11 Navier-Stokes equations and turbulence

So far, we have considered ideal gas dynamics governed by the Euler equations, where internal friction in the gas is assumed to be absent. Real fluids have internal stresses however, due to *viscosity*. The effect of viscosity is to dissipate relative motions of the fluid into heat.

11.1 Navier-Stokes equations

The Navier-Stokes equations are given by

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0 \tag{11.1}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla(\rho \mathbf{v}^T \mathbf{v} + P) = \nabla \mathbf{\Pi} \tag{11.2}$$

$$\frac{\partial}{\partial t}(\rho e) + \nabla[(\rho e + P)\mathbf{v}] = \nabla(\mathbf{\Pi} \mathbf{v}) \tag{11.3}$$

Here $\mathbf{\Pi}$ is the so-called viscous stress tensor, which is a material property. For $\mathbf{\Pi} = 0$, the Euler equations are recovered. To first order, the viscous stress tensor must be a linear function of the velocity derivatives. The most general tensor of rank-2 of this type can be written as

$$\mathbf{\Pi} = \eta \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3}(\nabla \cdot \mathbf{v}) \right] + \xi (\nabla \cdot \mathbf{v}) \mathbf{1} \tag{11.4}$$

Here η scales the traceless part of the tensor and describes the shear viscosity. ξ gives the strength of the diagonal part, and is the so-called bulk viscosity. Note that η and ξ can in principle be functions of local fluid properties, such as ρ, T, etc.

In the following however we shall assume constant viscosity coefficients. Also, we specialize to incompressible fluids with $\nabla \cdot \mathbf{v} = 0$, which is a particularly important case in practice. Let's see how the Navier-Stokes equations simplify in this case. Obviously, ξ is then unimportant and we only need to deal with shear viscosity. Now, let us consider one of the components of the viscous shear force:

$$\frac{1}{\eta}(\nabla \mathbf{\Pi})_x = \frac{\partial}{\partial x}\left(2\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \tag{11.5}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) v_x = \nabla^2 v_x \tag{11.6}$$

where we made use of the $\nabla \cdot \mathbf{v} = 0$ constraint. If we furthermore introduce the *kinematic viscosity* ν as

$$\nu \equiv \frac{\nu}{\rho}, \tag{11.7}$$

we can write the momentum equation in the compact form

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{v}, \tag{11.8}$$

where the derivative on the left-hand side is the Lagrangian derivative. We hence see that the motion of individual fluid elements responds to pressure gradients and to viscous forces. This form of the equation is also often simply referred to as the Navier-Stokes equation.

11.2 Scaling properties of viscous flows

Consider the Navier-Stokes equations for some flow problem that is characterized by some characteristic length L_0, velocity V_0, and density scale ρ_0. We can then define dimensionless fluid variables of the form

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{V_0}, \qquad \hat{\mathbf{x}} = \frac{\mathbf{x}}{L_0}, \qquad \hat{P} = \frac{P}{\rho_0 V_0^2}. \tag{11.9}$$

Similarly, we define a dimensionless time, a dimensionless density, and a dimensionless nabla operator:

$$\hat{t} = \frac{t}{L_0/V_0}, \qquad \hat{\rho} = \frac{\rho}{\rho_0}, \qquad \hat{\nabla} = L_0 \nabla. \tag{11.10}$$

Inserting these definitions into the Navier-Stokes equation (11.8), we obtain the dimensionless equation

$$\frac{D\hat{\mathbf{v}}}{D\hat{t}} = -\frac{\hat{\nabla}\hat{P}}{\hat{\rho}} + \frac{\nu}{L_0 V_0} \hat{\nabla}^2 \hat{\mathbf{v}}. \tag{11.11}$$

Interestingly, this equation involves one number,

$$\mathrm{Re} \equiv \frac{L_0 V_0}{\nu}, \tag{11.12}$$

which characterizes the flow and determines the structure of the possible solutions of the equation. This is the so-called Reynolds number. Problems which have similar Reynolds number are expected to exhibit very similar fluid behavior. One then has *Reynolds-number similarity.* In contrast, the Euler equations ($\mathrm{Re} \to \infty$) exhibit always scale similarity because they are invariant under scale transformations.

One intuitive interpretation one can give the Reynolds number is that it measures the importance of inertia relative to viscous forces. Hence:

$$\mathrm{Re} \approx \frac{\text{inertial forces}}{\text{viscous forces}} \approx \frac{D\mathbf{v}/Dt}{\nu \nabla^2 \mathbf{v}} \approx \frac{V_0/(L_0/V_0)}{\nu V_0/L_0^2} = \frac{L_0 V_0}{\nu} \tag{11.13}$$

If we have $\mathrm{Re} \sim 1$, we are completely dominated by viscosity. On the other hand, for $\mathrm{Re} \to \infty$ viscosity becomes unimportant and we approach an ideal gas.

11.3 Turbulence

Fluid flow which is unsteady, irregular, seemingly random, and chaotic is called *turbulent*. Examples of such situations include the smoke from a chimney, a waterfall, or the wind field behind a fast car or airplane. The characteristic feature of turbulence is that the fluid velocity varies significantly and irregularly both in position and time. As a result, turbulence is a statistical phenomenon and is best described with statistical techniques.

