Chapter 2

Atmospheric turbulence: "Seeing"

The Earth's atmosphere can be regarded as a dielectric medium which affects the radiation that passes through it. Not only does it absorb and emit radiation, it also refracts radiation. The air at 0 Celcius at 1 bar has in the optical a refractive index of about $n = 1.00029$. It is very close to 1, so air is almost as good as vacuum, but not quite. Patches of air of different temperature can lead to slight under- or over-densities of the air, leading to slight refraction. Since these patches of air move, the refraction changes all the time. This leads to slight variations of the position of stars on the sky, often the appearance of multiple "copies" of the same star close together (called *speckles*) and to slight variations in the brightness, (called *scintillation*). The latter is what we know as the "twinkling" of stars at night. All these phenomena also are wavelength-dependent, which is why the twinkling of stars is also often accompanied by variations in their color. In this chapter we will discuss atmospheric turbulence and the effects it has on imaging with a telescope. The lecture material in this chapter is heavily based on lecture PPT slides of A. Quirrenbach and on the literature below.

Literature:

Landau & Lifshitz, "Fluid Mechanics", Volume 6 of the famous Landau & Lifshitz "Course of Theoretical Physics"

J.W. Hardy, "Adaptive Optics for Astronomical Telescopes", Oxford series in optical and imaging science, Oxford University Press

P. Léna, F. Lebrun, F. Mignard, "Observational Astrophysics"

D.L. Fried, in "Adaptive optics for Astronomy", p. 25

F. Roddier, in Progress in Optics XIX, p. 281

F. Roddier, in Diffraction-Limited Imaging with Very Large Telescopes, p. 33

V.I. Tatarski, "Wave Propagation in a Turbulent Medium"

2.1 The theory of turbulence in the Earth's atmosphere

The turbulence in the Earth's atmosphere is always subsonic. To a very good degree one can regard the velocity fields to be divergence-free. That means that the turbulence in the Earth's atmosphere is entirely carried by solenoidal motions: i.e. vorticity. It is caused by many different phenomena:

- *Convection:* If the lower parts of the atmosphere heat up, the atmosphere can become convectively unstable, leading to convective bubbles of gas rising to higher altitudes. Often these bubbles produce *cumulus clouds*, the strongest of which, the *cumulonimbus clouds*, cause lightning storms. But not all convection leads to clouds, and convection happens on all scales. For instance, if an asphalt road gets heated up by intense sunlight, the few centimeter thick layers of air above it convectively rise, while cooler air sinks. This leads to "seeing" effects one can often observe in those conditions.

- *Wind shear:* Different layers of the atmosphere may have different horizontal velocities. If the vertical gradient of this horizontal velocity dv_x/dz is strong enough, the *Kelvin-Helmholtz instability* can set in, leading to a turbulent layer. For this to happen, the *Richardson number* Ri$\equiv g/[\Delta h\ (dv_x/dz)^2]$, where g is the gravitational acceleration and Δh is the altitude difference between the adjacent layers, must obey Ri$\ll 1$.

- *Wind over objects:* If a wind flows over objects, such as a mountain or the telescope dome, then turbulence can also be induced.

In this chapter we will discuss the nature of this turbulence and how it affects images of celestial objects.

2.2 Kolmogorov turbulence

Turbulence is a very complex phenomenon. A precise simulation of turbulence still often exceeds the capacities of modern supercomputers, although much progress has been made over the last decades. However, certain statistical properties of turbulence can be derived and understood without much computational effort.

