Chapter 2

Hyperbolic equations

As we have seen, and will further argue below, the hydrodynamics equations are nothing more than signal-propagation equations. Equations of this kind are called *hyperbolic equations*. The equations of hydrodynamics are only a member of the more general class of hyperbolic equations, and there are many more examples of hyperbolic equations than just the equations of hydrodynamics. But of course we shall focus mostly on the application to hydrodynamics. In this chapter we shall study the mathematical properties of hyperbolic equations, and analytic methods how to solve them for simple linear problems. This background knowledge is not only important for understanding the nature of hydrodynamic flows, it also lies at the basis of numerical hydrodynamics algorithms. Hence we go into quite some detail here.

2.1 The simplest form of a hyperbolic equation: advection

Consider the following equation:

$$\partial_t q + u\partial_x q = 0 \tag{2.1}$$

where $q = q(x,t)$ is a function of one spatial dimension and time, and u is a velocity that is constant in space and time. This is called an *advection equation*, as it describes the time-dependent shifting of the function $q(x)$ along x with a velocity u (Fig. 2.1-left). The solution at any time $t > t_0$ can be described as a function of the state at time t_0:

$$q(x,t) = q(x-ut,0) \tag{2.2}$$

This is a so-called *initial value problem* in which the state at any time $t > t_0$ can be uniquely found when the state at time $t = t_0$ is fully given. The characteristics of this problem are straight lines:

$$x_{\rm char}(t) = x^{(0)}_{\rm char} + ut \tag{2.3}$$

This is a family of lines in the (x,t) plane, each of which is labeled by its own unique value of $x^{(0)}_{\rm char}$.

In this example the initial value problem is quite trivial, yet, as we will see below, this problem stands at the basis of numerical methods of hydrodynamics and is numerically surprisingly challenging to solve!

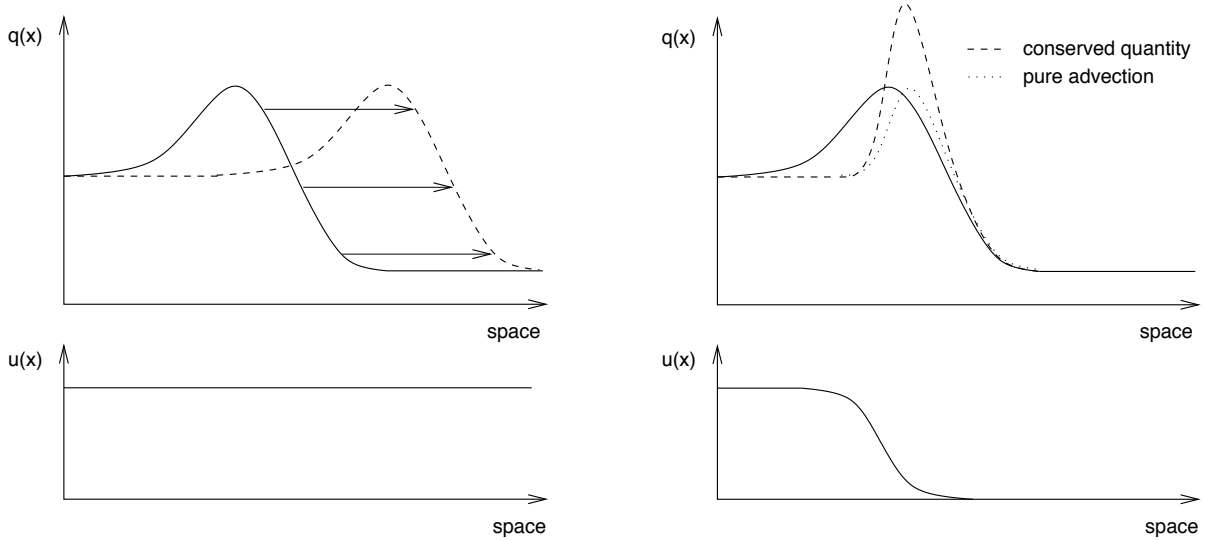

Figure 2.1. Advection of a function $q(x,t)$ with constant velocity u (left) and space-varying velocity $u(x)$ (right). The space-varying velocity problem comes in two versions: the conserved form (dashed) and the non-conserved simple advection form (dotted).

2.2 Advection with space-dependent velocity

Consider now that u is a function of space: $u = u(x)$. We get:

$$\partial_t q + u(x)\partial_x q = 0 \tag{2.4}$$

This is still an advection problem, but the velocity is not constant, and therefore the solution is somewhat more complex.

If we define the comoving derivative D_t as $D_t = \partial_t + u(x)\partial_x$, Eq.(2.4) translates into:

$$D_t q(x,t) = 0 \tag{2.5}$$

This simply tells that along each flow line (characteristic) the function q remains constant, i.e. the *comoving* derivative is zero.

