperature, pressure and number density N of the gas particles:

$$P = NkT \quad \leftrightarrow \quad P = \frac{\rho kT}{\mu} \quad (1.9)$$

where $k = 1.38 \times 10^{-16}$ erg/K is the Boltzmann constant. Another aspect of the ideal gas is the equation of state relating the pressure to the internal specific energy e

$$P = (\gamma - 1)\rho e \quad (1.10)$$

where γ is the adiabatic index of the gas. In numerical hydrodynamics this equation for the state of the gas is more relevant, and we will use this one typically, instead of Eq. (1.9). The adiabatic index γ of an ideal gas is derived from the number of degrees of freedom of each gas particle:

$$\gamma = \frac{f + 2}{f} \quad (1.11)$$

In case of an atomic hydrogen gas each particle has merely 3 degrees of freedom: the three translational degrees of freedom. So for atomic gas one has $\gamma = 5/3$. For diatomic molecular gas, such as most of the gas in the Earth's atmosphere (mainly N_2 and O_2), as well as for molecular hydrogen gas in interstellar molecular clouds (H_2) the gas particles have, in addition to the three translational degrees of freedom also 2 rotational ones, yielding $f = 5$ and thereby $\gamma = 7/5$. For adiabatic compression/expansion of an ideal gas with adiabatic index γ one can relate the pressure to the density at all times by

$$P = K\rho^\gamma \quad (1.12)$$

where K is a constant during the adiabatic process. The K is in some sense a form of entropy. In fluids without viscosity, nor shocks or heating or cooling processes K remains constant for any fluid package. The adiabatic sound speed C_s is by definition:

$$C_s^2 \equiv \frac{\partial P}{\partial \rho} = \gamma \frac{P}{\rho} = \gamma(\gamma - 1)e \quad (1.13)$$

For isothermal sound waves (in which K is not constant, but e is), the sound speed is:

$$c_s^2 = \frac{P}{\rho} = (\gamma - 1)e \quad (1.14)$$

1.2.1 Most used variables in numerical hydrodynamics

Of all the above quantities we typically use only a few. The most used symbols are:

$$\rho, \quad e, \quad h, \quad P \quad \text{and} \quad \vec{u} \quad (1.15)$$

1.3 The first law of thermodynamics

In compressible hydrodynamics a fluid or gas parcel undergoes many compression and decompression events. When a gas parcel is compressed, its temperature tends to rise and when it is decompressed the temperature drops. In the absence of any heat transfer to or from the parcel we know from the first law of thermodynamics that

$$dU = -PdV \quad (1.16)$$

V is the volume occupied by the gas parcel, $U = \int_V \rho e dV$ is the total thermal energy in the volume V and P is the pressure. Eq.(1.16) is valid for *adiabatic (de-)compression*, i.e. (de-)compression of the gas parcel without any heat exchange. With this expression we can derive Eqs. (1.10,1.12) from the assumption that $e \propto \rho^\alpha$, where α is for now some arbitrary constant which we shall constrain later. If we assume that the density in the parcel is constant and if we define the mass of the parcel $M = \rho V$ (which is constant) then we can rewrite Eq.(1.16):

$$M e d \lg e = M \frac{P}{\rho} d \lg \rho \quad (1.17)$$

yielding:

$$P = \rho e \frac{d \lg e}{d \lg \rho} = \alpha \rho e \quad (1.18)$$

Since $e \propto \rho^\alpha$, we see that $P \propto \rho^{\alpha+1}$. Since we define γ in Eq.(1.12) such that $P \propto \rho^\gamma$, we see that $\alpha = \gamma - 1$, which means we can write

$$P = (\gamma - 1) \rho e \quad (1.19)$$

proving Eq.(1.10), which is the equation of state for a polytropic gas.

The equation of state tells us what the pressure P is, given the density ρ and the internal heat energy e . We have seen here that upon adiabatic compression or decompression the ρ and e are related to each other. However, the precise proportionality of the relation is fixed by the constant K in Eq.(1.12). So, given the constant K , each density ρ has its own well-defined thermal energy e . Two gas parcels with the same K lie on the same adiabat. Upon compression and decompression of a parcel the K does not change, unless heat is transferred to or from the parcel.

The constant K is related to the *entropy* of the gas. The full first law of thermodynamics reads:

$$dU = dQ - PdV = TdS - PdV \quad (1.20)$$

where dQ is the added thermal heat to the system, and S is the entropy of the volume V which can be expressed in terms of the specific entropy s as $S = Ms$. Let us consider K now to be a variable instead of a constant, and work out Eq.(1.20). We use $P = K\rho^\gamma = (\gamma - 1)\rho e = \rho kT/\mu$ (note: $k \neq K$), $e = K\rho^{\gamma-1}/(\gamma - 1)$, $U = Me$ and $M = \rho V$ and we obtain

$$\frac{\mu}{k} ds = \frac{1}{\gamma - 1} d \lg e - d \lg \rho = \frac{1}{\gamma - 1} d \lg K \quad (1.21)$$

where we used $d \lg e = (\gamma - 1) d \lg \rho + d \lg K$. We can therefore write:

$$s = s_0 + \frac{k}{\mu(\gamma - 1)} \lg K \quad (1.22)$$

where s_0 is an arbitrary offset constant. This shows that the number K in Eq.(1.12) is another way of writing the entropy of the gas. The entropy of a gas parcel is an important quantity. We will see below that normal gas flow does not change the entropy of a gas parcel. Shock fronts, however, will increase the entropy, and heating/cooling through radiative processes does this also. Heat conduction can also do this. But these are all special conditions. Under normal conditions a parcel of ideal gas keeps a constant entropy. This is a property that can be both useful and problematic for the development of numerical hydrodynamic schemes, as we shall see later.

