Chapter 9

Modeling solids: Finite Element Methods

NOTE: I found errors in some of the equations for the shear stress. I will correct them later.

The static and time-dependent modeling of solids is different from the modeling of fluids, but there are also a number of similarities, both physically and technically. In Chapter 8 we have seen that when some of the signals (typically sound waves) move extremely fast compared to the signals we are interested in, and if these fast signals damp out quickly enough, then *implicit integration methods* are extremely powerful. They allow us to ignore the fast signals and compute the slower motions without excessive computational effort. In the limit of semi-infinite signal propagation (e.g. incompressible fluids) such implicit methods are unavoidable, as information from one side of the computational domain influences the motion on the other side. In other words: everything communicates instantly with everything, and this has to be represented in the method of modeling, typically involving matrices. We will find such matrix methods also very useful for the modeling of solids.

The mechanics of solids has many similarities to fluid mechanics. Also here one has sound waves, often propagating at very high speed compared to the motion we are interested in. But there is also a major difference: in solids the various solid elements are linked to their neighbors: if a red patch of material lies next to a blue patch, then these two patches will remain neighbors no matter how much the solid is deformed and moved, until the solid breaks. If the solid breaks, then the mechanics suddenly changes drastically, as then the two separated patches suddenly no longer communicate forces. But before the material breaks, neighboring patches stay neighbors and communicate forces among themselves. In fluids, on the other hand, patches of fluid change neighbors all the time, and forces are exchanged simply with the neighboring fluid elements at that time, which may be different at a future time.

There is also another difference between solids and fluids: the way that forces are exchanged. Pressure forces are exchanged in the same way for both solids and fluids. But shear stresses are different. For a viscous fluid, shear stress is proportional to gradients in the fluid *motion*. This means, if the fluid stands still, then the shear stress is zero. In solids, on the other hand, shear stresses are caused by *deformation*, i.e. by the difference of the positions of patches of solid from their original position. In solids, stress can be non-zero even if the solid is at rest. And the stress is not dependent on gradients in the velocity of the patches, only the location of the patches.

Yet, we will see that several of the methods we have used so far have also application to solids, albeit in a different form.

Literature:

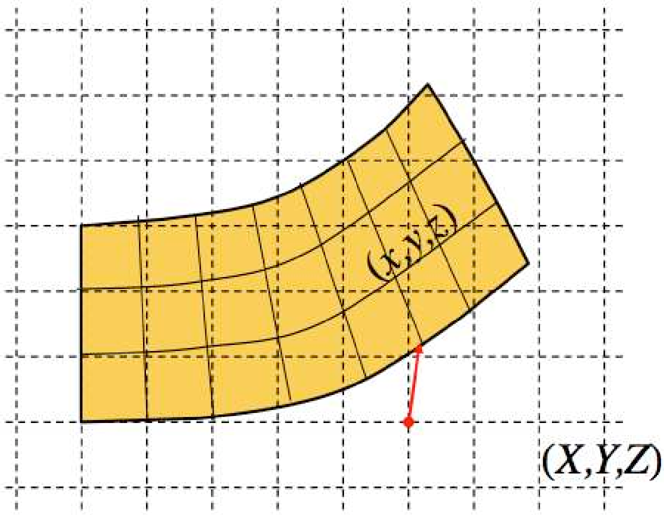

Figure 9.1. A picogram showing how the internal material coordinates (x, y, z) and the spatial coordinates (X, Y, Z) relate to each other in case of a material deformation.

