Chapter 1

Basics of radiation transfer theory

In most cases in astronomy we can regard radiation as a particle phenomenon: In this picture light consists of photons moving with the light speed along straight lines through space. They can be created and destroyed by interaction with matter. Hot gases or dust clouds can cool by emitting copious amounts of photons. It is typically this kind of radiation that we observe with our telescopes. But the radiation that is emitted in one region can be absorbed by other matter, which can thus be radiatively heated. In this way radiation can act a carrier of heat and/or momentum exchange between matter parcels that are otherwise too far apart to interact with each other. In other words: radiation is not only a diagnostic tool for us as astronomers, it is also (and perhaps even predominantly) a critical ingredient in the thermal balance of the objects we observe. A serious interpretation of observations therefore often forces us to learn about the emission, absorption and transport of radiation *inside* our objects of interest. The theory of "radiative transfer" (also called "radiation transport") is the theory of how radiation and matter interact based on the particle description of light. For most astrophysical purposes this particle description is sufficient to understand the production and transfer of radiation in/through astrophysical objects, at least at the macroscopic level. In this chapter we will discuss the general theory of radiative transfer in a nutshell.

Literature:

The book by Rybicki & Lightman "Radiation processes in Astrophysics", which emphasizes the various physical processes that produce, absorb and scatter radiation, but also has a bit of fundamental theory of radiative transfer in it.

Lecture notes by Rob Rutten "Radiative transfer in stellar atmospheres" (`http://www.astro.uu.nl/~rutten/Lecture_notes.html`): an excellent overview of radiative transfer theory, with of course a particular emphasis on stars.

1.1 Intensity and flux

So let us consider radiation as a movement of photons along straight lines. In empty space photons do not encounter any obstacles and photons do not interact. So if we wish to measure the amount of radiation at some point $\vec{r}$ in space, we also have to specify in which direction we are "looking". Let us denote this with a unit vector $\vec{\Omega}$[1]. Finally, we have to

[1] In a lot of literature the symbol $\vec{n}$ is used for the direction and $d\Omega$ for a differential solid angle. We use the Ω symbol for both.

specify at which wavelength λ we wish to measure the radiation field, or equivalently at which frequency $\nu = c/\lambda$. The quantity we thus measure is called *intensity*:

$$I(\vec{x}, \vec{\Omega}, \nu) \tag{1.1}$$

which has units erg cm−2 s^{-1} Hz^{-1} $ster^{-1}$. Let us examine these units. The erg cm−2 s^{-1} is a measure of flux. The cm−2 arises simply because if you have a larger telescope you pick up more photons and thus receive more energy (erg). The Hz^{-1} arises because our intensity is a monochromatic intensity. The $ster^{-1}$ arises because we are not interested in the total flux, but just the "flux per steradian".

The quantity "intensity" $I(\vec{x}, \vec{\Omega}, \nu)$ is a 6-dimensional function: 3 space dimensions, 2 direction dimensions and one frequency dimension (adding time would make it 7-dimensional, but we will not concern ourselves with this). This property alone already foreshadows why radiative transfer is so complex: just the storage of the intensity field alone in computer memory already poses a challenge because of the high dimensionality. But the difficulties of radiative transfer are much deeper than that, as we will see shortly.

But let us first analyze this "intensity" quantity a bit more, because when one first encounters it it can be a confusing entity. Most interestingly, in vacuum the intensity I is *constant along a ray*. To take an example: if we measure the intensity of the radiation of the sun at a distance of 1 astronomical unit (AU), and we redo the measurement at 5 AU, we get the same answer. Of course, if we would measure the solar *flux* (in units of erg cm−2 s^{-1} Hz^{-1}) at 5 AU, we would get a 25 times smaller value than at 1 AU, as one would expect. The flux is in fact a vectorial quantity and is related to the intensity by

$$\vec{F}(\vec{x}, \nu) = \oint I(\vec{x}, \vec{\Omega}, \nu)\vec{\Omega} d\Omega \tag{1.2}$$

If we approximate the sun as a disc of radius $R_\odot$ with constant brightness over its surface, then we can write

$$F = I\Delta\Omega = I\frac{\pi R_\odot^2}{d^2} \tag{1.3}$$

where d is the distance between the observer and the sun. The intensity I is in fact the *surface brightness* of the sun, which does not change with distance to the observer. The flux goes as $1/d^2$ because the solid angle of the sun changes with d as $1/d^2$. This shows that the intensity I is a quantity that, along a straight line through vacuum, remains constant:

$$\frac{dI}{ds} = 0 \tag{1.4}$$

where s is a measure of distance along the ray.