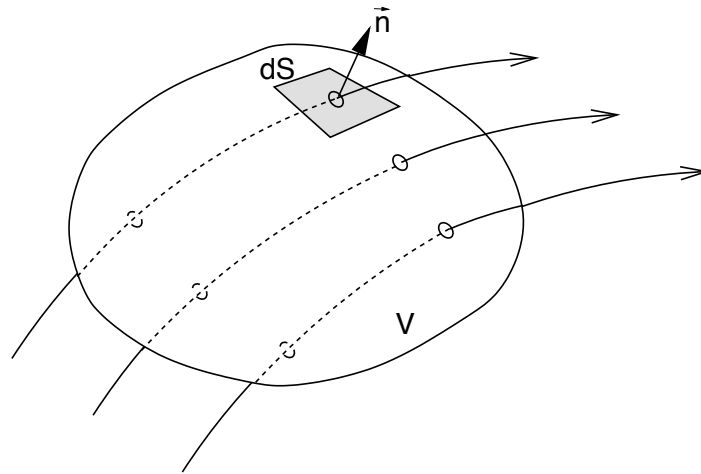


Figure 1.1. Flow through a volume V with surface $\partial V \equiv S$. The diagram shows Gauss's theorem by which the change of the total content within the volume equals the integral of the flux through its surface.

1.4 The Euler equations: the equations of motion of the gas

The motion of a gas is governed entirely by conservation laws: the conservation of matter, the conservation of momentum and the conservation of energy. These conservation laws can be written in the form of partial differential equations (PDEs) as well as in the form of integral equations. Both forms will prove to be useful for the numerical methods outlined in this lecture.

1.4.1 Conservation of mass

Consider an arbitrary volume V in the space in which our gas flow takes place. Its surface we denote as $\partial V \equiv S$ with the (outward pointing) normal unit vector at each location on the surface denoted as \vec{n} and differential surface element as dS (Fig. 1.1). The conservation of mass says that the variation of the mass in the volume must be entirely due to the in- or outflow of mass through ∂V :

$$\frac{\partial}{\partial t} \int \rho dV = - \int_{\partial V} \rho \vec{u} \cdot \vec{n} dS \quad (1.23)$$

Using Gauss's theorem, we can write this as:

$$\frac{\partial}{\partial t} \int \rho dV = - \int_V \nabla \cdot (\rho \vec{u}) dV \quad (1.24)$$

Since this has to be true for any volume V one chooses, we arrive at the following PDE:

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0 \quad (1.25)$$

where $\rho \vec{u}$ is the mass flux. This is also called the *continuity equation*.

Often it is useful to write such equations in *tensor form* (see appendix A):

$$\partial_t \rho + \partial_i (\rho u_i) = 0 \quad (1.26)$$

1.4.2 Conservation of momentum

The momentum density of the gas $\rho \vec{u}$ (which is equal to the mass flux). The total momentum in a volume V is therefore the volume integral over $\rho \vec{u}$. In principle we can do the same trick as

above, with a volume integral over $\rho\vec{u}$ and a surface integral over $\rho\vec{u}\vec{u} \cdot \vec{n}$. But here we must also take into account the forces that act on the surface by the gas surrounding the volume. At any position on the surface, the force acting by the gas outside the volume onto the gas inside the volume is $-P\vec{n}$. We can therefore write:

$$\frac{\partial}{\partial t} \int \rho\vec{u}dV = - \int_{\partial V} \rho\vec{u}\vec{u} \cdot \vec{n}dS - \int_{\partial V} P\vec{n}dS \quad (1.27)$$

Unfortunately, to use Gauss's theorem, one must have an inner product of something with \vec{n} at the surface, and $P\vec{n}$ is not. To solve this problem we need to introduce the unit tensor \mathbf{I} , which is in index notation the Kronecker-delta (see Appendix A)¹. With this we can write $P\vec{n}$ as $P\mathbf{I} \cdot \vec{n}$. Using Gauss's theorem we get:

$$\frac{\partial}{\partial t} \int \rho\vec{u}dV = - \int_V \nabla \cdot (\rho\vec{u}\vec{u} + \mathbf{I}P)dV \quad (1.28)$$

and arrive thus at the PDE:

$$\partial_t(\rho\vec{u}) + \nabla \cdot (\rho\vec{u}\vec{u} + \mathbf{I}P) = 0 \quad (1.29)$$

which is the same as

$$\partial_t(\rho\vec{u}) + \nabla \cdot (\rho\vec{u}\vec{u}) + \nabla P = 0 \quad (1.30)$$

The quantity $\rho\vec{u}\vec{u} + \mathbf{I}P$ is the *stress tensor* of the fluid.

