Numerical Integration

Problem sheet 1

28/04/2009

Euler Method

The time evolution of a function $f(t)$ is given by the differential equation

$$\frac{df(t)}{dt} = -\nu f(t) \tag{1}$$

1. Calculate the analytic solution $f(t)$
2. In "numerical notation" we denote the so called timestep by Δt and the value of the function f at the time $t = n\Delta t$ with $n \in \{0, 1, 2, ...\}$ by f^n. Calculate the numerical expression f^{n+1} by discretizing the time derivative operator in equation (1) via $df(t)/dt = (f^{n+1} - f^n)/\Delta t$ and choosing the right-hand-side value $f(t)$ at the
   (a) actual time $f(t) = f^n$(Forward Euler method/Explicit method)
   (b) future time $f(t) = f^{n+1}$(Backward Euler method/Implicit method)
3. Calculate a linear approximation for the time evolution in dependence of $f(t)$ and compare the results with problem 2. *Hint: Calculate $f(t + \Delta t)$ and use Taylor-Expansion of $exp(-\nu\Delta t)$ up to the first order.*
4. Write a code in your favorite programming language to calculate numerically the discretized time evolution from problem 2 from $t_0 = 0$ to $t_{max} = 100$ in both explicit and implicit method. Compare with the analytic solution. Choose $\nu = 0.1$.
   (a) Determine a reasonable timestep t_{opt}, which minimizes the deviation from the analytic solution, but is calculated in reasonable CPU-time.
   (b) Vary the timestep Δt in a wide range. What is the result? Give an Explanation.

Newton-Raphson Method

Explicit method calculates the state of a system at a later time from the current state of the system, i.e.

$$f^{n+1} = F(f^n).$$

Implicit method however evolves the system by involving both the current state and the later one, i.e.

$$f^{n+1} = F(f^n, f^{n+1}),$$

or in general form,

$$G(f^{n+1}, f^n) = 0. \tag{2}$$

It is clear that solving eq. (2) requires extra work. Implicit method is used simply because many problems arising in real life are often *stiff*, for which the use of explicit method requires impractical timestep to bound the error (numerical stability). This is actually a quite common situation in astrophysical environment, ex. a strong source term. Hence getting familiar with implicit method is practically useful.

To solve eq. (2) we need a root-finding algorithm, such as Newton's method (also known as Newton-Raphson method). Consider the following differential equation

$$\frac{df(t)}{dt} = -[f(t)]^2 \tag{3}$$

1. Find the algorithm of Newton's method from the reference books or websites (ex. Wikipedia)
2. Given initial condition $f(0) = 1$, solve eq. (3) analytically and numerically from $t = 0$ to $t = 100$. Try again with explicit and implicit method learned from the last excercise.
3. Apply Newton's method to problem 2.
4. Can you solve a more general equation like $df/dt = -\alpha f^2 - \beta f^5$ implicitly without root finder? Don't try to do it, it is very hard! This gives us a good reason to learn root-finding algorithm.

Higher Order method

Euler method – 1st order

A successful integration relies on how accurate we evaluate the value at the later time. The accuracy of forward Euler method is first order, meaning that it introduces an error proportional to timestep Δt. Prove this statement.

Midpoint method – 2nd order

An example of a second-order method with two stages is provided by the midpoint method. Given

$$y'(t) = f(t, y(t)),$$

numerically, integration is done by the following formula

$$y^{n+1} = y^n + \Delta t f(t^n + \frac{\Delta t}{2}, y^n + \frac{\Delta t}{2} f(t^n, y^n))$$

1. Give an geometric explanation to this method.
2. Apply this method to $dy(t)/dt = -[y(t)]^2$ with $y(0) = 1$. Try out Δt=0.1, 1.0 and 2.0. You should see that midpoint method is more stable and accurate than the forward Euler method.

Runge-Kutta method – higher order

Euler method and midpoint method are the special cases of a general category named after two German mathematicians C. Runge and M. W. Kutta. It is a multi-step method in order to achieve higher order accuracy and stability at the expense of integration speed. This method is used widely in astrophysics. Just for your information, we are not going deeper here.