1. Randall J. Leveque, "Finite Volume Methods for Hyperbolic Problems", Cambridge Textbooks in Applied Mathematics

2. Carlos A. Felippa, "Introduction to Finite Element Methods", `http://www.colorado.edu/engineering/CAS/courses.d/IFEM.d/`

9.1 Behavior of solids: strain and stress

9.1.1 Modeling a solid using displacement vectors

Let us find a method for modeling a solid block of material with size $L_x \times L_y \times L_z$. We assume that at rest the block of solid does not have any internal stresses. Let us define a cartesian coordinate system (X, Y, Z). We can now give a label to each point in the block with the coordinates at which that point is. We use lower-case letters for that: (x, y, z). The reason why we use lower case letters for labeling the points in the solid, while using upper case letters for the spatial coordinates is because only in the relaxed case the two are the same, while in the case of deformation, displacement or rotation, the two are different (see Fig.9.1).

In general, we can assign a spatial location (X, Y, Z) to each element of the block (x, y, z). So initially, in the relaxed state, we have

$$X(x, y, z) = x \tag{9.1}$$
$$Y(x, y, z) = y \tag{9.2}$$
$$Z(x, y, z) = z \tag{9.3}$$

But if we translate the block by a vector $(\Delta x, \Delta y, \Delta z)$, then we obtain:

$$X(x, y, z) = x + \Delta x \tag{9.4}$$
$$Y(x, y, z) = y + \Delta y \tag{9.5}$$
$$Z(x, y, z) = z + \Delta z \tag{9.6}$$

Rotation of the block around the z-axis by an angle θ would give

$$X(x,y,z) = x\cos\theta - y\sin\theta \tag{9.7}$$
$$Y(x,y,z) = x\sin\theta + y\cos\theta \tag{9.8}$$
$$Z(x,y,z) = z \tag{9.9}$$

Both translation and rotation do not induce any stresses in the block. For that, you need to deform the block. For instance, a bending of the block in y-direction could look like

$$X(x,y,z) = x \tag{9.10}$$
$$Y(x,y,z) = y + ax^2 \tag{9.11}$$
$$Z(x,y,z) = z \tag{9.12}$$

where a gives the degree of bending.

For mathematical convenience we introduce the *displacement vector field*

$$\delta_x(x,y,z) = X(x,y,z) - x \tag{9.13}$$
$$\delta_y(x,y,z) = Y(x,y,z) - y \tag{9.14}$$
$$\delta_z(x,y,z) = Z(x,y,z) - z \tag{9.15}$$

It is with these displacement vectors that we will do our calculations from now on.

9.1.2 Deformation of solids: strain

A non-zero displacement does not necessarily mean that the object is deformed. At every position (x,y,z) in the object we can calculate the *strain tensor*

$$\varepsilon_{kl} = \frac{1}{2}(\partial_k\delta_l + \partial_l\delta_k) \tag{9.16}$$

The strain tensor tells how deformed the object it. A zero strain tensor means that there is no deformation in the material at that position. Note that the strain does not yet tell anything about the internal forces in the material. As we shall see below, strain induces stresses, but which stresses it induces depends on material properties.

One can also define a *rotation tensor*

$$\Omega_{kl} = \frac{1}{2}(\partial_k\delta_l - \partial_l\delta_k) \tag{9.17}$$

This tensor merely gives the orientation of the local piece of object with respect to the original coordinate system (X,Y,Z). The gradient of the displacement vector can be written as

$$\partial_k\delta_l = \varepsilon_{kl} + \Omega_{kl} \tag{9.18}$$

Let us give an example in 2-D:

$$\varepsilon = \begin{pmatrix} \partial_x X - 1 & \frac{1}{2}(\partial_x Y + \partial_y X) \\ \frac{1}{2}(\partial_x Y + \partial_y X) & \partial_y Y - 1 \end{pmatrix} \tag{9.19}$$

and

$$\Omega = \begin{pmatrix} 0 & -\frac{1}{2}(\partial_x Y - \partial_y X) \\ \frac{1}{2}(\partial_x Y - \partial_y X) & 0 \end{pmatrix} \tag{9.20}$$

In the above example of the strain tensor the diagonal elements give the stretching or compression deformation. The off-diagonal elements give the shear strain.