Again here we can put this in index notation, which is particularly practical in the momentum equation as we are dealing with the stress tensor here:

$$\partial_t(\rho u_i) + \partial_k(\rho u_i u_k + \delta_{ik}P) = 0 \quad (1.31)$$

If we include a volume force on the gas, such as gravity, then we must add this as a source term. For gravity the force is the divergence of the gravitational potential Φ , so we obtain:

$$\partial_t(\rho u_i) + \partial_k(\rho u_i u_k + \delta_{ik}P) = -\rho\partial_i\Phi \quad (1.32)$$

1.4.3 Conservation of energy

Energy exists in many forms. Here we concentrate on the two most basic ones: the thermal (internal) specific energy e and the kinetic specific energy $e_{\text{kin}} = u^2/2$. So the total energy is the volume integral of $\rho(e + u^2/2)$, and the advection of energy through the control volume surface is the surface integral of $\rho(e + u^2/2)\vec{u} \cdot \vec{n}$. But in addition to this we also have the work that the exterior acts on the control volume according to the first law of thermodynamics ($dU = TdS - PdV$), which is the surface integral of $P\vec{u} \cdot \vec{n}$. So the conservation equation of energy is:

$$\frac{\partial}{\partial t} \int \rho \left(e + \frac{1}{2}u^2 \right) dV = - \int_{\partial V} \rho \left(e + \frac{1}{2}u^2 \right) \vec{u} \cdot \vec{n}dS - \int_{\partial V} P\vec{u} \cdot \vec{n}dS \quad (1.33)$$

Using Gauss theorem we then get:

$$\frac{\partial}{\partial t} \int \rho \left(e + \frac{1}{2}u^2 \right) dV + \int \nabla \cdot \left[\left(\rho e + \frac{1}{2}\rho u^2 + P \right) \vec{u} \right] = 0 \quad (1.34)$$

¹For those who are very familiar with tensor mathematics, this is in fact the contravariant metric.

Since this must be valid for all control volumes V we get the differential form of the energy conservation equation:

$$\partial_t(\rho e_{\text{tot}}) + \nabla \cdot [(\rho e_{\text{tot}} + P)\vec{u}] = 0 \quad (1.35)$$

or in index notation:

$$\partial_t(\rho e_{\text{tot}}) + \partial_k[(\rho e_{\text{tot}} + P)u_k] = 0 \quad (1.36)$$

This is the energy conservation equation. With the definition of h this can also be written as:

$$\partial_t(\rho e_{\text{tot}}) + \partial_k[\rho h_{\text{tot}}u_k] = 0 \quad (1.37)$$

1.5 Lagrange form of the hydrodynamics equations

The equations derived in Section 1.4 are the hydrodynamics equations in the form which we shall later numerically solve. But there exists another form of these equations which is a bit more intuitive, and also has some applications in numerical schemes. The idea is to follow a gas element along its path and see how it changes its direction of motion and how its density and pressure change along its way. This is called the *Lagrange form* of the equations. To derive this form of the equations we need to introduce the *comoving derivative* D_t as

$$D_t \equiv \partial_t + \vec{u} \cdot \vec{\nabla} \quad (1.38)$$

1.5.1 Continuity equation

With this definition the continuity equation then becomes:

$$D_t \rho = -\rho \vec{\nabla} \cdot \vec{u} \quad (1.39)$$

This form of the continuity equation has a physical meaning. It says that a gas parcel changes its density when the gas motion converges. In other words: when the gas motion is such that the parcel gets compressed. This compression is expressed by $-\vec{\nabla} \cdot \vec{u}$.

1.5.2 Momentum conservation equation

The momentum equation can be written as

$$\vec{\partial}_t \rho + \rho \partial_t \vec{u} + \vec{u} \vec{\nabla} \cdot [\rho \vec{u}] + \rho \vec{u} \cdot \vec{\nabla} u + \vec{\nabla} P = 0 \quad (1.40)$$

We now use Eq. (1.25), i.e. the continuity equation, to remove two of the terms, and with the definition of the comoving derivative we obtain

$$D_t \vec{u} = -\frac{\vec{\nabla} P}{\rho} \quad (1.41)$$

This form of the equation of momentum conservation also has a physical interpretation. It says that a gas parcel will be accelerated due to a force, which is the pressure gradient. Any other body force, such as gravity, can be easily added as a term on the right-hand-side. This is the advantage of the Lagrangian form of the equations.

1.5.3 Energy conservation equation

Finally, the energy equation can be manipulated in a similar manner, also using Eq. (1.25) and the definition of the comoving derivative, and we obtain

$$D_t e_{\text{tot}} = -\frac{P}{\rho} \vec{\nabla} \cdot \vec{u} - \frac{1}{\rho} \vec{u} \cdot \nabla P \quad (1.42)$$

Now with $e_{\text{tot}} = e + |\vec{u}|^2/2$ we can write (in index notation)

$$D_t e_{\text{tot}} = D_t e + u_i \partial_t u_i + u_k u_i \partial_k u_i = D_t e + u_i (\partial_t u_i + u_k \partial_k u_i) = D_t e + u_i D_t u_i \quad (1.43)$$

With the momentum equation (Eq. 1.41) we can replace the part in brackets, which yields $-\vec{u} \cdot \vec{\nabla} P/\rho$. Therefore we obtain for the energy equation:

$$D_t e = -\frac{P}{\rho} \vec{\nabla} \cdot \vec{u} \quad (1.44)$$

This also has physical meaning: the thermal energy of a gas parcel changes only as a result of adiabatic compression. Recall that the first law of thermodynamics, when expressed in ρ and e , reads

$$de = T ds - Pd \left(\frac{1}{\rho} \right) \quad (1.45)$$

where s is the specific entropy. If we replace d with D_t , and we use the continuity equation we obtain

$$D_t e = T D_t s - \frac{P}{\rho} \vec{\nabla} \cdot \vec{u} \quad (1.46)$$

So this means that another way of writing the Lagrange form of the energy equation is:

$$T D_t s = 0 \quad (1.47)$$

or to say: the entropy of a gas parcel does not change along its path of motion! The equations of hydrodynamics, at least the simplified forms we wrote down until now, conserve entropy. After its journey, a fluid parcel lies on the same adiabat as it started out.