The formula describing the characteristics for this non-constant velocity case is:

$$x(t) = x_0 + \int_0^t u(x(t'))dt' \tag{2.6}$$

Reversely, when we are at a point (x,t) in spacetime, we can also integrate the characteristic back in time to $t = 0$:

$$x(0) = x + \int_t^0 u(x(t'))dt' \tag{2.7}$$

(note that since we integrate from t to 0 the dt' will be negative). For the simplified case of a constant $u(x) = u$ this would lead to:

$$x(0) = x - ut \tag{2.8}$$

So the advection, which means looking back in time, means that

$$q(x,t) = q\left(x + \int_t^0 u(x(t'))dt', 0\right) \tag{2.9}$$

For the simplified case of a constant $u(x) = u$ this would lead to:

$$q(x,t) = q(x - ut, 0) \tag{2.10}$$

2.3 Advection of a conserved quantity with space-dependent velocity

Now consider yet another version of the advection equation:

$$\partial_t q + \partial_x [q u(x)] = 0 \tag{2.11}$$

This is very similar to Eq. (2.4), but this time with $u(x)$ inside the ∂_x operator. This is a *conservation equation*. It has the following property of conservation: if $u(x)$ is zero at both x_0 and x_1, then the integral $\int_{x_0}^{x_1} q dx$ is constant in time.

This equation can be written in the previous form, with $u(x)$ outside the operator:

$$\partial_t q + u(x)\partial_x q = -q\partial_x u(x) \tag{2.12}$$

This is an advection equation of the type of Eq. (2.4), but this time with a non-zero right-hand-side. This can be regarded as some kind of source term in the equation (even though strictly speaking it is not, as the system is conserved). The characteristics are the same as for the case without this term, but in this case the value of q is no longer constant along these characteristics:

$$D_t q(x,t) = -q(x,t)\partial_x u(x) \tag{2.13}$$

When the characteristics converge, q goes up, and when they diverge, q goes down. This is depicted in Fig. 2.1-right, dashed line. This is precisely what one expects to happen if q represents some kind of density.

2.4 Flux-form of conservation equation and the Jacobian

Let us stick with the equation (2.11), but write it in the following form:

$$\partial_t q + \partial_x f(q,x) = 0 \tag{2.14}$$

where $f(q,x)$ is called the flux of the quantity q. The flux f is entirely determined for any given value of the function q and position x. We can then use the chain rule to arrive at

$$\partial_t q + \left.\frac{\partial f}{\partial q}\right|_{x=\text{const}} \partial_x q = -\left.\frac{\partial f(q,x)}{\partial x}\right|_{q=\text{const}} \tag{2.15}$$

In our case we have:

$$\left.\frac{\partial f}{\partial q}\right|_{x=\text{const}} = u(x) \qquad \text{and} \qquad \left.\frac{\partial f}{\partial x}\right|_{q=\text{const}} = q\partial_x u(x) \tag{2.16}$$

In this way we also arrive at Eq. (2.12). It is this route to Eq. (2.12) that we will take in the cases of the more complex non-linear systems of equations to follow. The $\partial f/\partial q$ is the Jacobian of the system.

2.5 Coupled set of equations

All of the above can also be formulated for coupled sets of PDEs. Instead of a state scalar q we defined a state vector $Q = (q_1, \cdots, q_m)$. The advection equation of the non-conservative type is then

$$\partial_t Q + A\partial_x Q = 0 \tag{2.17}$$

where A is an $m \times m$ matrix. The advection equation of conservative type is

$$\partial_t Q + \partial_x (AQ) = 0 \tag{2.18}$$

The more general conservation equation is:

$$\partial_t Q + \partial_x F = 0 \tag{2.19}$$

where $F = F(Q, x)$ is the flux. Like in the scalar case we have

$$\partial_t Q + \left.\frac{\partial F}{\partial Q}\right|_{x=\text{const}} \partial_x Q = - \left.\frac{\partial F(q,x)}{\partial x}\right|_{Q=\text{const}} \tag{2.20}$$

where the $J \equiv \partial F / \partial Q$ is the Jacobian of the system, which is an $m \times m$ matrix. If F is a linear function of Q, then one can write $F = AQ$ where A is then the Jacobian, and we arrive at Eq. (2.18). If A is then also independent of x, then we arrive at Eq. (2.18).