The basic idea is that we start from some large scale stirring, inducing solenoidal motions of wave number $k_0 = 2\pi/L_0$ where L_0 is the spatial scale of these large scale motions. Now, as we know from stirring milk in a cup of coffee: the large motions tend to break up in ever smaller-scale motions. In fact, it is this property of turbulence that leads to the quick dissolution of sugar in a cup of tea. This down-scaling of the turbulence is called a *turbulent cascade*. Energy is transferred from large-scale motions to ever smaller scale motions, until one reaches a spatial scale $L_\nu \ll L_0$ which is the viscous dissipation scale: the spatial scale where the viscosity of the gas/fluid starts to prohibit further cascading. At this scale the kinetic energy of the gas motions is converted into heat. An important dimensionless number characterizing the turbulence is the *Reynolds number*:

$$\mathrm{Re} = \frac{L_0 V_0}{\nu} \tag{2.1}$$

where V_0 is the typical turbulent velocity at the largest scale and ν (in units of cm^2/s for CGS units) is the kinematic viscosity of the gas or fluid. For air at 0 degrees Celsius this is $\nu = 0.132\mathrm{cm}^2/\mathrm{s}$, and at -40 degrees Celcius it is $\nu = 0.104\mathrm{cm}^2/\mathrm{s}$. We can define a *turbulent viscosity*

$$\nu_{\mathrm{turb}} \equiv L_0 V_0 \tag{2.2}$$

so that $\nu_{\rm turb} = {\rm Re}\,\nu$. For Re$\gg 1$ the ratio $L_0/L_\nu \gg 1$, though Re and L_0/L_ν are not linearly related.

If we assume that there is a constant random stirring at the scale L_0, then this input energy cascades down to L_ν where it is dissipated. The turbulent cascade is thus characterized by an energy input rate per gram of gas ε in units of erg gram^{-1} sec^{-1} which in this stationary state is the same as the energy dissipation rate. In k-space this is the energy flow rate from small k to large k. Let us now try to characterize the energy stored in each mode k per gram of gas, written as $E(k)$. The quantity $E(k)dk$ has the dimension erg gram^{-1}. Since k has the dimension cm^{-1}, $E(k)$ has the dimension erg cm gram^{-1}.

Using these dimensions we can start a dimensional analysis. If we write erg = gram cm^2 sec^{-2} then we get the following dimensions:

$$k = \frac{1}{\rm length} \qquad \varepsilon = \frac{\rm length^2}{\rm time^3} \qquad E(k) = \frac{\rm length^3}{\rm time^2} \tag{2.3}$$

If $k_\nu \gg k_0$ then there is a large region in k-space, $k_0 \ll k \ll k_\nu$ where the function $E(k)$ must be independent on what is happening at k_0 or k_ν. The only "information" that it has in this intermediate-k region is k itself and the energy flow ε. Somehow we must be able to write $E(k) \propto \varepsilon^\alpha k^\beta$. Using the above dimensionalities you can see that this only makes dimensional sense if $\alpha = 2/3$ and $\beta = -5/3$, i.e.

$$E(k) = C\varepsilon^{2/3}k^{-5/3} \tag{2.4}$$

where C is a dimensionless constant. This is *Kolmogorov's law* for turbulence. It describes self-similar isotropic turbulence, and is valid for $k_0 \ll k \ll k_\nu$. The constant C is usually close to unity.

Note that $k \equiv |\vec{k}|$, so $E(k)dk$ is the energy for all modes $\vec{k}$ with $k \leq |\vec{k}| < k + dk$. In this way one could say that $E(k) = 4\pi|\vec{k}|^2 E(\vec{k})$, or

$$E(\vec{k}) \propto |\vec{k}|^{-11/3} \tag{2.5}$$

Let us continue our use of dimensional analysis to determine the typical magnitude of velocity excursions belonging to some scale $l = 1/k$.

$$l = {\rm length} \qquad \varepsilon = \frac{\rm length^2}{\rm time^3} \qquad v(l) = \frac{\rm length}{\rm time} \tag{2.6}$$

The only combination of l and ε that gives the right dimensions is:

$$v \propto (\varepsilon l)^{1/3} \propto (\varepsilon/k)^{1/3} \tag{2.7}$$

Let's do a consistency check: we know how $E(k)$ as well as $v(k)$ scale with k. Are they mutually consistent? The dimension of $E(k)dk$ is cm^2/s^2, i.e. $E(k)dk$ must be proportional to $v(k)^2$.

$$E(k)dk \propto \varepsilon^{2/3}k^{-5/3}dk \propto \varepsilon^{2/3}k^{-2/3} \propto (\varepsilon/k)^{2/3} \propto v^2 \tag{2.8}$$

which indeed shows that it is self-consistent. In other words: we could also have derived Kolmogorov's energy scaling law via the expression for $v(l)$.

