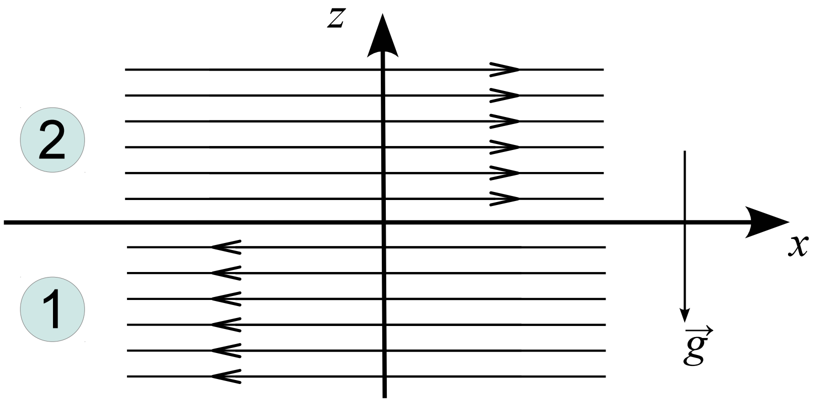
9 Fluid Instabilities

9.1 Stability of a shear flow

In many situations, gaseous flows can be subject to fluid instabilities in which small perturbations can rapidly flow, thereby tapping a source of free energy. An important example for this are Kelvin-Helmholtz and Rayleigh-Taylor instabilities, which we discuss in this chapter.

We consider a flow in the x-direction, which in the lower half-space $z < 0$ has velocity U_1 and density ρ_1, whereas in the upper half-space the gas streams with U_2 and has density ρ_2. In addition there can be a homogeneous gravitational field $\mathbf{g}$ pointing into the negative z-direction.



Let us assume the flow can, at least approximately, be treated as an incompressible potential flow. Let the velocity field in the upper and lower halves be given by

$$\text{upper half:} \quad \mathbf{v}_2 = \nabla\Phi_2 \quad \text{for } z > 0 \tag{9.1}$$

$$\text{lower half:} \quad \mathbf{v}_1 = \nabla\Phi_1 \quad \text{for } z < 0 \tag{9.2}$$

The equation of motion for an incompressible gas with constant density can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{g} - \nabla\left(\frac{P}{\rho}\right). \tag{9.3}$$

If we use the identity $(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\mathbf{v}^2/2 - \nabla\times\mathbf{v}$, together with the assumption of a potential flow $\mathbf{v} = \nabla\Phi$ (and hence $\nabla\times\mathbf{v} = 0$), we can write the equation of motion as

$$\nabla\frac{\partial \Phi}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{v}^2\right) - \mathbf{g} + \nabla\left(\frac{P}{\rho}\right) = 0 \tag{9.4}$$

Writing the gravitational acceleration as $\mathbf{g} = -g\hat{\mathbf{e}}_z$, this implies

$$\frac{\partial \Phi}{\partial t} + \left(\frac{1}{2}\mathbf{v}^2\right) + gz + \frac{P}{\rho} = const. \tag{9.5}$$

which is Bernoulli's theorem, a useful result that we will exploit later on.

We now assume for the velocity potentials in the upper and lower half

$$\Phi_1 = U_1\, x + \phi_1 \tag{9.6}$$
$$\Phi_2 = U_2\, x + \phi_2 \tag{9.7}$$

where ϕ_1 and ϕ_2 are infinitesimal perturbations. Note that these must fulfill $\nabla^2\phi_1 = 0$ and $\nabla^2\phi_2 = 0$ because the density is supposed to be constant in each of the two regions. Let us further introduce a function $\xi(x,t) = z$ that describes the z-location of the interface. The total time derivative of this equation hence describes the velocity of the interface in the z-direction. This must match the fluid velocities in the z direction of the two phases, yielding for example for the 1-side:

$$\frac{\partial \xi}{\partial x}\frac{\partial \Phi_1}{\partial x} + \frac{\partial \xi}{\partial t} = \frac{\partial \Phi_1}{\partial z}. \tag{9.8}$$

This gives first

$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\left(U_1 + \frac{\partial \phi_1}{\partial x}\right) = \frac{\partial \phi_1}{\partial z}. \tag{9.9}$$

The second term in brackets on the left side can be dropped to leading order. The same condition also holds for the other phase, hence we obtain the two equations:

$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}U_1 = \frac{\partial \phi_1}{\partial z}, \tag{9.10}$$
$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}U_2 = \frac{\partial \phi_2}{\partial z}. \tag{9.11}$$

We can now relate the perturbed interface to the unperturbed state via the Bernoulli equation. For example, for phase 1, we can write

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{2}(U_1 + \frac{\partial \phi_1}{\partial x})^2 + g\xi + \frac{P_1}{\rho_1} = \frac{1}{2}U_1^2 + \frac{\overline{P}}{\rho_1}, \tag{9.12}$$

where the left hand side is the perturbed state, the right hand side is the unperturbed state, with the initial pressure $\overline{P}$. An analogous equation can also be written down for phase 2, with an equal initial pressure $\overline{P}$. In addition, the pressures P_1 and P_2 must be equal. Equating the two pressures from the two Bernoulli equations and keeping only leading order terms leads to