1.2 Formal radiative transfer equation

The constancy of intensity in vacuum is a property that can be very conveniently used to describe the interaction with matter, for if space is not a vacuum but filled with some material with extinction coefficient α (in units of 1/cm) the equation of radiative transfer becomes:

$$\frac{dI}{ds} = -\alpha I \tag{1.5}$$

If we change variables to optical depth τ, where $d\tau = \alpha ds$, we obtain

$$\frac{dI}{d\tau} = -I \tag{1.6}$$

with solution

$$I = I(0)e^{-\tau} \tag{1.7}$$

But matter does not only extinct radiation: it can also emit radiation. So let us add an emissivity coefficient j (in units of erg cm−3 s^{-1} Hz^{-1} ster^{-1}) to obtain

$$\frac{dI}{ds} = j - \alpha I \tag{1.8}$$

Changing variables again with $d\tau = \alpha ds$ we get

$$\frac{dI}{d\tau} = \frac{j}{\alpha} - I \tag{1.9}$$

The ratio j/α is, in radiative transfer theory, called a *source function*, denoted with S (capital S, to distinguish it from the path length s). We thus get, along a ray through a medium,

$$\frac{dI}{d\tau} = S - I \tag{1.10}$$

Now suppose that $S(s) = S(\tau)$ =constant, then the solution of this equation is

$$I = I(0)e^{-\tau} + S(1 - e^{-\tau}) \tag{1.11}$$

This shows that in a medium with an optical depth τ sufficiently large the original input intensity (before the ray entered the medium) $I(0)$ is gradually replaced with the intensity $I = S$ inside the medium.

Equation (1.10) is called the *formal radiative transfer equation* and is clearly easy to solve along each single ray through a medium. So why is radiative transfer considered to be so complex then? The reason is that, in most circumstances, the source function S is unknown in advance, and depends on the outcome of the transfer equation itself. We will not go into this complexity here. For our purposes throughout this lecture we assume that the source function S is known everywhere inside our astrophysical objects of interest. In this way the entire problem of radiative transfer is reduced to a relatively simple problem: that of integrating the formal transfer equation, Eq. (1.10).

1.3 Planck function: blackbody radiation

The source function S is the value that the intensity acquires in a homogeneous medium when $\tau \gg 1$. From thermodynamics we know that this radiation field must be a thermal radiation field. Indeed, in a thermalized medium $S_\nu = B_\nu(T)$ where $B_\nu(T)$ is the Planck function at temperature T. Note that we put the frequency index as a subscript, as is standard notation in radiative transfer theory. The Planck function is given by

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1} \tag{1.12}$$

In fact, the relation $j_\nu = \alpha_\nu B_\nu(T)$ (which is another way of saying $S_\nu = B_\nu(T)$) is called *Kirchhoff's law*, after Gustav Kirchhoff (1824-1887) who was a professor at the University of Heidelberg.

1.4 Emission and absorption spectral lines

One very basic thing can already be understood from the above basic radiative transfer theory: how spectral lines form. Suppose $I_\nu(0) = B_\nu(T_0)$, and this radiation goes through a layer with $S_\nu = B_\nu(T_1)$ with $T_1 \neq T_0$. Suppose also that α_ν is given by

$$\alpha_\nu = A \exp\left[-\left(\frac{\nu - \nu_l}{\Delta\nu}\right)^2\right] \tag{1.13}$$

for some value of A, where ν_l is the frequency of the spectral line and $\Delta\nu$ the width of the line. Here we assume a simple Gaussian line profile. Suppose that at line center ($\nu = \nu_l$) the optical depth $\tau_{\nu_l} = 10$ while far outside the line the optical depth is of course nearly zero. According to the formal line transfer equation (Eq. 1.10) you can now see that at line center the outcoming radiation is $I_{\nu=\nu_l} \simeq B_\nu(T = T_1)$ while far outside the line it is $I_{\nu=\nu_l} \simeq B_\nu(T = T_0)$. If $T_1 > T_0$ this thus produces an *emission line* while if $T_1 < T_0$ it produces an *absorption line*.