The above scaling of $v(l)$ also leads directly to a typical time scale for eddy motions at spatial scale l:

$$\tau_{\rm eddy}(l) = l/v(l) \propto \varepsilon^{-1/3}\, l^{2/3} \tag{2.9}$$

This means that, as expected, the time scale of small-scale eddy motions is shorter than that of large scale eddies.

Let us, finally, find a scaling law for the dissipation scale $L_\nu = 2\pi/k_\nu$. Using the same type of dimensional analysis we find:

$$L_\nu = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \tag{2.10}$$

Also here we can do a self-check: The scale L_ν must be such that if $l = L_\nu$ the Reynolds number Re of that scale is unity:

$$1 = \mathrm{Re}(l) = \frac{l\,v(l)}{\nu} \propto \frac{\varepsilon^{1/3}\,l^{4/3}}{\nu} \tag{2.11}$$

which gives $l \propto (\nu^3/\varepsilon)^{1/4}$ as the scale at which Re=1, which is the definition of L_ν.

2.3 Turbulence structure function

In Appendix B we defined and derived some statistical tools to describe stochastic signals. Let us apply these to turbulence. The structure function of the velocity field in turbulence is

$$D_{\vec{v}}(\vec{x}_1,\vec{x}_2) = E[|\vec{v}(\vec{x}_1) - \vec{v}(\vec{x}_2)|^2] \tag{2.12}$$

where, compared to the appendix, we use here the position $\vec{x}$ instead of time t as the arguments. We can in fact use time to compute an estimate of this based on an average over a time span T:

$$D_{\vec{v}}(\vec{x}_1,\vec{x}_2) \simeq \langle|\vec{v}(\vec{x}_1,\tau) - \vec{v}(\vec{x}_2,\tau)|^2\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} |\vec{v}(\vec{x}_1,\tau) - \vec{v}(\vec{x}_2,\tau)|^2 d\tau \tag{2.13}$$

Let us now do another dimensional analysis, again with the Kolmogorov idea in mind. The dimension of $D_{\vec{v}}$ is cm^2 sec^{-2}. For isotropic turbulence it must depend on $|\vec{x}_2 - \vec{x}_1|$. So let us write $D_{\vec{v}}$ as

$$D_{\vec{v}}(\vec{x}_1,\vec{x}_2) = \alpha f(|\vec{x}_2 - \vec{x}_1|/\beta) \tag{2.14}$$

where the function $f()$ is a dimensionless function to be specified by our dimensional analysis (α and β are different here from the previous section). The dimension of α must be that of $D_{\vec{v}}$, that is: cm^2 sec^{-2}. The argument of the dimensionless function $f()$ must also be dimensionless, so β must have dimension cm. Our problem as a whole is defined by two quantities: the molecular kinematic viscosity ν (dimension cm^2 sec^{-1}) and the energy dissipation rate per gram of matter ε (dimension cm^2 sec^{-3}). The only combination with the right dimensions is clearly: $\alpha \propto \nu^{1/2}\varepsilon^{1/2}$ and $\beta \propto \nu^{3/4}\varepsilon^{-1/4}$. The next question is, what is the functional form of the function $f()$? The argument goes that in the intermediate scale regime the structure function should not depend on the viscosity ν. Only if $f(h) \propto h^{2/3}$ will ν drop out of the expression for $D_{\vec{v}}$, as you can verify. Therefore we have proven that $D_{\vec{v}}(\vec{x}_1,\vec{x}_2) \propto |\vec{x}_2 - \vec{x}_1|^{2/3}$, or quantitatively we can write

$$D_{\vec{v}}(\vec{x}_1,\vec{x}_2) = C_{\vec{v}}^2|\vec{x}_2 - \vec{x}_1|^{2/3} \tag{2.15}$$