We should, however, be careful with this statement. It is only true in regions of smooth flow. As we shall see below, gas flows tend to form shocks, which do not conserve entropy.

1.6 Properties of the hydrodynamic equations

1.6.1 Isentropic flow

In Section 1.4.3 we have derived the energy equation for a generic gas flow. But there are idealized situations where this equation becomes redundant. We have seen in Section 1.5.3 that a gas parcel does not change its entropy as it flows through some region. If we follow the motion of a gas parcel, then we do not need the energy equation: we can simply remember the initial entropy s (as represented, for instance, by the parameter K in the equation $P = \rho^\gamma$) and at the end of its journey we can compute the temperature and pressure immediately using this same s (or K) that we started out with. This is indeed true for Lagrangian systems, but if we use (as we shall do often below) the original form of the equations (Section 1.4), then we cannot keep track of which gas parcel is which. Therefore usually the energy equation is retained in numerical hydrodynamics.

However, if *all* of the gas in this region of interest has the same specific entropy, i.e. if the system is *isentropic*, then we do not need to keep track of parcels, since all gas lies on the same adiabat. In that case we can, like in the Lagrangian case, drop the energy equation and use the globally constant s or K to compute the pressure P from ρ whenever this is needed. This is the advantage of isentropic flow.

1.6.2 Isothermal flow

So far we have always assumed that the gas cannot radiate away its heat, nor get heated externally by radiation. Let us here consider the other extreme: the case of extreme heating/cooling: the situation where the gas is at any given time immediately cooled/heated to some ambient temperature. This can be true if the heating/cooling time scale of the gas is much shorter than the dynamic time scales, which is sometimes the case for astrophysical flows. In such a case we can also drop the energy equation, since we always know what e is: it is a global constant. In that case we can always say (for ideal gases): $P = (\gamma - 1)e\rho$, where $(\gamma - 1)e$ is a global constant. We see that P is then linearly proportional to ρ .

Note that this linear relation between P and ρ is similar as taking $\gamma = 1$ in the polytropic gas equation of state $P = K\rho^\gamma$. However, for $\gamma = 1$ we have $P = 0$, as $(\gamma - 1) = 0$ in the equation $P = (\gamma - 1)e\rho$. Some hydrodynamicists have in the past, however, modeled isothermal flows with $\gamma = 1.01$ or thereabout, so as to simulate roughly a linear relation between pressure and density, without killing the equations.

1.6.3 Pressureless flow

Virtually all the properties and peculiarities of the motion of gases arises because of the effect that the pressure P has. If we were to assume that the pressure is small, what would then happen? First of all, we need to define what is small. The pressure appears as a second term next to $\rho u_i u_k$ in the momentum equation. So if $P \ll \rho |\vec{u}|^2$ we can reasonably safely assume $P \simeq 0$. The momentum conservation equation then becomes

$$\partial_t(\rho u_i) + \partial_k(\rho u_i u_k) = 0 \quad (1.48)$$

The continuity equation remains unchanged. The energy equation becomes irrelevant because $e \simeq 0$. So the continuity equation together with Eq. (1.48) form the equations for a pressureless gas. It is more precise to say: for an extremely supersonic gas as no gas is perfectly pressureless. We will later examine the properties of *Burger's equations* (Section 2.13) which are mathematically equivalent to these equations.

1.7 Sound waves

As we will show rigorously lateron, the dynamics of inviscid fluids, as described by the Euler equations, is in fact purely a matter of the propagation of signals. There are two kinds of signals: sound waves and fluid movements. One can describe the entire dynamics of the fluid in terms of these kinds of signals. For 1-D fluids one can make diagrams of these signals in the (x, t) plane, and the signals propagate along lines called *characteristics*. In this section we will derive the propagation of sound waves in a static and in a moving gas with constant (background-) density, and we will therewith demonstrate the concept of characteristics. Since this example is for infinitesimal perturbations on a constant density medium, it will only give an impression of the principle, but it will be very important for what follows. In fact, the entire applied mathematics

of numerical hydrodynamics is based on these concepts, and they therefore stand at the basis of this lecture.