2.6 The wave equation in vector notation: eigenvalues and eigenvectors

In Section 1.7.1 we derived the wave equation for perturbations in an otherwise steady constant-density constant-velocity background medium. Let us now define:

$$Q \equiv \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_0 u_1 \end{pmatrix} \tag{2.21}$$

We can then write Eqs. (1.72,1.73) as:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} u_0 & 1 \\ C_s^2 & u_0 \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \tag{2.22}$$

We can also write this in flux conservative form:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \partial_x \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \tag{2.23}$$

with the flux $F = (f_1, f_2)$ given as:

$$F \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} u_0 & 1 \\ C_s^2 & u_0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u_0 q_1 + q_2 \\ C_s^2 q_1 + u_0 q_2 \end{pmatrix} \tag{2.24}$$

or in other words, the Jacobian matrix is:

$$J_{ik} = \frac{\partial f_i}{\partial q_k} = \begin{pmatrix} u_0 & 1 \\ C_s^2 & u_0 \end{pmatrix} \tag{2.25}$$

One of the advantages of writing the wave equation, and lateron many other equations, in the form of a matrix equation like Eq. (2.22), is that we can use some aspects of linear algebra to solve the equations in an elegant way, which will later turn out to have a very powerful application in numerical methods of hydrodynamics.

let us look again at the Jacobian matrix of the wave equation, Eq. (2.25). This matrix has the following eigenvectors and eigenvalues:

$$e_{-1} = \begin{pmatrix} 1 \\ -C_s \end{pmatrix} \qquad \text{with} \qquad \lambda_{-1} = u_0 - C_s \tag{2.26}$$

$$e_{+1} = \begin{pmatrix} 1 \\ +C_s \end{pmatrix} \qquad \text{with} \qquad \lambda_{+1} = u_0 + C_s \tag{2.27}$$

If we include the passive tracer of Eq. (1.7.2) we have:

$$Q \equiv \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_0 u_1 \\ \varphi \end{pmatrix} \tag{2.28}$$

and the matrix form of the equation of motion becomes:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} u_0 & 1 & 0 \\ C_s^2 & u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0 \tag{2.29}$$

This has the following set of eigenvectors and eigenvalues:

$$e_{-1} = \begin{pmatrix} 1 \\ -C_s \\ 0 \end{pmatrix} \qquad \text{with} \qquad \lambda_{-1} = u_0 - C_s \tag{2.30}$$

$$e_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{with} \qquad \lambda_{-1} = u_0 \tag{2.31}$$

$$e_{+1} = \begin{pmatrix} 1 \\ +C_s \\ 0 \end{pmatrix} \qquad \text{with} \qquad \lambda_{+1} = u_0 + C_s \tag{2.32}$$

We can now decompose any vector q into eigenvectors:

$$Q = \tilde{q}_{-1} e_{-1} + \tilde{q}_0 e_0 + \tilde{q}_{+1} e_{+1} \tag{2.33}$$

Then the equation of motion becomes:

$$\partial_t \begin{pmatrix} \tilde{q}_{-1} \\ \tilde{q}_0 \\ \tilde{q}_{+1} \end{pmatrix} + \begin{pmatrix} u_0 - C_s & 0 & 0 \\ 0 & u_0 & 0 \\ 0 & 0 & u_0 + C_s \end{pmatrix} \partial_x \begin{pmatrix} \tilde{q}_{-1} \\ \tilde{q}_0 \\ \tilde{q}_{+1} \end{pmatrix} = 0 \tag{2.34}$$

The matrix is here diagonal. This has the advantage that we have now decomposed the problem into three *scalar advection equations*:

$$\partial_t \tilde{q}_i + \lambda_i \partial_x \tilde{q}_i = 0 \tag{2.35}$$

for any $i = -1, 0, +1$. For these equations we know the solution: they are simply shifts of an initial function (see Sections 2.1 and 2.2). In the case at hand here we are fortunate that the C_s and u_0 are constant, so we get:

$$\tilde{q}_i(x,t) = \tilde{q}_i(x - \lambda_i\, t, 0) \tag{2.36}$$

for any $i = -1, 0, +1$. The vector-notation and the decomposition into eigenvectors and eigenvalues stands at the basis of much of the theory on numerical algorithms to follow.

The system of equations described here is a *hyperbolic set of equations*, which is another way of saying that they describe the motion of signals. We will define hyperbolicity more rigorously below.

2.7 Hyperbolic sets of equations: the linear case with constant Jacobian

Let us consider a set of linear equations that can be written in the form:

$$\partial_t Q + A\partial_x Q = 0 \tag{2.37}$$

where Q is a vector of m components and A is an $m \times m$ matrix.

