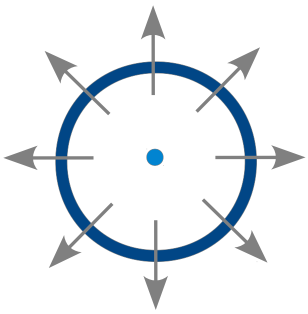
10 Point explosion

The sudden release of a large amount of energy E into a background fluid of density ρ_1 creates a strong explosion, characterized by a strong shock wave (a 'blast wave') emanating from the point where the energy was released. Such explosions occur for example in astrophysics in the form of supernova explosions.



But how fast will the shock wave travel and what is left behind? The problem of the point explosion is also known as Sedov-Taylor explosion, after the two scientists that first solved it by analytic (and in part numerical) means in the context of atomic bomb explosions. Today, the problem can provide a useful test to validate a hydrodynamical numerical scheme, because an analytic solution for it can be computed which can then be compared to numerical results. Also, the problem serves as a good example to demonstrate the power of dimensional analysis and scale-free solutions.

10.1 A rough estimate

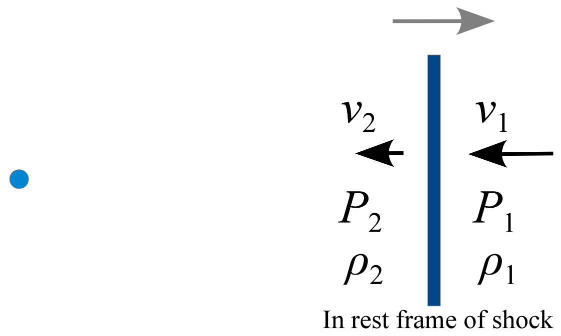
Let's begin by deriving an order of magnitude estimate for the radius $R(t)$ of the shock as a function of time. The mass of the swept up material is of order $M(t) \sim \rho_1 R^3(t)$. The fluid velocity behind the shock will be of order the mean radial velocity of the shock, $v(t) \sim R(t)/t$. We further expect

$$E_{\rm kin} \sim \frac{1}{2} M v^2 \sim \rho_1 R^3 \frac{R^2}{t^2} = \rho_1 \frac{R^5}{t^2} \tag{10.1}$$

What about the thermal energy in the bubble created by the explosion? This should be of order

$$E_{\rm therm} \sim \frac{3}{2} P V \tag{10.2}$$

where P is the postshock pressure. To find this pressure, we need to recall the jump conditions across a shock. If the shock moves to the right with velocity $v_1 = v(t)$, then in the rest-frame of the shock the background gas streams with velocity v_1 to the left, and comes out of the shock with a higher density ρ_2, higher pressure P_2, and with a lower velocity v_2.



The Rankine-Hugonoit relations for the shock tell us

$$\frac{\rho_1}{\rho_2} = \frac{v_2}{v_1} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)\mathcal{M}^2} \tag{10.3}$$

where

$$\mathcal{M} = \frac{v_1}{c_1} \tag{10.4}$$

is the Mach number of the shock. For a strong explosion, the sound-speed of the background medium is negligibly small, so that the Mach number will tend to infinity in this limit. For the pressure, the Rankine-Hugonoit relation is

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1} \tag{10.5}$$

As the background pressure is $P_1 = \rho_1 c_1^2/\gamma$, we then obtain in the limit of a *strong shock*:

$$P_2 \simeq \frac{2\rho_1 v_1^2}{\gamma + 1} \tag{10.6}$$

With this postshock pressure, we can now estimate the thermal energy in the shocked bubble:

$$E_{\text{therm}} \sim P_2 R^3 \sim \rho_1 v_1^2 R^3 \sim \rho_1 \frac{R^5}{t^2} \tag{10.7}$$

This suggests that the thermal energy is of the same order as the kinetic energy, and scales in the same fashion with time. Hence also for the total energy E, which is a conserved quantity, we expect

$$E = E_{\text{kin}} + E_{\text{therm}} \sim \rho_1 \frac{R^5}{t^2} \tag{10.8}$$

Solving for the radius $R(t)$, we get the expected dependence

$$R(t) \propto \left(\frac{E\, t^2}{\rho_1}\right)^{\frac{1}{5}} \tag{10.9}$$

10.1.1 Dimensional analysis

Another powerful approach to make a similar kind of estimate is through dimensionless analysis. If we assume the postshock pressure is always much larger than the preshock pressure, $P_2 \gg P_1$, then the value of P_1 plays no role. The only parameters of the problem are then E and ρ_1, and all quantities appearing in the solution can only be a combination of E, ρ_1, and t. We can hence use dimensional analysis to identify some of the expected dependencies.

