Exercises for
Numerical Fluid Mechanics (WS2012/13)

Volker Springel & Cornelis Dullemond

Exercise sheet 2 *(duration: 2 weeks)*

A first simple advection program

The goal of this exercise is to program a simple first order advection method. We will numerically solve the following advection equation

$$\partial_t q(x,t) + u\partial_x q(x,t) = 0 \tag{5}$$

with $u = 1$. We discretize this between $x = 0$ and $x = 10$ on M grid points with spacing $\Delta x = 10/(M-1)$, with $x_0 = 0$ and $x_{M-1} = 10$ (in programming languages in which array indices start at 0). As a left boundary condition we simply set $q_0 = 1$, and only update $q_1 \cdots q_{M-1}$ every time step. As an initial condition let us take:

$$q(x,0) = \begin{cases} 1 & \text{for } x \leq 3 \\ 0 & \text{for } x > 3 \end{cases} \tag{6}$$

1. **The analytic solution**
   Give the analytic solution for $q(x,t)$ at time $t = 4$.

2. **Centered differencing**
   Let us discretize this equation using the *centered differencing scheme*:

   $$\frac{y_i^{n+1} - y_i^n}{\Delta t} + u\frac{y_{i+1}^n - y_{i-1}^n}{2\Delta x} = 0 \tag{7}$$

   Note: This algorithm will be unstable, but let us try anyway.

   (a) Write y_i^{n+1} explicitly as a function of y_{i-1}^n, y_i^n and y_{i+1}^n.

   (b) On the left side (i.e. for $i = 0$) we already decided to impose the boundary condition $q = 1$; but what should we do with the gridpoint $i = M-1$ (i.e. the right-most one)? Hint: There is no perfect solution; just explain why this point has to be treated separately, and give a solution that you think is reasonable.

   (c) How many time steps must we do in order to integrate from $t = 0$ to $t = 4$ when we specify a fixed time step Δt? Think carefully about what you should do if $4/\Delta t$ is not an integer (e.g. if $\Delta t = 0.3$): How can we make sure to end up still exactly at $t = 4$ and not at $t > 4$? Note that there are various possible ways: any method that works is fine.

   (d) Write a computer program that integrates the equations from time $t = 0$ to time $t = 4$ for a pre-defined Δt, using the centered differencing discretization scheme. Make a plot for the case $M = 100$, $\Delta t = 0.04$ after 100 time steps (i.e. at exactly $t = 4$).

   (e) Try out different time steps Δt and show that they all produce oscillations that grow exponentially in time.

3. **One-sided differencing: The upwind method**
Let us now discretize the equation using the *upwind differencing scheme*:

$$\frac{y_i^{n+1} - y_i^n}{\Delta t} + u\frac{y_i^n - y_{i-1}^n}{\Delta x} = 0 \tag{8}$$

   (a) Program this in a computer program.
   (b) Do we need to treat the gridpoint $i = M - 1$ specially?
   (c) Make again a plot for the case $M = 100$, $\Delta t = 0.04$ after 100 time steps (i.e. at exactly $t = 4$). Overplot the analytical solution.
   (d) Make another plot for the same problem, but after only 50 time steps (i.e. at exactly $t = 2$). Compare the smearing out of the jump: what do you see?
   (e) Experimentally find out at which Δt the algorithm becomes unstable.
   (f) Try out $\Delta t = 0.101$. You'll be surprised! Explain what happens.

4. **One-sided differencing: The downwind method**
Let us now discretize the equation using the *downwind differencing scheme*:

$$\frac{y_i^{n+1} - y_i^n}{\Delta t} + u\frac{y_{i+1}^n - y_i^n}{\Delta x} = 0 \tag{9}$$

   (a) Program this in a computer program.
   (b) Make again a plot for the case $M = 100$, $\Delta t = 0.04$ after 100 time steps (i.e. at exactly $t = 4$). What do you see?
   (c) Explain why no signal is transported to the right *at all*.

5. **A non-constant velocity**
Now let us assume that the velocity u is a function of x, and that the advection equation is conservative:

$$\partial_t q(x,t) + \partial_x[u(x)q(x,t)] = 0 \tag{10}$$

Let us take for the velocity profile:

$$u(x) = \begin{cases} 1 & \text{for } x \leq 4 \\ \frac{2}{3}\exp(4-x) + \frac{1}{3} & \text{for } x > 4 \end{cases} \tag{11}$$

Let us take the following form of upwind discretization scheme:

$$\frac{y_i^{n+1} - y_i^n}{\Delta t} + \frac{u_i y_i^n - u_{i-1}y_{i-1}^n}{\Delta x} = 0 \tag{12}$$

   (a) Program this in a computer program.
   (b) Make a plot for the case $M = 100$, $\Delta t = 0.04$ after 100 time steps. Explain what you see.

For all exercises, please always do the following:

- Make an electronic document (DOC or PDF) which includes your text concerning the exercises, as well as figures belonging to it.
- Upload your document *and your computer program* to the Moodle.