This system is called *hyperbolic* if the matrix A is diagonalizable with real eigenvalues. The matrix is diagonalizable if there exists a complete set of eigenvectors e_i, i.e. if any vector can be written as:

$$Q = \sum_{i=1}^{m} \tilde{q}_i e_i \tag{2.38}$$

In this case one can write

$$AQ = \sum_{i=1}^{m} \lambda_i \tilde{q}_i e_i \tag{2.39}$$

We can define a matrix in which each column is one of the eigenvectors:

$$R = (e_1, \cdots, e_m) \tag{2.40}$$

Then we can transform Eq. (2.37) into:

$$R^{-1}\partial_t Q + R^{-1}ARR^{-1}\partial_x Q = 0 \tag{2.41}$$

which with $\tilde{Q} = R^{-1}Q$ then becomes:

$$\partial_t \tilde{Q} + \tilde{A}\partial_x \tilde{Q} = 0 \tag{2.42}$$

where $\tilde{A} = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$. Not all λ_i must be different from each other.

This system of equations has in principle m sets of characteristics. But any set of characteristics that has the same characteristic velocity as another set is usually called the same set of characteristics. So in the case of 5 eigenvalues, of which three are identical, one typically says that there are three sets of characteristics.

2.8 Boundary conditions (I)

So far we have always assumed that space is infinite. In real-life applications the domain of interest is always bound. In some cases these boundaries are real (like a wall or a piston) but in other cases they have to be somewhat artificially imposed because computing power is not as infinite as space is and one is limited to a finite volume. It is therefore important to know how spatial boundary conditions are set. In the numerical chapters we will go into this in far more detail than here, often going into more practical matters. Here we are concerned with the mathematical issue.

We look at the general hyperbolic equation

$$\partial_t Q(x,t) + A\partial_x Q(x,t) = 0 \tag{2.43}$$

where Q is an vector with m components and A is an $m \times m$ matrix. Let us define as our domain of interest the space between x_0 and $x_1 > x_0$. At these boundaries we wish to impose the necessary and sufficient boundary conditions. Since this is a coupled set of m linear equations we need m boundary conditions. In systems of hyperbolic equations one usually sets *Dirichlet boundary conditions*, i.e. one speficies the values of the components of Q at these boundaries. The best way to do this is to decompose into the eigenvalues of A:

$$\partial_t \tilde{q}_i(x,t) + \lambda_i \partial_x \tilde{q}_i(x,t) = 0 \tag{2.44}$$

For each $i \in [1, \cdots, m]$ one can then set:

$$\tilde{q}_i(x = x_{0|1}, t) = \tilde{q}_i^{(\mathrm{bc})}(t) \tag{2.45}$$

The $x_{0|1}$ means that one can *either* specify that condition for $\tilde{q}_i$ at $x = x_0$ *or* at $x = x_1$. For instance, for $m = 2$ one could specify one condition at x_0 for $\tilde{q}_1$ and one condition at x_1 for $\tilde{q}_2$. In total one can impose m conditions, divided over the left and right boundaries. Note, by the way, that the boundary conditions may be time-dependent!

The question now is: for some $\tilde{q}_i$, at which boundary should we impose the Dirichlet boundary condition? Suppose $\lambda_i > 0$. It is then clear that signals propagate from left to right. Therefore it is clear that the boundary condition, for $\tilde{q}_i$, must be set at the left boundary, at $x = x_0$. The condition specifies the *inflow* of the signal into the system through the left boundary. If one were to specify it at the other boundary (which in a numerical simulation one could simply try to do, and see what happens), then the installed boundary value is immediately advected off the domain again and will not influence the solution within the domain at all. In contrast, setting $\tilde{q}_i$ at the left boundary means that this value is advected into the domain and affects the solution there.

In general one can say that at boundary $x = x_0$ one specifies $\tilde{q}_i(x = x_0, t) = \tilde{q}_i^{(\mathrm{bc})}(t)$ for all i for which $\lambda_i > 0$ and at boundary $x = x_1$ one specifies $\tilde{q}_i(x = x_1, t) = \tilde{q}_i^{(\mathrm{bc})}(t)$ for all i for which $\lambda_i < 0$.

Another possible kind of boundary conditions are *periodic boundary conditions*. Here we set, at every time t, for $\lambda_i > 0$:

$$\tilde{q}_i(x = x_0, t) := \tilde{q}_i(x = x_1, t) \tag{2.46}$$

In this way, the information flowing out of the right boundary flows back into the left boundary. The system is in this way closed and depends now only on the initial conditions.