If the turbulent motions are subsonic, the flow can often be approximately treated as being incompressible, even for an equation of state that is not particularly stiff. Then only solenoidal motions that are divergence free can occur, or in other words, only shear flows can occur. We have already seen that such flows are subject to fluid instabilities such as the Kelvin-Helmholtz instability, which can easily produce swirling motions on many different scales. Such vortex-like motions, also called *eddies*, are the conceptual building blocks of Kolmogorov's theory of incompressible turbulence, which yields a surprisingly accurate description of the basic phenomenology of turbulence, even though many aspects of turbulence are still not fully understood.

11.3.1 Kolmogorov's theory of incompressible turbulence

We consider a fully turbulent flow with characteristic velocity V_0 and length scale L_0. We assume that a quasi-stationary state for the turbulence is achieved by some kind of driving process on large scales, which in a time-averaged way injects an energy ϵ per unit mass. We shall also assume that the Reynolds number Re is large.

We imagine that the turbulent flow can be considered to be composed of eddies of different size l, with characteristic velocity $v(l)$, and associated timescale $\tau(l) = l/v(l)$.

For the largest eddies, $l \sim L_0$ and $v(l) \sim V_0$, hence viscosity is unimportant for them. But large eddies are unstable und break up, transferring their energy to somewhat smaller eddies. This continues to yet smaller scales, until

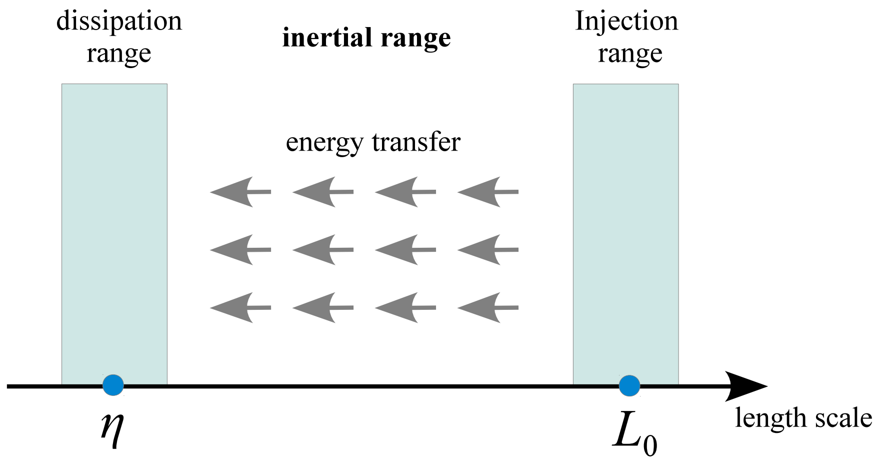
$$\mathrm{Re}(l) = \frac{l v(l)}{\nu} \tag{11.14}$$

reaches of order unity. For these eddies, viscosity will be very important so that their kinetic energy is dissipated away. We will see that this transfer of energy to smaller scales gives rise to the *energy cascade* of turbulence.

But several important questions are still unanswered.

1. What is the actual size of the smallest eddies that dissipate the energy?

2. How do the velocities $v(l)$ of the eddies vary with l when the eddies become smaller?



Kolmogorov conjectured a number of hypothesis that can answer these questions. In particular, he hypothesized:

- For high Reynolds number, the small-scale turbulent motions ($l \ll L_0$) become statistically isotropic. Any memory of large-scale boundary conditions and the original creation of the turbulence on large scales is lost.
- For high Reynolds number, the statistics of small-scale turbulent motions has a universal form and is only determined by ν and the energy injection rate per unit mass, ϵ.

From ν and ϵ, one can construct characteristic Kolmogorov length, velocity and timescales. Of particular importance is the *Kolmogorov length*:

$$\nu \equiv \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \tag{11.15}$$

Velocity and timescales are given by

$$v_\nu = (\epsilon\nu)^{1/4}, \qquad \tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2} \tag{11.16}$$

We then see that the Reynolds number at the Kolmogorov scales is

$$\mathrm{Re}(\eta) = \frac{\eta v_\eta}{\nu} = 1, \tag{11.17}$$

showing that they describe the dissipation range.

Kolmogorov has furthermore made a second similarity hypothesis, as follows:

- For high Reynolds number, there is a range of scales $L_0 \gg l \gg \eta$ over which the statistics of the motions on scale l take a universal form, and this form is *only* determined by ϵ, *independent* of ν.