The dimensions of the principal quantities are:

$$[\rho_1] = \frac{\mathsf{M}}{\mathsf{L}^3} \tag{10.10}$$

$$[E] = \mathsf{M}\frac{\mathsf{L}^2}{\mathsf{T}^2} \tag{10.11}$$

$$[t] = \mathsf{T} \tag{10.12}$$

The only quantity of dimension length we can construct from this is

$$\left[\left(\frac{Et^2}{\rho_1}\right)^{\frac{1}{5}}\right] = \mathsf{L} \tag{10.13}$$

Hence any radius relevant for the problem, in particular the shock radius, must depend on these variables through this combination. This also motivates us to define a *similarity variable* η as

$$\eta \equiv \frac{r}{(Et^2/\rho_1)^{1/5}} \tag{10.14}$$

At fixed time, this is simply a scaled radial coordinate. The shock will be at this time at some position η_s in this variable. Now, at a different time, the shock will be *at the same* value of η_s. The 'self-similar' solution becomes stationary in the appropriately scaled variables. For the shock position we can hence write

$$R(t) = \eta_s \left(\frac{Et^2}{\rho_1}\right)^{1/5} \propto t^{1/5}, \tag{10.15}$$

where η_s is a constant of order unity. The shock velocity follows via time differentiation as

$$v_s(t) = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\frac{R(t)}{t} \propto t^{-3/5} \tag{10.16}$$

10.1.2 Obtaining the exact solution

In order to obtain the numerical value of η_s, and the detailed structure of the interior solution in the explosion bubble, we can try to further exploit the scale-similarity of the solution (apart from assuming spherical symmetry, of course). We first recall that outside of the shock, the solution is $v = 0$, $\rho = \rho_1$, and $P = P_1 \simeq 0$. Inside the shock, for $0 < r < R(t)$, we can make the ansatz

$$\rho(r,t) = \rho_2 A(\eta) \tag{10.17}$$

because all physical quantities inside the shock can only depend on η. In this ansatz, we have already incorporated the postshock density

$$\rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1, \tag{10.18}$$

of a strong shock, so that we know $A(\eta_s) = 1$. Similarly, we can make the ansatz

$$P(r,t) = P_2 \left(\frac{\eta}{\eta_s}\right)^2 B(\eta), \tag{10.19}$$

so that at the shock we get $B(\eta_s) = 1$. The extra factor $(\eta/\eta_s)^2$ is here introduced for convenience later on, but it could also be absorbed into a redefinition of B. Finally, we can adopt

$$v(r,t) = v_2^{\rm lab} \left(\frac{\eta}{\eta_s}\right) C(\eta) \tag{10.20}$$

to describe the run of the radial velocity, with $C(\eta_s) = 1$. We need to be a bit careful about the meaning of $v_2^{\rm lab}$, which is not the postshock velocity v_2 in the shock's restframe that we considered earlier. Rather, $v_2^{\rm lab}$ is relative to the restframe of the background medium (the lab frame). If the shock moves with v_s to the right, we have

$$v_1^{\rm lab} = v_s - v_1 \tag{10.21}$$

$$v_2^{\rm lab} = v_s - v_2 \tag{10.22}$$

where positive v_1 and v_2 describe motion to the left, as in our sketch. Since $v_1^{\rm lab} = 0$, it follows $v_1 = v_s$, and from the jump conditions we have

$$v_2^{\rm lab} = v_1 - v_1\frac{\gamma-1}{\gamma+1} = \frac{2}{\gamma+1}v_s \tag{10.23}$$

We have now written $\rho(r,t)$, $P(r,t)$ and $v(r,t)$ in self-similar form in terms of three functions $A(\eta)$, $B(\eta)$ and $C(\eta)$. To determine these functions, we use the Euler equations in spherical symmetry:

$$\frac{\partial}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v\rho) = 0 \tag{10.24}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} \tag{10.25}$$

$$\frac{\partial}{\partial t}\left[\rho(u + \frac{1}{2}v^2)\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\rho v(u + \frac{P}{\rho} + \frac{1}{2}v^2)\right] = 0 \tag{10.26}$$

This is augmented with the ideal gas equation of state,

$$P = (\gamma - 1)\rho u. \tag{10.27}$$

We can substitute into these equations our three self-similar ansatz functions. Whenever the thermal energy appears, it can be replaced through a combination of pressure and density, $u = P/[(\gamma - 1)\rho]$.

In addition, we also want to carry out a change of variables, from (r, t) to (η, t). This can be accomplished by noting that:

$$\left.\frac{\partial}{\partial t}\right|_r = \left(\frac{\partial \eta}{\partial t}\right)_r \left.\frac{\partial}{\partial \eta}\right|_t + \left(\frac{\partial t}{\partial t}\right)_r \left.\frac{\partial}{\partial t}\right|_\eta = -\frac{2\eta}{5t}\left.\frac{\partial}{\partial \eta}\right|_t + \left.\frac{\partial}{\partial t}\right|_\eta \tag{10.28}$$

$$\left.\frac{\partial}{\partial r}\right|_t = \left(\frac{\partial \eta}{\partial r}\right)_t \left.\frac{\partial}{\partial \eta}\right|_t + \left(\frac{\partial t}{\partial r}\right)_t \left.\frac{\partial}{\partial t}\right|_\eta = \frac{\eta}{r}\left.\frac{\partial}{\partial \eta}\right|_t \tag{10.29}$$