2.9 Hyperbolic sets of equations: the linear case with variable Jacobian

So what happens if the matrix A depends on x, i.e. $A \to A(x)$? In this case also the eigenvalues change with position, like described in Section 2.2. But in addition to this also the eigenvectors can depend on spatial position x, and hence the diagonalization of the matrix. In other words, the transformation matrix becomes a local matrix: $R \to R(x)$. This means that on cannot decompose Q into characteristic modes globally, but this has to be done locally. This also means that the modes get mixed. Consider the following example:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} \cos x & \sin x \\ \sin x & -\cos x \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \tag{2.47}$$

on a domain limited by $x_0 = 0$ and $x_1 = \pi$. The eigenvalues of the matrix are always ± 1, but the eigenvectors change with x. At both $x = x_0 = 0$ and $x = x_1 = \pi$ the eigenvectors can be written as $e_1 = (1, 0)$ and $e_2 = (0, 1)$, while at, for example, $x = \pi/2$ we have $e_1 = (1, 1)/\sqrt{2}$ and $e_2 = (-1, 1)/\sqrt{2}$. Or more general:

$$e_1 = \begin{pmatrix} \cos(x/2) \\ \sin(x/2) \end{pmatrix} \tag{2.48}$$

$$e_2 = \begin{pmatrix} -\sin(x/2) \\ \cos(x/2) \end{pmatrix} \tag{2.49}$$

(the eigenvectors at x_1 are now minus those of x_0, but there is no difference in the meaning as the norm of the eigenvalues are irrelevant except in the definition of the norm of $\tilde{q}_i$).

If we set $\tilde{q}_1(x_0) = q_1(x_0) = f(t)$ where $f(t)$ is some function of time t, and we set $\tilde{q}_2(x_1) = 0$ and the initial value of $\tilde{q}$ at $\tilde{q}_i(x, t = t_0)$, then the signal put into mode $\tilde{q}_1$ at the left boundary initially propagates from left to right, but as it gets to larger x it starts to mix with the $\tilde{q}_2$ mode, which moves in opposite direction.

This example shows that although hyperbolic equations are about signal propagation, this does not mean that the signals are simply pure waves moving across the domain, but can instead interact with each other even if the equations are linear. However, *locally* one can always uniquely divide the state vector Q up into the characteristic modes moving each at their own characteristic speeds.

2.10 Boundary conditions (II)

When the Jacobian $m \times m$ matrix A depends on x, the question of how and where to impose boundary conditions can become, in some circumstances, a bit more difficult. The limitation that one must impose precisely m boundary conditions is no longer strictly true. It turns out to depend on the number of inward-pointing characteristics at each of the boundaries. Consider the simple example of scalar advection problem between $x_0 = -1$ and $x_1 = 1$ with advection speed $A(x) = u(x) = x$:

$$\partial_t q + x\partial_x q = 0 \tag{2.50}$$

In this case, at $x = x_0 = -1$ the characteristic is pointing out of the domain. But the same is true at $x = x_1 = 1$! So neither at x_0 nor at x_1 must one determine boundary conditions. The opposite example

$$\partial_t q - x\partial_x q = 0 \tag{2.51}$$

requires a Dirichlet boundary condition to be set at *both* boundaries. Here the information (signal) flows into the domain on both sides and piles up near $x = 0$.

In general when the sign of an eigenvalue flips one has the risk that the number of boundary conditions deviates from m. The physical interpretation of sign-flips of eigenvalues can be many. For instance, as we shall see later, the sign of one of the eigenvalues flips in the case of a standing shock in hydrodynamics. Indeed, signals pile up on both sides of the shock and a catastrophy can only be avoided by the minute, but essential viscosity in the shock front which (only very locally) 'erases' these converging signals again in a *non-hyperbolic* manner. But these issues will be discussed in the chapter on supersonic flows and shocks, chapter **??**

2.11 Hyperbolic equations versus elliptic equations

As mentioned above, if the Jacobian matrix can be diagonalized and the eigenvalues are real, then the system is hyperbolic. So what if the eigenvalues are imaginary? Consider the following system, similar to the equations for waves in hydrodynamics:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \tag{2.52}$$

defined on a domain $x \in [x_0, x_1]$. This clearly can be written as:

$$\partial_t^2 q_1 - \partial_x^2 q_1 = 0 \tag{2.53}$$

which is a wave equation with eigenvalue $\lambda_{-1} = -1$ and $\lambda_{+1} = 1$. If the state Q is given at time $t = t_0$, and the boundary conditions are specified at $x = x_0$ and $x = x_1$ in the way explained in Section 2.8, then the function $Q(x, t)$ within the domain can be computed for all $t > t_0$ (in fact, also backward in time).

Now consider the following equation:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \tag{2.54}$$

defined on a domain $x \in [x_0, x_1]$. This can be written as:

$$\partial_t^2 q_1 + \partial_x^2 q_1 = 0 \tag{2.55}$$

However, this is not a hyperbolic equation. The eigenvalues of the Jacobian matrix are $\pm i$. It is clear that we cannot do the same trick with eigenvectors and eigenvalues here, because it does not make sense to move something with a speed $\pm i$. The nature of this equation is therefore entirely different even though it is merely one minus sign in the Jacobian matrix. In fact, Eq. (2.55) can be recognized as the Laplace equation. It needs the specification of boundary conditions at $x = x_0, x = x_1, t = t_0$ and $t = t_1$. In other words: the state at some time t depends not only on the past but also on the future. Clearly this makes not much sense, and the Laplace equation is usually more used in two (or more) spatial directions instead of space and time.

