

Riemann Solver

Problem sheet 7/8

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We have seen that the exact Riemann solver is an expensive method and in more general case, such as MHD equations, Riemann invariants are actually very difficult to acquire.

Problem sheet 7/8 is a series of problems that help to develop a 1D HLL-type Riemann solver that is practically useful and can be easily extended to solve multi-dimensional equations. Here is the plan:

1. We first solve the jump condition and Riemann invariants for isothermal equation.
2. We derive and develop Riemann solver based on HLL.
3. We solve a Riemann problem numerically and compare the results with the “exactly” solution from 1. Here, we are not trying to really overlap the exact solution on the numerical results. But we do know the values and speeds of the breaking points which divide the wave structures into different regions.
4. Implement the higher order scheme based on MUSCL-Hancock method.

1 Riemann problem (analytic part)

We are going to solve the following hydrodynamic equation

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + c_s^2 \rho \end{bmatrix}_x = 0. \quad (1)$$

Assuming the state vector, \mathbf{q} , and flux vector, \mathbf{f} ,

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}$$
$$\mathbf{f} = \begin{bmatrix} \rho u \\ \rho u^2 + c_s^2 \rho \end{bmatrix},$$

equation (1) can be recast into

$$\mathbf{q}_t + \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \mathbf{q}_x = 0, \quad (2)$$

with $\frac{\partial \mathbf{f}}{\partial \mathbf{q}}$ being the Jacobian matrix as a function of state vector.

1. Find the two eigenvalues $\lambda_1 < \lambda_2$ and the corresponding eigenvectors e_1, e_2 .
2. Apply the Rankine-Hugoniot jump condition to the two characteristic families. Derive the following relations:

$$u_l = u_r \pm c_s \left(\sqrt{\frac{\rho_l}{\rho_r}} - \sqrt{\frac{\rho_r}{\rho_l}} \right), \quad (3)$$

with '+' corresponds to the 1-shock and '-' the 2-shock. Jump condition tells us how the state vector changes in the shock discontinuity.

3. Since the solution is self-similar, with the auxiliary variable $\xi = x/t$, eq. (2) reads

$$\frac{\partial \mathbf{f}}{\partial \mathbf{q}} \mathbf{q}'(\xi) = \xi \mathbf{q}'(\xi). \quad (4)$$

It shows that $\mathbf{q}'(\xi) \propto e_{1,2}$ and ξ represents the corresponding eigenvalues. Eq. (4) tells us how the state vector changes in the smooth region (if it is a simple wave). Solve the following differential equation and obtain the Riemann invariants for the two characteristic families.

$$\mathbf{q}'(\xi) = e_{1,2}.$$

It should look like

$$\begin{cases} u_l + c_s \ln \rho_l = u_r + c_s \ln \rho_r & \text{1-rarefaction} \\ u_l - c_s \ln \rho_l = u_r - c_s \ln \rho_r & \text{2-rarefaction} \end{cases} \quad (5)$$

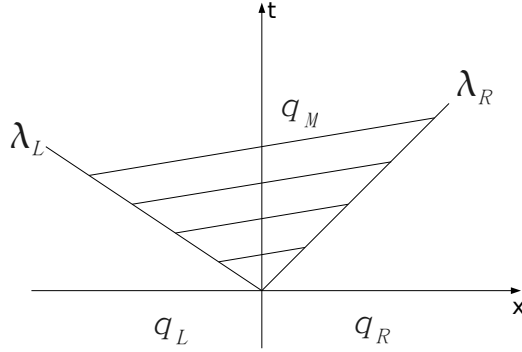


Figure 1: 2-wave HLL solver

2 HLL Solver

H(arten)-L(ax)-van L(eer) Riemann solver is a 2-wave solver and can be applied either to hydrodynamic or MHD equations. One retains only the 2 'fastest' waves (e.g. in general, the 2 fast magneto-acoustic waves) and then assume that between the 2 waves there is a uniform state \mathbf{q}_M as shown in Fig. 1.