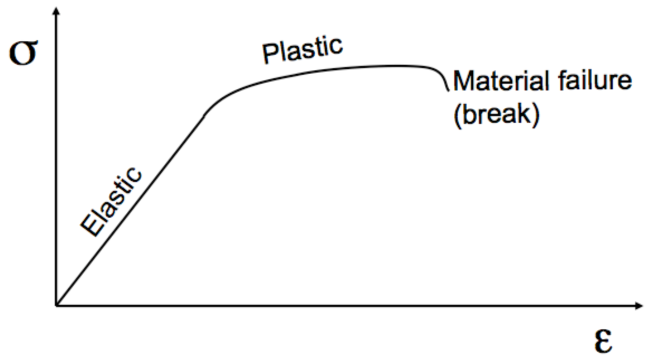

Figure 9.2. Stress response regimes of a solid material to an imposed strain.

9.1.3 Material reaction to strain: stress

Due to the internal structure of solids, a strain imposed on a solid body will induce internal stresses. A stress is like a force per unit surface. We will denote the stress as a tensor σ_{ij}. In the linear regime, i.e. for small enough strain ε_{ij}, one can write

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{9.21}$$

where C_{ijkl} is a rank four tensor which contains the material properties such as stiffness and shear strength. This linear relation only holds for strains that are small enough. As an example consider a block of granite: it is extremely strong (inducing very large σ_{ij} for very modest ε_{kl}), but if stretched to more than a percent or so, it will break. Most materials have a linear response to small strains (the so-called "elastic" regime), then beyond a certain strain threshold the respons weakens (the so-called "plastic" regime), and then beyond an even larger strain threshold the material will break (see Fig. 9.2).

9.1.4 Stress-strain relation for isotropically elastic material

For simple elastic materials with no internal preferential directions and isotropic response, *Hooke's law* holds for small enough strains:

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11} \tag{9.22}$$

where E is called *Young's modulus*. In reaction to a stretch in e.g. 1-direction (i.e. x-direction) the material counters with a force that tries to counteract this stretch. One can also invert this statement: when a given force per unit surface σ_{11} acts in x-direction on a block of material, then in response the block will deform according to $\varepsilon_{11} = \sigma_{11}/E$. Typically a material will also deform in the perpendicular directions:

$$\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11} = -\frac{\nu}{E}\sigma_{11} \tag{9.23}$$

When we stretch a material in x-direction, it will typically get thinner in y and z direction. The parameter ν is the *Poisson ratio*. If it is 0.5, the material is incompressible, i.e. it maintains a constant density. If it is negative, the material expands in y and z direction when it is stretched in x-direction. There are materials that have this pecular behavior, but it is rare. For most materials it is between 0 and 0.5. For *all* materials it is between -1 and 0.5. A few examples: most metals have $\nu \sim 0.3 \cdots 0.4$, concrete has $\nu \simeq 0.2$ and cork has $\nu \simeq 0$.

In general one can thus write:

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{pmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\end{pmatrix} \tag{9.24}$$

For the shear stress-strain relation one has simpler expressions:

$$\varepsilon_{12} = \mu\sigma_{12}, \qquad \varepsilon_{13} = \mu\sigma_{13}, \qquad \varepsilon_{23} = \mu\sigma_{23} \tag{9.25}$$

where μ is the *shear modulus*. For perfectly elastic materials one has:

$$\mu = \frac{E}{2(1+\nu)} \tag{9.26}$$

In matrix notation we can thus summarize:

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}1/E & -\nu/E & -\nu/E & 0 & 0 & 0\\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0\\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0\\ 0 & 0 & 0 & 1/2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 1/2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 1/2\mu\end{pmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{pmatrix} \tag{9.27}$$

The inverse of this relation is:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{pmatrix} \tag{9.28}$$

with

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{9.29}$$

9.1.5 Equations of motion

With the above knowledge of material behavior, we can now write down the time-dependent equations of motions for a solid. Let us define the following velocity components:

$$u = \frac{\partial\delta_x}{\partial t}, \quad v = \frac{\partial\delta_y}{\partial t}, \quad w = \frac{\partial\delta_z}{\partial t} \tag{9.30}$$