2.12 Hyperbolic equations: the non-linear case

The above definition for linear hyperbolic sets of equations can be generalized to non-linear sets of equations. Let us focus on the general conservation equation:

$$\partial_t Q + \partial_x F = 0 \tag{2.56}$$

where, as ever, $Q = (q_1, \cdots, q_m)$ and $F = (f_1, \cdots, f_m)$. In general, F is not always a linear function of Q, i.e. it cannot always be formulated as a matrix A times the vector Q (except if A is allowed to also depend on Q, but then the usefulness of writing $F = AQ$ is a bit gone). So let us assume that F is some non-linear function of Q. Let us, for the moment, assume that $F = F(Q, x) = F(Q)$, i.e. we assume that there is no explicit dependence of F on x, except through Q. Then according to Eq. (2.20) we get

$$\partial_t Q + \frac{\partial F}{\partial Q} \partial_x Q = 0 \tag{2.57}$$

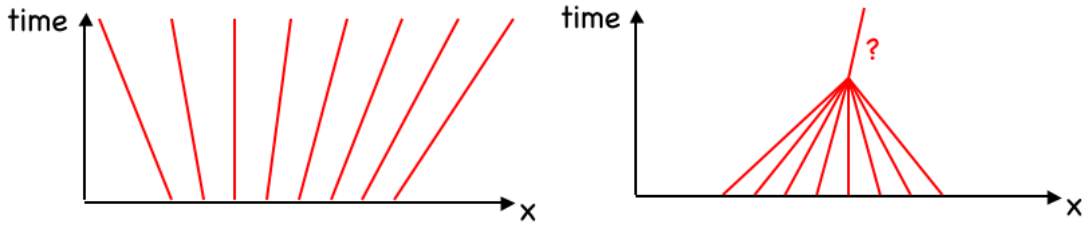

Figure 2.2. Characteristics of Burger's equation. Left: case of diverging characteristics. Right: case of converging characteristics with the formation of a singularity. Beyond the time of the creation of the singularity the solution is ill defined unless a recipe is given how to treat the singularity.

where $\frac{\partial F}{\partial Q}$ is the Jacobian matrix, which depends, in the non-linear case, on Q itself. We can nevertheless decompose this matrix in eigenvectors (which depend on Q) and we obtain

$$\partial_t \tilde{Q} + \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} \partial_x \tilde{Q} = 0 \tag{2.58}$$

Here the eigenvalues $\lambda_1, \cdots, \lambda_m$ and eigenvectors (and hence the meaning of $\tilde{Q}$) depends on Q. In principle this is not a problem. The characteristics are now simply given by the state vector Q itself. The state is, so to speak, self-propagating. We are now getting into the kind of hyperbolic equations like the hydrodynamics equations, which are also non-linear self-propagating.

2.13 Example of non-linear conservation equation: Burger's equation

Consider

$$\partial_t q + \frac{1}{2}\partial_x (q^2) = 0 \tag{2.59}$$

This is called *Burger's equation*. It is a conservation equation in q, with flux $f(q) = q^2/2$. The flux is only dependent on x through q. We have $\partial f/\partial q = q$, so we can write the above equation as

$$\partial_t q + q\partial_x q = 0 \tag{2.60}$$

So the advection velocity is, in this important example, the to-be-advected quantity q itself! The quantity propagates itself with $u = q$. If we use the comoving derivative, then we obtain the following equation:

$$D_t q(x,t) = 0 \tag{2.61}$$

which appears to be identical to Eq. (2.5). The difference lies in the definition of D_t which is defined with respect to a given function $u(x)$ in Eq. (2.5) and with respect to $q(x,t)$ in Eq. (2.61).

Eq. (2.61) shows that along a characteristic the value of q does not change, or in other words: the characteristic speed (slope in the x,t-plane) does not change along a characteristic. This means that the characteristics are straight lines in the (x,t)-plane, as in the case of the example of Section 2.3. The difference to that example lies in the fact that in this case not all straight line characteristics are parallel.

The interpretation of Burger's equation is that of the *motion of a pressureless fluid*. Often it is said to be the equation describing the motion of dust in space, as dust clouds do not have

pressure and each dust particle moves along a straight line. This is only partially correct, as we shall show below, but it does describe roughly the point.

Since any non-parallel straight lines in (x,t) *must* have a crossing at some point in space we can immediately derive that for converging flows there will be a point at which Burger's equations break down.