In other words, this also means that viscous effects are unimportant over this range of scales, which is called the *inertial range*. Given an eddy size l in the inertial range, one can construct its characteristic velocity and timescale just from l and ϵ:

$$v(l) = (\epsilon l)^{1/2}, \qquad \tau(l) = \left(\frac{l^2}{\epsilon}\right)^{1/3}. \tag{11.18}$$

One further consequence of the existence of the inertial range is that we expect that here the energy transfer rate

$$T(l) \sim \frac{v^2(l)}{\tau(l)} \tag{11.19}$$

of eddies to smaller scales is expected to be scale-invariant. Indeed, putting in the expected characteristic scale dependence we get $T(l) \sim \epsilon$, i.e. $T(l)$ is equal to the energy injection rate. This also implies that we have

$$\epsilon \sim \frac{V_0^2}{L_0}. \tag{11.20}$$

With this result we can also work out what we expect for the ratio between the characteristic quantities of the largest and smallest scales:

$$\frac{\eta}{L_0} \sim \left(\frac{\nu^3}{\epsilon L_0^4}\right)^{1/4} = \left(\frac{\nu^3}{V_0^3 L_0^3}\right)^{1/4} = \mathrm{Re}^{-\frac{3}{4}} \tag{11.21}$$

$$\frac{v_\eta}{V_0} \sim \left(\frac{\epsilon \nu}{V_0^4}\right)^{1/4} = \left(\frac{V_0^3 \nu}{L_0 V_0^4}\right)^{1/4} = \mathrm{Re}^{-\frac{1}{4}} \tag{11.22}$$

$$\frac{\tau_\eta}{\tau} \sim \left(\frac{\nu V_0^2}{\epsilon L_0^2}\right)^{1/2} = \left(\frac{\nu V_0^2 L_0}{V_0^3 L_0^2}\right)^{1/2} = \mathrm{Re}^{-\frac{1}{2}} \tag{11.23}$$

This shows that the Reynolds number directly sets the dynamic range of the inertial range.

11.3.2 Energy spectrum of Kolmogorov turbulence

Eddy motions on a length-scale l correspond to wavenumber $k = 2\pi/l$. The kinetic energy ΔE contained between two wave numbers k_1 and k_2 can be described by

$$\Delta E = \int_{k_1}^{k_2} E(k)\,\mathrm{d}k, \tag{11.24}$$

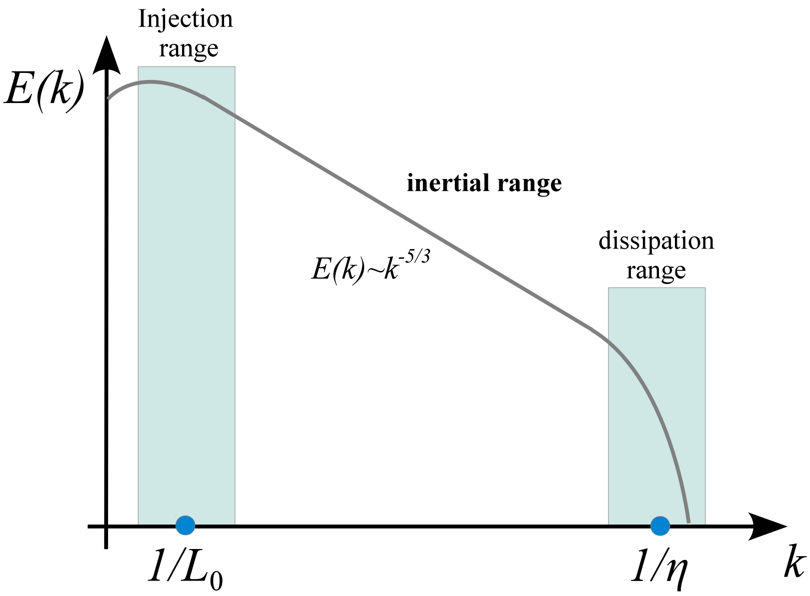
where $E(k)$ is the so-called energy spectrum. For the inertial range in Kolmogorov's theory, we know that $E(k)$ is a universal function that only depends on ϵ and k. Hence $E(k)$ must be of the form

$$E(k) = C\,\epsilon^a\,k^b, \tag{11.25}$$

where C is a dimensionless constant. Through dimensional analysis it is easy to see that one must have $a = 2/3$ and $b = -5/3$. We hence obtain the famous $-5/3$ slope of the Kolmogorov energy power spectrum:

$$E(k) = C\,\epsilon^{2/3}\,k^{-5/3}. \tag{11.26}$$

The constant C is universal in Kolmogorov's theory, but cannot be computed from first principles. Experiment and numerical simulations give $C \simeq 1.5$.



Actually, if we recall Kolmogorov's first similarity hypothesis, it makes the stronger claim that the statistics for all small scale motion is universal. This means that also the dissipation part of the turbulence must have a universal form. To include this in the description of the spectrum, we can for example write

$$E(k) = C\,\epsilon^{2/3}\,k^{-5/3} f_\eta(k\eta), \tag{11.27}$$

where $f_\eta(k\eta)$ us a universal function with $f_\eta(x) = 1$ for $x \ll 1$, and with $f_\eta(x) \to 0$ for $x \to \infty$. This function has to be determined experimentally or numerically. A good fit to different results is given by

$$f_\eta(x) = \exp\left(-\beta[(x^4 + c^4)^{1/4} - c]\right), \tag{11.28}$$

with $\beta_0 \sim 5.2$ and $c \sim 0.4$.