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + U_1\frac{\partial \phi_1}{\partial x} + g\xi\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + U_2\frac{\partial \phi_2}{\partial x} + g\xi\right). \tag{9.13}$$

We now seek solutions for the three functions ϕ_1, ϕ_2 and ξ, fulfilling the three equations (9.10), (9.11) and (9.13). To this end we make an eigenmode analysis. Consider the ansatz

$$\phi_1 = \phi_1(z)\exp[i(kx - \omega t)] \tag{9.14}$$

which respects the symmetry of the problem. Because ϕ_1 fulfills the Laplace equation, we get

$$\frac{\partial^2 \phi_1}{\partial z^2} = k^2\,\phi_1, \tag{9.15}$$

which has solutions $\phi_1(z) \propto \exp(kz)$ and $\phi_1(z) \propto \exp(-kz)$. However, the latter solution can be discarded because of boundary conditions, since for $z \to -\infty$ we need to have an unperturbed state with $\phi_1 \to 0$. In a similar way, we can make an ansatz for ϕ_2 and conclude that its z-dependence can only go as $\phi_2(z) \propto \exp(-kz)$. This therefore leads to the following three equations for a single Fourier mode:

$$\phi_1 = \hat{\phi}_1 \exp(kz)\exp[i(kx - \omega t)] \tag{9.16}$$

$$\phi_2 = \hat{\phi}_2 \exp(-kz)\exp[i(kx - \omega t)] \tag{9.17}$$

$$\xi = \hat{\xi}\exp[i(kx - \omega t)] \tag{9.18}$$

Here $\hat{\phi}_1$, $\hat{\phi}_2$, and $\hat{\xi}$ are the corresponding mode amplitudes. Inserting these mode equations into the differential equations (9.10), (9.11) and (9.13) yields three algebraic equations:

$$-i\omega\hat{\xi} + U_1 ik\hat{\xi} = k\hat{\phi}_1 \tag{9.19}$$

$$-i\omega\hat{\xi} + U_2 ik\hat{\xi} = -k\hat{\phi}_2 \tag{9.20}$$

$$\rho_1(-i\omega\hat{\phi}_1 + U_1 ik\hat{\phi}_1 + g\hat{\xi}) = \rho_2(-i\omega\hat{\phi}_2 + U_2 ik\hat{\phi}_2 + g\hat{\xi}) \tag{9.21}$$

Non-trivial solutions with $\hat{\xi} \neq 0$ are possible for

$$\omega^2(\rho_1 + \rho 2) - 2\omega k(\rho_1 U_1 + \rho_2 U_2) + k^2(\rho_1 U_1^2 + \rho_2 U_2^2) + (\rho_2 - \rho 1)kg = 0, \tag{9.22}$$

which is the *dispersion relation.* Unstable, exponentially growing mode solutions appear if there are solutions for ω with negative imaginary part. Below, we examine the dispersion relation for a few special cases.

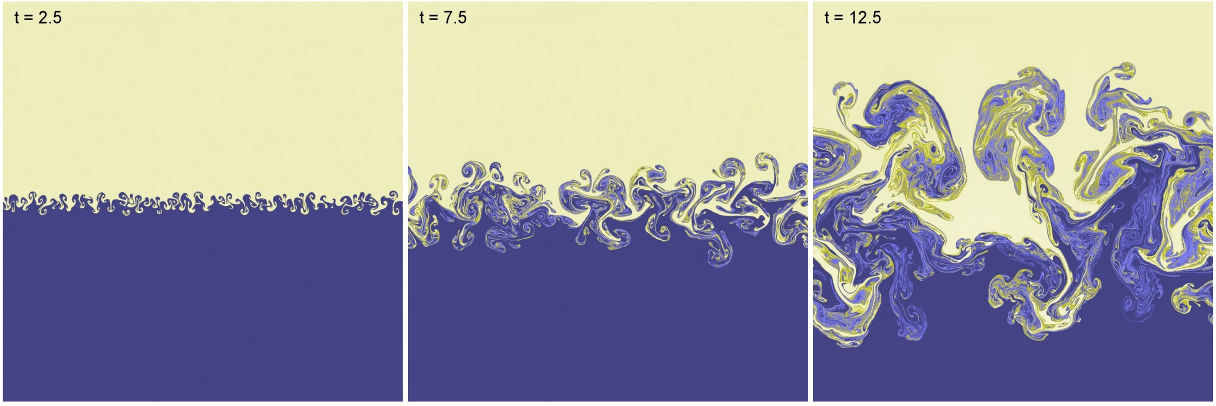
9.2 Rayleigh-Taylor instability

Let us consider the case of a fluid at rest, $U_1 = U_2 = 0$. The dispersion relation simplifies to

$$\omega^2 = \frac{(\rho_1 - \rho_2)kg}{\rho_1 + \rho_2}. \tag{9.23}$$

We see that for $\rho_2 > \rho_1$, i.e. the denser fluid lies on top, unstable solutions with $\omega^2 < 0$ exist. This is the so-called Rayleigh-Taylor instability. It is in essence buoyancy driven and leads to the rise of lighter material in a stratified atmosphere. The free energy that is tapped here is the potential energy in the gravitational field. Also notice that for an ideal gas, arbitrary small wavelengths are unstable, and those modes will also grow fastest.