To obtain the middle state \mathbf{q}_M , let us consider a volume of control V , i.e. an area of surface S in yz and delimited by $-\Delta x$ and Δx in x as shown in Fig. 2.

1. at $t = 0$, calculate averaged value of $\mathbf{q}(0)$ within V .
2. at $t = \Delta t$, the left and the right waves have reached: $x = \lambda_L \Delta t$ and $x = \lambda_R \Delta t$. Calculate $\mathbf{q}(\Delta t)$.
3. We also have the relation

$$S \times 2\Delta x \times (\mathbf{q}(\Delta t) - \mathbf{q}(0)) = (F_L - F_R)\Delta t. \quad (6)$$

Combine Eq. (6) and the results from 1 and 2, we obtain

$$\mathbf{q}_M = \frac{F_L - F_R + \lambda_R \mathbf{q}_R - \lambda_L \mathbf{q}_L}{\lambda_R - \lambda_L}.$$

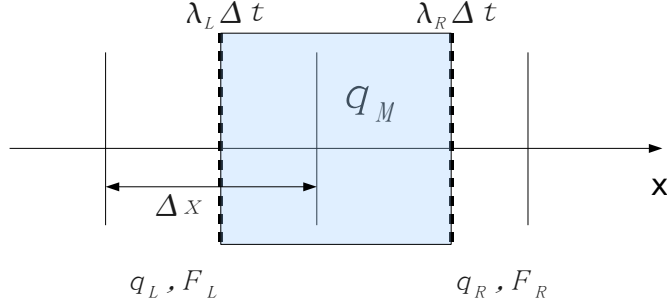


Figure 2: Waves in volume of control.

4. Assuming that $\lambda_L < 0$ and $\lambda_R > 0$ (as in Fig. 2), let us consider the volume of control delimited by $x = -\Delta x$ and $x = 0$. Based on conservation law in the volume of control, prove that

$$F_M = F_L + \lambda_L(\mathbf{q}_M - \mathbf{q}_L) = \frac{\lambda_R F_L - \lambda_L F_R + \lambda_L \lambda_R (\mathbf{q}_R - \mathbf{q}_L)}{\lambda_R - \lambda_L}.$$

Note that this expression is symmetrical in $R \leftrightarrow L$ showing that we will get the same result if we consider the volume of control delimited by $x = \Delta x$ and $x = 0$.

5. Now the flux, F_{HLL} , at the interface can be evaluated by

$$F_{HLL} = \begin{cases} F_L & \lambda_L > 0, \lambda_R > 0 \\ F_M & \lambda_L < 0, \lambda_R > 0 \\ F_R & \lambda_L < 0, \lambda_R < 0 \end{cases}$$

6. The final thing we need to decide is the value of λ_L and λ_R . Let us follow the estimate proposed by Davis(1988):

$$\lambda_L = \min[\lambda_1(\mathbf{q}_L), \lambda_1(\mathbf{q}_R)]$$

$$\lambda_R = \max[\lambda_2(\mathbf{q}_L), \lambda_2(\mathbf{q}_R)].$$

7. Implement a 1D hydrodynamic code based on HLL.

3 Riemann problem (numerical part)

Consider a Riemann problem bounded by $x = [0, 1]$ with outflow boundary condition:

$$\mathbf{q} = \begin{cases} (1, 0) & x \leq 0.5 \\ (0.125, 0) & x > 0.5 \end{cases}.$$

1. Use the code just developed to evolve this Riemann problem, run till $t = 0.15$.
2. Compare the numerical \mathbf{q}_M with the analytic solution. Hint: The wave structure is 1-rarefaction and 2-shock. Combine Eq.(3) and Eq.(5) to evaluate the analytic \mathbf{q}_M .
3. Are the wave speeds correct in the numerical result?

4 Higher Order scheme – MUSCL-Hancock

Apply MUSCL-Hancock method to the new code with whatever slope limiter you like. Solve the Riemann problem again and compare the result with the first order scheme.