1.7.1 Derivation of wave equation

Consider a constant density medium with a small space- and time-dependent perturbation. Let us restrict ourselves to 1-D. The density is then

$$\rho(x, t) = \rho_0 + \rho_1(x, t) \quad (1.49)$$

with $\rho_1 \ll \rho_0$. The fluid is assumed to move with a speed $u(x, t)$ which is given by

$$u(x, t) = u_0 + u_1(x, t) \quad (1.50)$$

The equations of motion in 1-D are:

$$\partial_t \rho + \partial_x(\rho u) = 0 \quad (1.51)$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0 \quad (1.52)$$

With $P = K\rho^\gamma$ the second equation becomes

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \gamma \frac{P}{\rho} \partial_x \rho = 0 \quad (1.53)$$

With Eqs.(1.49,1.50) we get to first order in the perturbations:

$$\partial_t \rho_1 + \rho_0 \partial_x u_1 + u_0 \partial_x \rho_1 = 0 \quad (1.54)$$

$$u_0 \partial_t \rho_1 + \rho_0 \partial_t u_1 + 2\rho_0 u_0 \partial_x u_1 + u_0^2 \partial_x \rho_1 + \gamma \frac{P_0}{\rho_0} \partial_x \rho_1 = 0 \quad (1.55)$$

If we insert Eq.(1.54) into Eq. (1.55) then we get:

$$\partial_t \rho_1 + \rho_0 \partial_x u_1 + u_0 \partial_x \rho_1 = 0 \quad (1.56)$$

$$\rho_0 \partial_t u_1 + \rho_0 u_0 \partial_x u_1 + \gamma \frac{P_0}{\rho_0} \partial_x \rho_1 = 0 \quad (1.57)$$

If, just for now, we set $u_0 = 0$ (static background density), then we obtain:

$$\partial_t \rho_1 + \rho_0 \partial_x u_1 = 0 \quad (1.58)$$

$$\rho_0 \partial_t u_1 + \gamma \frac{P_0}{\rho_0} \partial_x \rho_1 = 0 \quad (1.59)$$

These two equations can be combined to

$$\partial_t^2 \rho_1 - \gamma \frac{P_0}{\rho_0} \partial_x^2 \rho_1 = 0 \quad (1.60)$$

This is a wave equation with waves travelling at velocity $+\sqrt{\gamma P_0/\rho_0}$ and $-\sqrt{\gamma P_0/\rho_0}$. Hence the definition of $C_s^2 = \gamma P/\rho$ in Section 1.2.

Now let us go back to the full equations with $u_0 \neq 0$. Let us define a *comoving derivative* D_t of some function $q(x, t)$ as:

$$D_t q(x, t) \equiv \partial_t q(x, t) + u_0 \partial_x q(x, t) \quad (1.61)$$

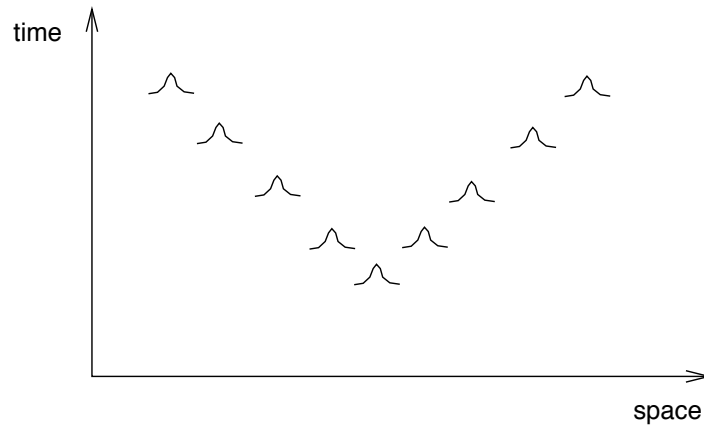


Figure 1.2. The movement of the signal of a perturbation imposed at some point x_0 and time t_0 in a 1-D spacetime, for the case of zero background velocity, i.e. $u_0 = 0$.

With this definition we can write Eqs. (1.56,1.56) as:

$$D_t \rho_1 + \rho_0 \partial_x u_1 = 0 \quad (1.62)$$

$$\rho_0 D_t u_1 + \gamma \frac{P_0}{\rho_0} \partial_x \rho_1 = 0 \quad (1.63)$$

As before, these two equations can be combined to

$$D_t^2 \rho_1 - \gamma \frac{P_0}{\rho_0} \partial_x^2 \rho_1 = 0 \quad (1.64)$$

This can be interpreted as a wave equation *in the comoving frame of the background medium*. So the two waves propagate with velocities $u_0 + \sqrt{\gamma P_0 / \rho_0}$ and $u_0 - \sqrt{\gamma P_0 / \rho_0}$.

The solution is:

$$\rho_1(x, t) = A e^{ikx - i\omega t} \quad (1.65)$$

$$u_1(x, t) = \frac{A}{\rho_0} \sqrt{\gamma \frac{P_0}{\rho_0}} e^{ikx - i\omega t} \quad (1.66)$$

with the dispersion relation

$$\frac{\omega}{k} = u_0 \pm \sqrt{\gamma \frac{P_0}{\rho_0}} \quad (1.67)$$

1.7.2 Characteristics

The fact that linear perturbations move as waves can be depicted in the (x, t) plane. Consider a sudden point-like perturbation at some place x_0 at time t_0 . For $t < t_0$ there were no perturbations, and at time t_0 this point-like perturbation is initiated. What will happen? One wave will propagate to the left at velocity $u_0 - \sqrt{\gamma P_0 / \rho_0}$, and one wave will propagate to the right at velocity $u_0 + \sqrt{\gamma P_0 / \rho_0}$. Of course, if $u_0 > \sqrt{\gamma P_0 / \rho_0}$ then the leftward moving wave is also moving to the right as it is dragged along. In this case the background velocity u_0 is *supersonic*. In any case, we can plot the exact location of the perturbation at any time $t > t_0$ in the (x, t) diagram (Fig. 1.2).