The complete set of equations of motion is then:

$$\partial_t\delta_x = u \tag{9.31}$$
$$\partial_t\delta_y = v \tag{9.32}$$
$$\partial_t\delta_z = w \tag{9.33}$$
$$\rho\partial_t u = \partial_x\sigma_{11} + \partial_y\sigma_{12} + \partial_z\sigma_{13} \tag{9.34}$$
$$\rho\partial_t v = \partial_x\sigma_{21} + \partial_y\sigma_{22} + \partial_z\sigma_{23} \tag{9.35}$$
$$\rho\partial_t w = \partial_x\sigma_{31} + \partial_y\sigma_{32} + \partial_z\sigma_{33} \tag{9.36}$$

At each time step the stress tensor σ must be determined from the strain tensor ε, which follows from the derivatives of the displacement vector $(\delta_x, \delta_y, \delta_z)$. The six variables to solve are: $\delta_x, \delta_y, \delta_z, u, v, w$.

We could use implicit methods for solving this set of equations. But if we instead want to use the methods we derived in earlier chapters for hyperbolic sets of equations (which the above are), then we encounter a problem. Although the above equations are hyperbolic, they are second order. So it is not possible to use methods derived for advection equations to these equations.

To make the equations first order one can switch from solving for the displacement vector $(\delta_x, \delta_y, \delta_z)$ to solving for the strain tensor ε. We thus obtain:

$$\partial_t \varepsilon_{11} = \partial_x u \tag{9.37}$$
$$\partial_t \varepsilon_{22} = \partial_y v \tag{9.38}$$
$$\partial_t \varepsilon_{33} = \partial_z w \tag{9.39}$$
$$\partial_t \varepsilon_{12} = \tfrac{1}{2}(\partial_x v + \partial_y u) \tag{9.40}$$
$$\partial_t \varepsilon_{13} = \tfrac{1}{2}(\partial_x w + \partial_z u) \tag{9.41}$$
$$\partial_t \varepsilon_{23} = \tfrac{1}{2}(\partial_y w + \partial_z v) \tag{9.42}$$
$$\rho \partial_t u = \partial_x \sigma_{11} + \partial_y \sigma_{12} + \partial_z \sigma_{13} \tag{9.43}$$
$$\rho \partial_t v = \partial_x \sigma_{21} + \partial_y \sigma_{22} + \partial_z \sigma_{23} \tag{9.44}$$
$$\rho \partial_t w = \partial_x \sigma_{31} + \partial_y \sigma_{32} + \partial_z \sigma_{33} \tag{9.45}$$

This is a linear set of equations which one can solve using the techniques of Chapters 2, 3 and 4.

Note that we now have 3 more equations and 3 more variables, yielding in total 9 equations for 9 variables. In principle we only have 6 independent variables, 3 for space and 3 for velocity. The 6 spatial variables of the strain tensor are therefore not independent. This means that strictly speaking, at each time step of the time-dependent integration we should assure that the 6 components of the stress tensor are still self-consistent. In fact, in the end we wish to see what is the *shape* of an object, so we must then reconstruct the displacement vector $(\delta_x, \delta_y, \delta_z)$ from the components of the strain tensor, ε_{11}, ε_{22}, ε_{33}, ε_{12}, ε_{13} and ε_{23}. If this is done at each time step, then it is guaranteed that the equations remain self-consistent. More details on how to solve these equations can be found in the book by LeVeque.