Using these relations and our ansatz substitutions in the Euler equations leads to a set of coupled ordinary differential equations in η. From the mass conservation equation, we obtain

$$-\eta\frac{\mathrm{d}A}{\mathrm{d}\eta} + \frac{2}{\gamma+1}\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta A C) + \frac{4}{\gamma+1}AC = 0 \tag{10.30}$$

Similarly, one gets from the momentum equation

$$-C - \frac{2}{5}\eta\frac{\mathrm{d}C}{\mathrm{d}\eta} + \frac{4}{5(\gamma+1)}\left(C^2 + C\eta\frac{\mathrm{d}C}{\mathrm{d}\eta}\right) = -\frac{2}{5}\frac{\gamma-1}{\gamma+1}\frac{1}{A}\left(2B + \eta\frac{\mathrm{d}B}{\mathrm{d}\eta}\right) \tag{10.31}$$

and from the energy equation:

$$-2(B{+}AC^2) - \frac{2}{5}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}(B{+}AC^2) + \frac{4}{5(\gamma+1)}\left(5C(\gamma B + AC^2) + \eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left[C(\gamma B + AC^2)\right]\right) = 0 \tag{10.32}$$

The equations (10.30), (10.31), and (10.32) are three 1st order, non-linear, coupled differential equations for the functions $A(\eta)$, $B(\eta)$ and $C(\eta)$. They can in principle be easily numerically integrated from the point $A(\eta_s) = B(\eta_s) = C(\eta_s) = 1$ to $\eta = 0$. There is only one catch here – we don't actually know the starting value η_s yet!

If we simply guess a value for η_s, we can carry out the integration of the three functions and will get some kind of solution. The trouble is, another guess will also give us a solution, but a different one. However, for the right solution, an additional constraint must hold. The total energy in the solution must be equal to the total energy E released by the explosion. This corresponds to the constraint

$$\int_0^{R(t)} \left(\frac{P}{\gamma - 1} + \frac{1}{2}\rho v^2\right) 4\pi r^2 \mathrm{d}r = E. \tag{10.33}$$

Inserting our self-similar ansatz, this becomes

$$\frac{32\pi}{25(\gamma^2 - 1)} \int_0^{\eta_s} (B + AC^2)\eta^4 \,\mathrm{d}\eta = 1 \tag{10.34}$$

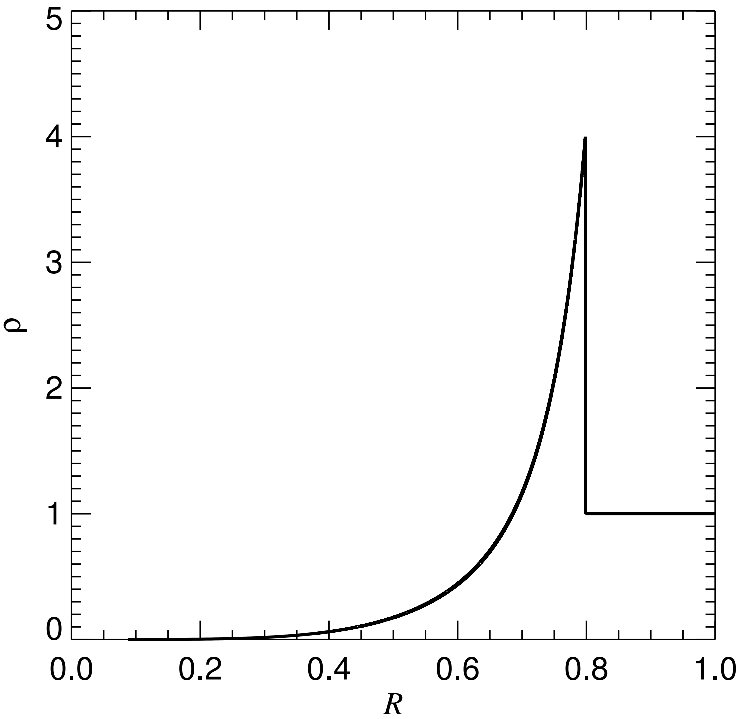
The solution strategy is therefore as follows:

1. Guess a value for η_s.
2. Calculate a numerical solution for $A(\eta)$, $B(\eta)$ and $C(\eta)$ based on equations (10.30), (10.31), and (10.32).
3. Check how well equation (10.34) integrates to 1. Adjust η_s accordingly and repeat until convergence.

Sedov actually managed to make considerably further progress in analytically solving these equations, deriving algebraic expressions for the shapes of the functions A, B and C. But even in this approach, there is however still one numerical integration left that is needed to determine η_s.

10.2 Structure of the solution

For a $\gamma = 5/3$ gas, η_s is given by $\eta_s \simeq 1.15$. The density structure described by $A(\eta)$ at time $t = 0.4$ for an explosion with energy $E = 1$ in a cold background medium with unit density $\rho = 1$ looks as follows:



We see that a large part of the mass in the bubble is indeed swept up close to the shock, with the density approaching zero at the very center. Interestingly, it turn out that the pressure is almost constant in the bubble; it drops slightly from the post-shock value and is then almost constant. The velocity profile on the other is approximately rising linearly from the explosion site to the edge of the shock.