One sees that this forms a wedge and, dependent on u_0 , this wedge is slanted or not. In case of supersonic background flow the wedge tops over. The lines shown in these diagrams

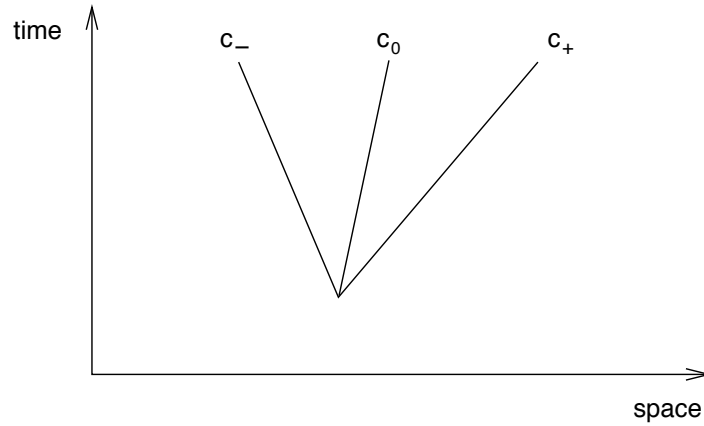


Figure 1.3. The three characteristics c_- , c_0 , c_+ of the hydrodynamics equations in 1-D, belonging to the characteristic velocities λ_{-1} , λ_0 , λ_{+1} .

are called *characteristics*. They do not necessarily have to be defined as belonging to the origin (x_0, t_0) . Any line in the (x, t) diagram following the possible trajectory of a signal is called a characteristic. In this case we have plotted the sound-characteristics.

There is also another set of characteristics. This is not seen in the above analysis. However, it can be shown if we give the fluid a ‘color’. Suppose that at $t = t_0$ we dye all gas left of x_0 blue and all gas right of x_0 as red. We now introduce a special function $\varphi(x, t)$ which gives the color. If it is 0, it means blue, if 1 it means red. The equation of this *passive tracer* is:

$$\partial_t \varphi + u \partial_x \varphi = 0 \quad (1.68)$$

We now insert $u = u_0 + u_1$:

$$\partial_t \varphi + u_0 \partial_x \varphi + u_1 \partial_x \varphi = 0 \quad (1.69)$$

We immediately see that the last term is negligible compared to the first two, so we obtain approximately:

$$\partial_t \varphi + u_0 \partial_x \varphi = 0 \quad (1.70)$$

This signal evidently propagates with velocity u_0 . This gives the third set of characteristics, describing the movement of the fluid itself. This shows that in total we have, for this 1-D example, three sets of characteristics, moving at speeds:

$$\lambda_{-1} = u_0 - \sqrt{\gamma P_0 / \rho_0} \quad (1.71)$$

$$\lambda_0 = u_0 \quad (1.72)$$

$$\lambda_{+1} = u_0 + \sqrt{\gamma P_0 / \rho_0} \quad (1.73)$$

This example shows that, at least for linear perturbations of an otherwise steady constant-density background, the hydrodynamics equations amount to the propagation of signals at three different speeds. Two signals are sound signals, while a third signal is the movement of mass (Fig. 1.3). This third signal may sound a bit as a cheat, since it is simply the passive co-movement with the background fluid, and has no dynamical character of its own as the sound waves do. However, in the non-linear evolution of hydrodynamic flows this third characteristic plays an essential role and is no longer a passive tracer. We will see this later, when we view the hydrodynamics equations as a hyperbolic set of equations.

1.8 Viscosity

1.8.1 Bulk and shear viscosity

So far we have assumed that the only force on a fluid parcel is the force due to a pressure gradient. There are also other forces that can be involved in the exchange of momentum between fluid parcels. Here we focus on the force of viscosity. A force between two adjacent fluid parcels A and B can be seen as a flux of momentum from fluid parcel A to B, or the reverse flux from B to A. Let us recall Eq.(1.31) and write it in a special way:

$$\partial_t(\rho u_i) + \partial_k(\rho u_i u_k + \delta_{ik} P) = 0 \quad (1.74)$$

$$= \partial_t(\rho u_i) + \partial_k(\rho u_i u_k + \Pi_{ki}) = 0 \quad (1.75)$$

$$= \partial_t(\rho u_i) + \partial_k T_{ki} = 0 \quad (1.76)$$

where T_{ki} is the momentum flux density tensor and Π_{ki} the pressure tensor². The T_{ki} tensor is the flux of i momentum in k direction. It is the spatial part of the stress-energy tensor familiar from relativity theory ($T^{\mu\nu} = \rho u^\mu u^\nu + g^{\mu\nu} P$, for those who are familiar with it). Both T_{ki} and Π_{ki} are always a symmetric tensor, i.e. it could also describe the flux of k momentum in i direction. For the particular case at hand, where Π_{ki} represents merely the pressure force, this tensor has two special properties:

1. Momentum is exchanged only in the direction where the transported momentum points. In other words: Π_{ki} is a diagonal tensor. X-momentum is transported in X-direction, Y-momentum in Y-direction etc.
2. The forces are always perfectly ‘elastic’ in the sense that the force does not depend on the speed of compression or decompression of a fluid element. These (de-)compressions are therefore reversible processes.