9.2 Finite Element Methods (FEM) for solids

The equations and methods described in Section 9.1 showed the basic principles of how solids are modeled. But the method for solving the motion and deformation of solid objects in that section is rarely used, because it can not naturally treat complex structures. In practice engineers want to be able to solve for the motion or the static structure of complex objects such as bridges, houses, ships, cars, airplanes etc. What they then often use is a "Finite Element Method" (FEM). With this method the object can be constructed out of individual blocks of varying shapes, glued together to form a full object. See Fig.9.3 for the most basic 1-D, 2-D and 3-D finite elements. This does not mean that it would not be possible to introduce more complex finite elements. On the contrary: one can certainly also include squares, cubes etc. But the advantage of rods, triangles and tetraeders is that in those simple finite elements the stress tensor is constant throughout the element (at least in the linear regime).

For the 1-D finite element (the rod) we have two displacement vectors (see Fig.9.3). If we apply this finite element to 1-D space, each displacement vector contains one number. We

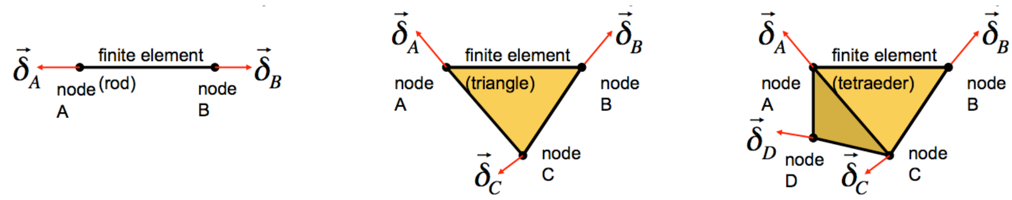

Figure 9.3. The most basic finite elements: 1-D rod, 2-D triangle and 3-D tetraeder.

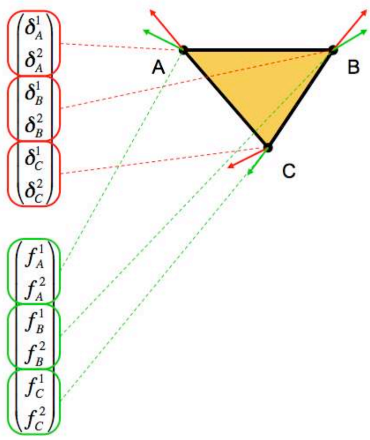

Figure 9.4. Illustration of displacements and resulting forces on the nodes of a triangular finite element in 2-D space.

have one translational degree of freedom, so this leaves one variable describing the deformation (strain). When applying it do 3-D space, each displacement vector has 3 components, yielding 6 variables. Three of those take care of translational freedom, two for rotational freedom. This leaves again one variable describing the deformation.

For the 2-D finite element (the triangle) we have three displacement vectors (see Fig.9.3). If we apply this finite element to 2-D space, each displacement vector contains two numbers. We have two translational degrees of freedom and one rotational degree of freedom, so this leaves 3 variables describing the deformation (strain). When applying it do 3-D space, each displacement vector has 3 components, yielding 9 variables. Three of those take care of translational freedom, three for rotational freedom. This leaves again 3 variables describing the deformation.

For the 3-D finite element (the tetraeder) we have four displacement vectors (see Fig.9.3). Each displacement vector has 3 components, yielding 12 variables. Three of those take care of translational freedom, three for rotational freedom. This leaves 6 variables describing the deformation.

9.2.1 Direct stiffness method for statics

A method for computing the equilibrium shape of objects under stress is the *Direct Stiffness Method* (DSM). Let us discuss this in a 2-D example, and let us focus on a single triangular finite element (see Fig. 9.4). We have displacements of the nodes A, B and C, which we write as δ_A^1, δ_B^2 for node A, and similar for B and C. These displacements induce a strain inside the triangle, which induces a stress. If this triangle is glued to other finite elements or to a boundary, then we must find a way to communicate these stresses to these other elements in terms of forces that act on them. There are various ways of doing this, but the easiest is to assume that each element only communicates with neihboring elements via forces on their common nodes (in this case A, B or C). We write these forces as δ_A^1, δ_B^2 for node A, and similar for B and C. For static problems, where we assume that all motions vanish, we can already know that the net sum of all the forces must be zero, eliminating 2 of the 6 independent variables (remember, we assume here to be in 2-D). Also, no net torque must act on the finite element, eliminating yet another variable. So of the 6 force variables, only 3 are independent. Indeed, of the 6 displacement variables, also only 3 are related to the deformation, so that fits.