A growing Rayleigh-Taylor instability:



If on the other hand we have $\rho_1 > \rho_2$, then the interface is stable and will only oscillate when perturbed.

9.3 Kelvin-Helmholtz instability

If we set the gravitational field to zero, $g = 0$, we have the situation of a pure shear flow. In this case, the solutions of the dispersion relation are given by

$$\omega_{1/2} = \frac{k(\rho_1 U_1 + \rho_2 U_2)}{\rho_1 + \rho_2} \pm i \frac{\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2} |U_1 - U_2| \tag{9.24}$$

Interestingly, in an ideal gas there is an imaginary growing mode component for every $|U_1 - U_2| > 0$! This means that a small wave-like perturbation at an interface will grow rapidly into large waves that take the form of characteristic Kelvin-Helmholtz "billows". In the non-linear regime reached during the subsequent evolution of this instability the waves are rolled up, leading to the creation of vortex like structures. As the instability grows fastest for small scales (high k), with time the billows tend to get larger and larger.

As the Kelvin-Helmholtz instability basically means that any sharp velocity gradient in a shear flow is unstable in a freely streaming fluid, this instability is particularly important for the creating of fluid turbulence.

Under certain conditions, some modes can however be stabilized against the instability. This happens for example if we consider shearing with $U_1 \neq U_2$ in a gravitational field $g > 0$. Then the dispersion relation has the solutions

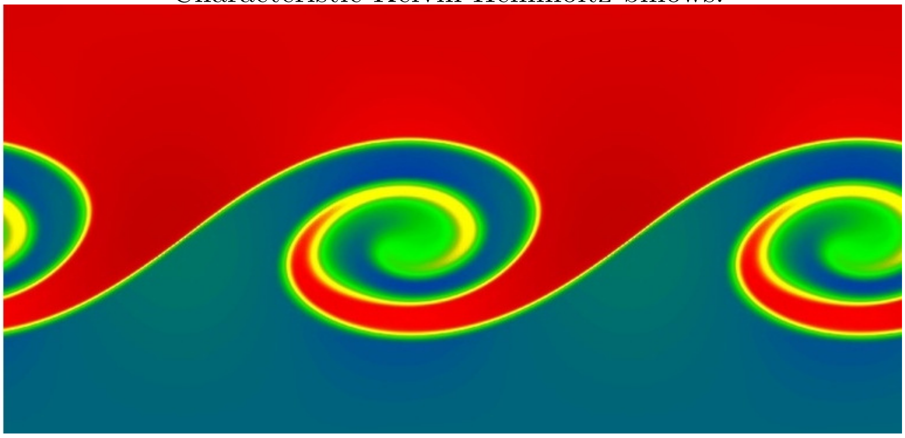
$$\omega = \frac{k(\rho_1 U_1 + \rho_2 U_2)}{\rho_1 + \rho_2} \pm \frac{\sqrt{-k^2 \rho_1 \rho_2 (U_1 - U_2)^2 - (\rho_1 + \rho_2)(\rho_2 - \rho_1) k g}}{\rho_1 + \rho_2}. \tag{9.25}$$

Stability is possible if two conditions are met. First, we need $\rho_1 > \rho_2$, i.e. the lighter fluid needs to be on top (otherwise we would have in any case a Rayleigh-Taylor instability). Second, the condition

$$(U_1 - U_2)^2 < \frac{(\rho_1 + \rho_2)(\rho_1 + \rho_2) g}{k \rho_1 \rho_2} \tag{9.26}$$

must be fulfilled. Compared to the ordinary Kelvin-Holmholtz instability without a gravitational field, we hence see that sufficiently small wavelengths are stabilized below a threshold wavelength. The larger the shear becomes, the further this threshold moves to small scales.

Characteristic Kelvin-Helmholtz billows:



The Rayleigh-Taylor and Kelvin-Helmholtz instabilities are by no means the only fluid instabilities that can occur in an ideal gas. For example, there is also the Richtmyer-Meshov instability, which can occur when an interface is suddenly accelerated, for example due to the passage of a shock wave. In self-gravitating gases, there is the Jeans instability, which occurs when the internal gas pressure is not strong enough to prevent a positive density perturbation from growing and collapsing under its own gravitational attraction. This type of instability is particularly important in cosmic structure growth and star formation. If the gas dynamics is coupled to external sources of heat (e.g. through a radiation field), a number of further instabilities are possible. For example, a thermal instability can occur when a radiative cooling function has a negative dependence on temperature. If the temperature drops somewhere a bit more through cooling than elsewhere, the cooling rate of this cooler patch will increase such that it is cooling even faster. In this way, cool clouds can drop out of the background gas.