Now to come back at the difference of Burger's equation with the motion of dust. In Burger's equation the assumption is that at any time and any position there exists only one velocity. In case of dust flows this is not necessary: since the particles do not interact, at any given time and position one can have dust particles flowing left, right at various velocities. If two dust cloud approach, in Burger's equation the equations produce shocks (i.e. a breakdown of the pure inviscid equation). In the case of dust the particles would not feel each other and the cloud simply go through each other.

Perhaps a more adequate physical interpretation of Burger's equation is that of a pressureless fluid. If the characteristics converge a shock will happen and Burger's equation will no longer be valid and will be replaced by the more generally valid hydrodynamics equation *with* pressure.

2.14 Isothermal hydrodynamic equations

The topic of this lecture is hydrodynamics, so let's express the equations of hydrodynamics in the above form. Let's take the isothermal equations for simplicity. We have

$$\partial_t \rho + \partial_x(\rho u) = 0 \tag{2.62}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + \rho c_s^2) = 0 \tag{2.63}$$

Let us define

$$q_1 \equiv \rho \qquad , \qquad q_2 \equiv \rho u \tag{2.64}$$

Then we can write the above equations as

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \partial_x \begin{pmatrix} q_2 \\ \frac{q_2^2}{q_1} + q_1 c_s^2 \end{pmatrix} = 0 \tag{2.65}$$

or in other words: $Q = (q_1, q_2)$ and $F = (q_2, q_2^2/q_1 + q_1 c_s^2)$. This can be written with the Jacobian:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \left(c_s^2 - \frac{q_2^2}{q_1^2}\right) & 2\frac{q_2}{q_1} \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \tag{2.66}$$

The eigenvalues are

$$\lambda_\pm = \frac{q_2}{q_1} \pm c_s = u \pm c_s \tag{2.67}$$

and the eigenvectors are:

$$e_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix} \tag{2.68}$$

One sees that both the eigenvalues and the eigenvectors depend on the state vector (q_1, q_2) itself and are therefore space- and time-dependent. The state vector determines for itself how it should be decomposed. Modes mix in two different ways: a) the eigenvectors change in space and time, and b) each mode influences the other mode due to the non-linearity.

→ **Exercise:** In Section 2.6 we derived the set of eigenvectors and eigenvalues for the perturbation equation of hydrodynamics with an adiabatic equation of state. If we replace $\gamma P_0/\rho_0$ with the isothermal sound speed c_s^2, then we obtain the results for isothermal waves. The funny thing is, however, that the Jacobain matrix in that case (Eq. 2.25) does not appear to be the linearized form of the Jacobian matrix derived in the present section (the one used in Eq. 2.66). Explain this in terms of how the linearization is done in Section 2.6.

2.15 Non-isothermal hydrodynamic equations

Now let us turn to the generalization of the isothermal hydrodynamics equations: the non-isothermal hydrodynamics equations. Note that this is *not* equal to the adiabatic hydrodynamics equations, because the adiabatic hydrodynamics equations assume that all of the gas lies on the same adiabat, or in other words: that the gas is isentropic. In constrast, we would now like to relax any assumption of the entropy of the gas, and allow the entropy of the gas to vary arbitrarily in space. This means necessarily that we must include a third equation: the energy equation. Now the equations become significantly more complex.

We have now:

$$\partial_t \rho + \partial_x(\rho u) = 0 \tag{2.69}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0 \tag{2.70}$$

$$\partial_t(\rho e_{\rm tot}) + \partial_x[(\rho e_{\rm tot} + P)u] = 0 \tag{2.71}$$

Let us define

$$q_1 \equiv \rho \quad , \quad q_2 \equiv \rho u \quad , \quad q_3 \equiv \rho e_{tot} \tag{2.72}$$

Then we can write the above equations as

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} + \partial_x \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = 0 \tag{2.73}$$

in which

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ (\rho e_{\rm tot} + P)u \end{pmatrix} = \begin{pmatrix} q_2 \\ (\gamma-1)q_3 + \left(\frac{3-\gamma}{2}\right)\frac{q_2^2}{q_1} \\ \gamma\frac{q_3 q_2}{q_1} + \left(\frac{1-\gamma}{2}\right)\frac{q_2^3}{q_1^2} \end{pmatrix} \tag{2.74}$$

where we used

$$u = q_2/q_1 \tag{2.75}$$

$$P = (\gamma-1)\left(q_3 - \frac{1}{2}\frac{q_2^2}{q_1}\right) \tag{2.76}$$

Eq. 2.73 with the above expressions for (f_1, f_2, f_3) can be written with the Jacobian:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\frac{q_2^2}{q_1} & (3-\gamma)\frac{q_2}{q_1} & (\gamma-1) \\ -\left\{\gamma\frac{q_3 q_2}{q_1^2} + (\gamma-1)\frac{q_2^3}{q_1^3}\right\} & \left\{\gamma\frac{q_3}{q_1} + \frac{3}{2}(1-\gamma)\frac{q_2^2}{q_1^2}\right\} & \gamma\frac{q_2}{q_1} \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0 \tag{2.77}$$