We now have to find a way to convert the stress tensor σ_{ij} into these forces. The entire process, starting with displacements, via strain, stress and ending with nodal forces can be summarized using the co-called *stiffness matrix*:

$$\begin{pmatrix} f_A^1 \\ f_A^2 \\ f_B^1 \\ f_B^2 \\ f_C^1 \\ f_C^2 \end{pmatrix} = \begin{pmatrix} K_{AA}^{11} & K_{AA}^{12} & K_{AB}^{11} & K_{AB}^{12} & K_{AC}^{11} & K_{AC}^{12} \\ K_{AA}^{21} & K_{AA}^{22} & K_{AB}^{21} & K_{AB}^{22} & K_{AC}^{21} & K_{AC}^{22} \\ K_{BA}^{11} & K_{BA}^{12} & K_{BB}^{11} & K_{BB}^{12} & K_{BC}^{11} & K_{BC}^{12} \\ K_{BA}^{21} & K_{BA}^{22} & K_{BB}^{21} & K_{BB}^{22} & K_{BC}^{21} & K_{BC}^{22} \\ K_{CA}^{11} & K_{CA}^{12} & K_{CB}^{11} & K_{CB}^{12} & K_{CC}^{11} & K_{CC}^{12} \\ K_{CA}^{21} & K_{CA}^{22} & K_{CB}^{21} & K_{CB}^{22} & K_{CC}^{21} & K_{CC}^{22} \end{pmatrix} \begin{pmatrix} \delta_A^1 \\ \delta_A^2 \\ \delta_B^1 \\ \delta_B^2 \\ \delta_C^1 \\ \delta_C^2 \end{pmatrix} \tag{9.46}$$

Since translation and rotation give zero force, the matrix must have eigenvectors with zero eigenvalue, or in other words: it has a zero determinant.

9.2.2 DSM: A 1-D finite element

Let us now make an example, going to even lower dimension: 1-D. If we have a 1-D rod in 1-D space, then we have two displacement variables: δ_A and δ_B, only the combination of which is an independent variable because we are not interested in translation. We then have

$$\begin{pmatrix} f_A \\ f_B \end{pmatrix} = \begin{pmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{pmatrix} \begin{pmatrix} \delta_A \\ \delta_B \end{pmatrix} = K \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_A \\ \delta_B \end{pmatrix} \tag{9.47}$$

where K is the stiffness of the cylindrical rod:

$$K = E\pi r^2/L \tag{9.48}$$

where E is Young's modulus, r is the radius of the cylinder and L its length.

9.2.3 DSM: Constructing a triangle with three rods

Using the simple rod model above we can now start constructing a simple triangle from three individual rods and compute the stiffness matrix for that triangle. See Fig.9.5 for the geometry. We leave it as an exercise for the reader to show that we obtain

$$\begin{pmatrix} f_A^1 \\ f_A^2 \\ f_B^1 \\ f_B^2 \\ f_C^1 \\ f_C^2 \end{pmatrix} = K \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & -1 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} \delta_A^1 \\ \delta_A^2 \\ \delta_B^1 \\ \delta_B^2 \\ \delta_C^1 \\ \delta_C^2 \end{pmatrix} \tag{9.49}$$

with again $K = E\pi r^2/L$ with L defined in Fig.9.5.

The result can be interpreted in two ways: in one sense we have constructed a small object from three rods. We could add more rods and create more complex objects. In another sense we have created a model for a 2-D triangular finite element. However, this model is just one model, and is not a generally valid model for triangular finite elements.