If we have viscosity, then these properties do no longer hold. Viscosity is a force that acts whenever there are velocity gradients in the flow, or to be precise, whenever there is shear flow. In the presence of viscosity, Π_{ki} reads:

$$\Pi_{ki} = \delta_{ki} P - \sigma'_{ki} \quad (1.77)$$

where σ'_{ki} is the viscous stress tensor:

$$\sigma'_{ki} = \eta \left(\partial_i u_k + \partial_k u_i - \frac{2}{3} \delta_{ki} \partial_l u_l \right) + \zeta \delta_{ki} \partial_l u_l \quad (1.78)$$

where η is the coefficient of shear viscosity and ζ is the coefficient of bulk viscosity. These are also called *the* coefficient of viscosity and the *second* viscosity respectively. The *kinematic viscosity* coefficient ν is defined as

$$\nu = \eta / \rho \quad (1.79)$$

The equations of motion are then:

$$\partial_t(\rho u_i) + \partial_k(\rho u_i u_k + \delta_{ik} P - \sigma'_{ik}) = 0 \quad (1.80)$$

or, in Lagrange form:

$$\rho D_t u_i = -\partial_i P + \partial_k \sigma'_{ki} \quad (1.81)$$

²Here we deviate from the Landau & Lifshitz notation.

This equation is called the *Navier-Stokes equation*.

To understand what these equations mean we can do two experiments, which we do in 3-D Cartesian coordinates (x, y, z) :

1. Take a velocity field $u_i = (y, 0, 0)$. This is a typical shear flow without compression. We obtain:

$$\sigma'_{ki} = \begin{pmatrix} 0 & \eta & 0 \\ \eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ki} \quad (1.82)$$

When inserting this into Eq. (1.81) we see that this transports negative x -momentum in positive y -direction. The way to see this is that anything in Eq. (1.81) that is being taken the divergence of should be considered as a *flux*. In this case it is a flux of momentum, and for the divergence of σ'_{ki} it is the viscous flux of momentum. This is what we are familiar with as friction. It may seem strange that although we have a clearly non-zero viscosity, the divergence of this viscosity is zero in this particular example, and therefore the change of velocity is zero. This is, however, an artifact of the global symmetry.

2. Take a velocity field $u_i = (-x, -y, -z)$. This is a typical compressive flow without shear. We obtain:

$$\sigma'_{ki} = \begin{pmatrix} -3\zeta & 0 & 0 \\ 0 & 0 & -3\zeta \\ 0 & 0 & -3\zeta \end{pmatrix}_{ki} \quad (1.83)$$

When inserting this into Eq. (1.81) we see that this transports positive x -momentum in positive x -direction. But if we would flip the sign of u_i (i.e. have decompression), then we have the force also flip sign: we it will transports negative x -momentum in positive x -direction. In other words: bulk viscosity acts against both compression and decompression. Or we can say: whatever one does (compression or decompression), it always costs energy. This is what bulk viscosity does.

- **Exercise:** Show that the viscous force vanishes for solid-body rotation, and explain why this must be so.
- **Exercise:** Assume a constant density flow with a velocity field of $u_i = (\sin(y), 0)$ at time $t = 0$. Compute the $\partial_t u_i$ at time $t = 0$.
- **Exercise:** Consider a nearly infinitely long pipe along the z -axis, with radius r . Let us assume that, for the length of the pipe that we are interested in, the x and y velocities are zero, while $v_z = r^2 - x^2 - y^2$. With a non-zero shear viscosity coefficient η , what does the pressure gradient in z direction have to be to keep the flow steady? This is the classical “flow in a pipe” solution.

1.8.2 How important is viscosity?

Some flows are very little affected by viscosity, for instance the flow of air around obstacles, or the flow of water through a river. Other flows are very much affected by viscosity, for instance the flow of sirup or Nutella that one smears on bread, or the flow of a glacier. A quantity to tell the difference is the *Reynolds number*:

$$Re = \frac{\rho u L}{\eta} = \frac{u L}{\nu} = \frac{\text{inertial forces}}{\text{viscous forces}} \quad (1.84)$$

where u is the typical velocity in the fluid flow and L is a typical spatial scale corresponding to the size of the fluid patterns we are interested in. One sees that the Reynolds number is not a very strictly defined quantity, because u may vary in the flow and L is just a ‘typical’ length scale. But it does an indication of the importance of viscosity in the flow:

$Re \gg 1$ → Very inviscid flows. These flows can often easily become turbulent.

$Re \ll 1$ → Very viscous flows. These flows tend to be very laminar.

We will see in later chapters that:

1. High Re flows are difficult to model numerically because the numerical algorithm artificially lowers Re (‘numerical viscosity’)
2. High Re flows often become turbulent (though not always). The flow patterns in turbulent flows have a range of scales L , going from the largest turbulent eddies, down to very small scales. This is called a *turbulent cascade*. The smallest scale of turbulence is the scale L where $Re = uL/\nu$ starts to exceed unity. Therefore, the Reynolds number of turbulent flows is always a matter of definition which L to use.
3. Turbulence can act as an ‘effective viscosity’ in itself.
4. Even laminar flows of high Re are sometimes affected by viscosity through *boundary layers*: thin layers between a high- Re flow and some solid surface with a thickness such that in this layer $Re \gtrsim 1$.

