Heat Equation

Problem Sheet 9

7/7/2009

Heat-diffusion equation constitutes the prototype for a parabolic PDE:

$$\frac{\partial Q}{\partial t} = \kappa \nabla^2 Q, \tag{1}$$

with $\kappa > 0$ being the thermal diffusivity. Given the initial condition and the boundary condition, heat equation can be solved numerically either in explicit way or in implicit way. This exercise is aiming at solving eq. (1) in 1D analytically and numerically.

1 The Analytical Solution

A 1D box delimited in a heat reservoir with left-boundary $x = -L$ and right-boundary $x = L$ will be solved analytically.

- Recast eq. (1) into 1-dimensional form.
- The boundary condition:

  $$\begin{aligned} Q(x = -L, t) &= 3 \\ Q(x = L, t) &= 1 \end{aligned}$$

  and the initial condition:

  $$Q(x, t = 0) = 3 - 2\Theta(x)$$

  with the definition of $\Theta(x)$ being

  $$\Theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

- Without solving equation, could you imagine the solution of steady-state? Write down the solution.

- Solve the full analytical solution for this problem.
  hint: the solution looks like:

$$Q(x,t) = 2 - \frac{x}{L} + \sum_{n=1}^{\infty} \exp(-\kappa \lambda_n^2 t) \left(\frac{-2}{n\pi} \right) \sin(\lambda_n x) \tag{2}$$

  with $\lambda_n = n\pi/L$. You can check its consistency with the boundary condition, the initial condition and the heat equation.

- Explain why the higher order modes damp out faster by physical intuition and by mathematical language.

2 The Explicit Method

- Discretize the 1D heat equation derived in the last exercise in explicit form.
- Every explicit method yields to an upper limit of the timestep Δt_{max}. Derive this upper limit analytically. (Hint: Use the Von Neumann stability analysis learned in Chapter 3. The result should look like, $\Delta t_{max} < \Delta x^2/(2\kappa)$.
- Implement the explicit numerical method and solve the 1D heat equation of the last problem.
- Compare your result with the analytical solution. How does the solution look like in frequency domain?

3 The Implicit Method

- Discretize the 1D heat equation derived in the first exercise in implicit form.
- Prove that the implicit method is unconditionally stable.
- Write the equation in matrix form with the new state of the future (timestep $n+1$) at the left-hand side and the current state (timestep n) at the right-hand side. Hint: It looks like a tridiagonal matrix.
- Implement the implicit method and solve the heat equation in the first problem either by the IDL routine trisol() or the routines from Numerical recipes.
- Compare your result with the analytical solution.