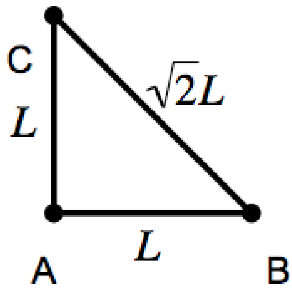

Figure 9.5. The geometry of our simple triangle constructed from three rods.

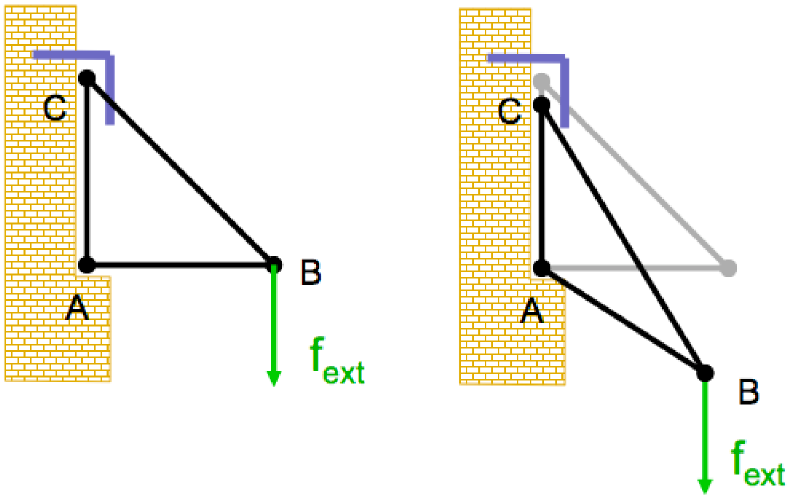

Figure 9.6. The static problem we wish to solve. Left: before application of force. Right: after application of force.

9.2.4 DSM: Solving for the shape of a construction made out of rods

Let us now demonstrate how we can use the above stiffness matrix formulation to solve an actual problem. Let us assume that we install the triangle against a wall in the way depicted in Fig.9.6. Point A is fixed, point C is allowed to move up and down, but not left or right and point B is completely free. A force $f_{\rm ext}$ is applied downward on point B. We now wish to solve for the displacements of these points. The stiffness equation becomes:

$$K\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} \end{pmatrix}\begin{pmatrix} \delta_A^1 \\ \delta_A^2 \\ \delta_B^1 \\ \delta_B^2 \\ \delta_C^1 \\ \delta_C^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -f_{\rm ext} \\ 0 \\ 0 \end{pmatrix} \tag{9.50}$$

where we have replaced rows 1,2 and 5 of the stiffness matrix of Eq.(9.49) with the condition that $\delta_A^1 = 0$, $\delta_A^2 = 0$ and $\delta_C^1 = 0$, which are the boundary conditions of the problem. Note that such boundary conditions are essential, because otherwise the matrix has zero determinant and the problem is ill-determined. By replacing these three lines with the boundary conditions the problem now becomes solvable. Note that we can simplify the above matrix equation as:

$$\begin{pmatrix} 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} \delta_B^1 \\ \delta_B^2 \\ \delta_C^2 \end{pmatrix} = \frac{1}{K} \begin{pmatrix} 0 \\ -f_{\text{ext}} \\ 0 \end{pmatrix} \tag{9.51}$$

The solution is:

$$\delta_B^1 = -\frac{f_{\text{ext}}}{K} \tag{9.52}$$

$$\delta_C^2 = -\frac{f_{\text{ext}}}{K} \tag{9.53}$$

$$\delta_B^2 = -2(1+\sqrt{2})\frac{f_{\text{ext}}}{K} \tag{9.54}$$

→ **Exercise:** Write a computer program (for instance in IDL) that solves for the static structure of a simple bridge made out of rods, and plot the result for different values of K.