1.9 Shock waves

Under normal conditions the equations of hydrodynamics ensure that a fluid parcel keeps its entropy constant, as we have seen in Section 1.5.3. However, even in the case of inviscid fluids ($Re \rightarrow \infty$) there are conditions under which this is no longer true: when a *shock wave* occurs. The formation of a shock wave can be understood in various ways. One way is to consider a very strong sound wave. At the top of the wave the temperature is higher than at the valley of the wave. Since the sound speed is proportional to the square root of the temperature, it is to be expected that the top of the wave moves faster than the valley. The wave therefore has the tendency to steepen like waves on the beach. This will necessarily lead to the formation of a discontinuity called a shock wave. One sees that formally even normal sound waves eventually develop shock waves. These will, however, be very very weak ones, and they behave very much like a normal sound wave. In other words: a very weak shock wave is like a sound wave. The shock wave emerging from an explosion will, as it spherically expands, weaken in strength and eventually be a strong jump-like sound wave. Also the shock wave emerging from a supersonic airplane eventually is heard by people on the ground as a ‘sonic boom’, i.e. a sound wave.

The nature of a shock wave is given by its *jump conditions*. It gives which state the gas is in after going through a shock wave. Let us assume that we move along with the shock, so that in our laboratory frame the shock is standing still and the gas (before and after the shock) is moving. We assume a steady state in this frame and we define ρ_1 and ρ_2 (and idem for all other variables) such that the gas moves from region 1 to region 2. This means that mass flux,

momentum flux and energy flux through the shock must be constant:

$$\rho_2 u_2 = \rho_1 u_1 \quad (1.85)$$

$$\rho_2 u_2^2 + P_2 = \rho_1 u_1^2 + P_1 \quad (1.86)$$

$$\rho_2 h_{\text{tot}} u_2 = \rho_1 h_{\text{tot}} u_1 \quad (1.87)$$

Now let us define the specific volumes $V_2 = 1/\rho_2$ and $V_1 = 1/\rho_1$ and the mass flux $j = \rho_2 u_2 = \rho_1 u_1$. We then get

$$u_2 = jV_2 \quad u_1 = jV_1 \quad (1.88)$$

and

$$P_2 + j^2 V_2 = P_1 + j^2 V_1 \quad (1.89)$$

which yields

$$j^2 = (P_1 - P_2)/(V_2 - V_1) \quad (1.90)$$

In principle this equation allows for two solutions, one in which the post-shock medium has a higher pressure than the pre-shock medium ($P_2 > P_1$), and one in which the post-shock medium has a lower pressure than the pre-shock medium ($P_2 < P_1$). The latter will turn out to be an unphysical solution. In practise such a solution will quickly smear out into a *rarefaction wave*, but this topic will be discussed in chapter ???. We focus here on the case $P_2 > P_1$ which is what we call a shock wave.

We now substitute $u_2 - u_1 = j(V_2 - V_1)$ into Eq.(1.90) and obtain

$$u_1 - u_2 = \sqrt{(P_2 - P_1)(V_1 - V_2)} \quad (1.91)$$

where we took only the positive root because that is the physical one. Now let us turn to the energy equation

$$\rho_2 (h_2 + \frac{1}{2}u_2^2)u_2 = \rho_1 (h_1 + \frac{1}{2}u_1^2)u_1 \quad (1.92)$$

Because of mass conservation this turns into

$$h_2 + \frac{1}{2}u_2^2 = h_1 + \frac{1}{2}u_1^2 \quad (1.93)$$

and further into

$$h_2 - h_1 = \frac{1}{2}j^2(V_1^2 - V_2^2) \quad (1.94)$$

and with Eq. (1.90) into

$$h_2 - h_1 = \frac{1}{2}(V_1 + V_2)(P_2 - P_1) \quad (1.95)$$

Now replace $h = e + PV$ and we obtain

$$e_2 - e_1 = \frac{1}{2}(V_1 - V_2)(P_2 + P_1) \quad (1.96)$$

This is the relation between the state of the gas before and after the shock. It is called the *Rankine-Hugoniot shock adiabat*.

For polytropic gases we can derive the following expressions (see e.g. Landau & Lifshitz):

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \quad (1.97)$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1} \quad (1.98)$$

$$M_2^2 = \frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)} \quad (1.99)$$

where $M_1 \equiv |u_1|/C_s$ and $M_2 \equiv |u_2|/C_s$ are the *Mach numbers* on the pre- and post-shock regions.

Some properties of shocks:

- For mono-atomic gas ($\gamma = 5/3$) the maximum compression of the gas (for $M_1 \rightarrow \infty$) is $\rho_2/\rho_1 \rightarrow 4$, and for diatomic gas ($\gamma = 7/5$) it is $\rho_2/\rho_1 \rightarrow 6$.
- The Mach number before the shock is always $M_1 > 1$ and the Mach number behind the shock is always $M_2 < 1$.
- When a gas parcel goes through a shock, its entropy is increased. For the integral form of the hydrodynamics equations without viscosity, shocks are the only regions in space where gas parcels increase their entropy.