It can be useful to rewrite the Jacobian using the primitive variables:

$$\partial_t \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\rho u^2 & (3-\gamma)u & (\gamma-1) \\ -\{\gamma e_{\rm tot} u + (\gamma-1)u^3\} & \{\gamma e_{\rm tot} + \frac{3}{2}(1-\gamma)u^2\} & \gamma u \end{pmatrix} \partial_x \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0 \tag{2.78}$$

The eigenvalues are

$$\lambda_- = u - C_s \tag{2.79}$$
$$\lambda_0 = u \tag{2.80}$$
$$\lambda_+ = u + C_s \tag{2.81}$$
$$\tag{2.82}$$

with eigenvectors:

$$e_- = \begin{pmatrix} 1 \\ u - C_s \\ h_{\rm tot} - C_s u \end{pmatrix} \tag{2.83}$$
$$e_0 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix} \tag{2.84}$$
$$e_+ = \begin{pmatrix} 1 \\ u + C_s \\ h_{\rm tot} + C_s u \end{pmatrix} \tag{2.85}$$

where $h_{\rm tot} = e_{\rm tot} + P/\rho$ is the total specific enthalpy and $C_s = \sqrt{\gamma P/\rho}$ is the adiabatic sound speed.

2.16 Traffic flow equations

As already mentioned, hyperbolic equations are more general than only the equations of hydrodynamics. Here is an example of a model of traffic flow, as was first discussed in papers by Lighthill, Whitham and Richards (for the references, see book LeVeque from which this example is taken). Let us assume a single lane road with a density of cars q (the number of cars per car length). We assume that $0 \leq q \leq 1$ (because we cannot have more than 1 car per car length) and we verify a-posteriori if this condition is satified. The conservation equation is:

$$\partial_t q + \partial_x (qu) = 0 \tag{2.86}$$

where u is the speed of the cars at time t and position x. Suppose that there is a speed limit of $u_{\rm max}$ and that if the road is nearly empty, the cars drive at the speed limit $u = u_{\rm max}$. If this was all, then the advection equation simply moves the density of cars linearly toward larger x. However, if the road gets more crowded drivers naturally slow down. Let us for simplicity assume that

$$u(q) = u_{\rm max}(1 - q) \tag{2.87}$$

The flux of cars is then

$$f = q(1-q)u_{\rm max} \tag{2.88}$$

The flux of cars is greatest when $q = 1/2$. For lower q the flux is lower because the density of cars is lower, while for higher q the flux is lower because the speed of the cars goes down (congestion).

We can now write the traffic flow equation as

$$\partial_t q + u_{\max}(1 - 2q)\partial_x q = 0 \tag{2.89}$$

This shows that the characteristic velocity of the system is:

$$\lambda = u_{\max}(1 - 2q) \tag{2.90}$$

which can range from $-u_{\max}$ to $u_{\max}$. It is very important to note here that the characteristic speed is not equal to the speed of propagation of the cars! It is the speed at which information is propagating, not the speed of the advected quantity q itself. This is one of the peculiar features of non-linear hyperbolic equations, and it is very similar to the peculiarities of the Burger's equation. The traffic flow equation is a particularly nice example of a non-linear hyperbolic equation because it is a very simple equation, yet has very interesting solution properties.

2.17 Hyperbolic equations in 2-D and 3-D

So far we have done everything only in 1-D. But what we learned can also be generalized to higher dimensions, by use of the concept of *operator splitting*. In 2-D we get

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) \tag{2.91}$$

which can be written as:

$$\partial_t q_i + \frac{\partial f_i}{\partial q_k}\partial_x q_k + \frac{\partial g_i}{\partial q_k}\partial_y q_k \tag{2.92}$$

By operator splitting we can focus our attention to one of the space dimensions only. If we split it as

$$\partial_t q_i + \frac{\partial f_i}{\partial q_k}\partial_x q_k = -\frac{\partial f_i}{\partial q_k}\partial_y q_k \tag{2.93}$$

then we focus on the advection in x-direction, and consider the y-advection as a source term, while if we write

$$\partial_t q_i + \frac{\partial f_i}{\partial q_k}\partial_y q_k = -\frac{\partial f_i}{\partial q_k}\partial_x q_k \tag{2.94}$$

we focus on advection in y-direction and consider the x-direction as a source term. In numerical methods this operator splitting is often done to reduce the full 2-D or 3-D problem into consecutive 1-D problems which are much easier to handle.