Exercises belonging to lecture Observational Astronomy MKEP5 (SS 2011)

Sheet 8

Voluntary (!!!) exercises of radiative transfer

Please note: As always, all necessary information can be found either in the lecture notes or in the description of the exercises.

1. **Kirchhoff's law for lines**
For line emission/absorption, the equation for radiative transfer along a ray through some astrophysical cloud is (see lecture notes):

$$\frac{dI_\nu(s)}{ds} = j_\nu(s) - \rho(s)\kappa_\nu(s)I_\nu(s) \tag{1}$$

where s is the coordinate along the ray. For an atomic or molecular line transition from level i to level k of that atom/molecule the lecture notes gave expressions for the emissivity j_ν and $\rho\kappa_\nu$ in terms of the level populations n_i and n_k:

$$j_\nu = \frac{h\nu_{ik}}{4\pi} n_i A_{ik} \varphi_{ik}(\nu) \tag{2}$$

$$\rho\kappa_\nu = \frac{h\nu_{ik}}{4\pi}(n_k B_{ki} - n_i B_{ik})\varphi_{ik}(\nu) \tag{3}$$

where ν_{ik} is the line center frequency of the $i \to k$ transition. The Einstein coefficients A_{ik}, B_{ik} and B_{ki} are related via the Einstein relations, also given in the lecture. We assume that our cloud is in LTE at a temperature T.

   (a) Prove Kirchhoff's law for this problem, which states that

$$j_\nu = \rho\kappa_\nu B_\nu(T) \tag{4}$$

   where $B_\nu(T)$ is the Planck function at temperature T. *Hint:* The line width is very small, $\sigma \ll \nu_{ik}$, so that $B_\nu(T) \simeq B_{\nu_{ik}}(T)$ if ν is close to ν_{ik}.

2. **Line emission from a cloud**
Consider a spherical interstellar molecular cloud with a radius $R = 10^{13}$ cm and a temperature of $T = 30$ Kelvin[1]. The cloud of gas contains CO molecules at a number density of $n_{\rm CO} = 10$ cm^{-3}. We assume that the molecules are in LTE. We wish to calculate the strength of the microwave emission line from this cloud for the quantum transition from rotating CO molecules (denoted in quantum mechanics by $J = 1$) to non-rotating CO molecules ($J = 0$). The energy difference between these two states is $\Delta E_{1\to 0} = 7.6381004 \times 10^{-16}$ erg. The Einstein coefficient for this transition is $A_{1\to 0} = 7.203 \times 10^{-8}$ s^{-1}, and the level degeneracy of the $J = 1$ level is $g_1 = 3$

[1] We use CGS units in this exercise. In these units the Planck constant $h = 6.6262 \times 10^{-27}$ cm^{2}g/s, the speed of light is $c = 2.9979246 \times 10^{10}$ cm/s and Boltzmann's constant is $k = 1.3807 \times 10^{-16}$ cm^{2}g s^{-2}K^{-1}.

while that of the ground level[2] is $g_0 = 1$. We assume that there is no background radiation.

The goal is to compute the line intensity seen by an observer if he/she points his/her telescope toward the *center* of the cloud (so we are not looking at the edges of the cloud, only at the center!).

To arrive at our goal we have to first do some intermediate computations:

(a) Derive that the wavelength of this line is $\lambda_{1\rightarrow 0} = 0.2600758$ cm.

(b) What is the corresponding frequency $\nu_{1\rightarrow 0}$?

(c) Show that the line width is $\sigma = 5.114 \times 10^4$ Hz, assuming that the line is thermally broadened. Note that the weight of a CO molecule is 28 proton masses, which means $m_{\rm CO} = 28 m_p = 4.683 \times 10^{-23}$ gram.

(d) The partition function for CO at 30 K has the value 11.19 (this function is dimensionless). Show, with the number density of CO given above, that the number density of CO molecules in the $J = 0$ state is $n_0 \simeq 0.893$ cm^{-3} and the number density of CO molecules in the $J = 1$ state is $n_1 \simeq 2.229$ cm^{-3}.

(e) The line profile is given by:

$$\varphi(\nu) = \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{(\nu - \nu_{1\rightarrow 0})^2}{\sigma^2} \right) \tag{5}$$

Show that the peak value of the line profile function is 1.10325×10^{-5} Hz^{-1}.

(f) Show that in this model $B_{1\rightarrow 0} \simeq 3.189 \times 10^6$ cm^{2}Hz ster erg^{-1}, and $B_{0\rightarrow 1} \simeq 9.568 \times 10^6$ cm^{2}Hz ster erg^{-1}.

We now have gathered all the numbers we need in order to compute j_ν and $\rho\kappa_\nu$ at the center of the line.

(g) Show that at the center of the line we have $j_\nu \simeq 1.077 \times 10^{-28}$ erg sec^{-1} cm^{-3} Hz^{-1} ster^{-1}, and $\rho\kappa_\nu = 9.65 \times 10^{-16}$ cm^{-1}.

(h) Show that the optical depth τ_ν at line-center for a ray going through the center of the cloud is: $\tau_\nu \simeq 0.0193$.

The fact that $\tau_\nu \ll 1$ means that the cloud is "optically thin", in radiative transfer jargon. Now let us calculate the observed intensity.

(i) Calculate the line-center intensity as seen toward the center of the cloud, assuming that the cloud is perfectly optically thin (i.e. assume $\tau_\nu = 0$). Note that the background intensity is assumed to be zero (see above).

(j) If the number density of CO molecules is increased by a factor of 1000, both j_ν and $\rho\kappa_\nu$ will increase by a factor of 1000. Therefore we will have an optical depth of $\tau_\nu \simeq 19.3$. The observed intensity, however, will *not* be 1000 times larger than we calculated in the optically thin case. Explain why, and give the right value of the intensity (tip: $\tau_\nu = 19.3$ is a high optical depth: $e^{-19.3} \simeq 0.$).

[2] In the lecture the ground state was called state 1. Here we call it state 0. Just a nomenclature issue.