The quantity $C_{\vec{v}}^2$ determines the strength of the turbulence. It is called the *structure constant* for the velocity $\vec{v}$. In a way, Eq. (2.15) is a mathematically more elegant way of expressing what we already derived for the velocities $v(l)$ belonging to scale l in Section 2.2, namely that $v \propto l^{1/3}$. Likewise we can define the structure function and structure constant for the density

$$D_\rho(\vec{x}_1, \vec{x}_2) = C_\rho^2 |\vec{x}_2 - \vec{x}_1|^{2/3} \tag{2.16}$$

and for the temperature

$$D_T(\vec{x}_1, \vec{x}_2) = C_T^2 |\vec{x}_2 - \vec{x}_1|^{2/3} \tag{2.17}$$

They are all related, but we will not go into this here.

2.4 Wave front distortion by the atmosphere

If we would have turbulence, but without any temperature differences between the turbulent eddies, then the turbulence would have no influence at all on our observations[1]. After all, gas motion does not induce refraction. Only if you have pockets of gas at slightly different temperatures in the turbulent flow will you get the effect of seeing. This is not created by the temperature itself. But since the atmosphere is in approximate pressure equilibrium, a lower temperature means automatically a higher density and vice-versa. This leads to inhomogeneities in the refractive index $n(\vec{x}, t)$ which vary also in time, because

$$n(\vec{x}, t) \simeq 1 + 0.00029 \, \frac{\rho(\vec{x}, t)}{1.3 \times 10^{-3} \mathrm{g\,cm^{-3}}} \tag{2.18}$$

which is valid at 1 bar and 0 degrees Celcius. Here $1.3 \times 10^{-3} \mathrm{g\,cm^{-3}}$ is the density of air at 1 bar and 0 degrees Celsius. The structure function for the refractive index is, not surprisingly,

$$D_n(\vec{x}_1, \vec{x}_2) = C_n^2 |\vec{x}_2 - \vec{x}_1|^{2/3} \tag{2.19}$$

One can then derive, assuming isobaric perturbations,

$$C_n = \frac{8 \times 10^{-5} P[\mathrm{mb}]}{T^2[\mathrm{K}]} C_T \tag{2.20}$$

(Léna, Lebrun & Mignard, page 58).

Now let us look at a wave of light at some instant t

$$E \propto \psi(\vec{x}) \equiv e^{i\phi(\vec{x})} \tag{2.21}$$

where $\phi(\vec{x})$ is the phase. Let us suppose that this wave travels vertically downward through the atmosphere with a wave number k. Now let us have a look at a *turbulent layer* between $z = z_0$ and $z = z_1$ with $z_1 - z_0 \ll z_0$, where $z = 0$ is the location of our telescope. The phase shift $\delta\phi(x, y)$ as a result of this layer of inhomogeneities is:

$$\delta\phi(x, y) = k \int_{z_0}^{z_1} (n(x, y, z, t) - 1) dz \tag{2.22}$$

with $k = 2\pi/\lambda$. We ignore the fact that k in air is not exactly equal to its value in vacuum.

[1]Assuming we have only solenoidal turbulence, i.e. no compressional components of the turbulence.

We can now define the *coherence function* of the wave front $B(r)$, which is the auto-correlation function of $\psi(\vec{x}, t)$:

$$B_\psi(\vec{r}) = \langle \psi(\vec{x} + \vec{r})\psi^*(\vec{x}) \rangle \tag{2.23}$$

If we assume that $\psi(\vec{x})$ is a Gaussian signal, then we can write:

$$\begin{aligned}
B_\psi(\vec{r}) &= \langle \psi(\vec{x} + \vec{r})\psi^*(\vec{x}) \rangle & (2.24)\\
&= \langle \exp[i(\phi(\vec{x}) - \phi(\vec{x} + \vec{r}))] \rangle & (2.25)\\
&= \exp\left[-\frac{1}{2} \langle |\phi(\vec{x}) - \phi(\vec{x} + \vec{r})|^2 \rangle \right] & (2.26)\\
&= \exp\left[-\frac{1}{2} D_\phi(\vec{r}) \right] & (2.27)
\end{aligned}$$

where we have made use of the results of Appendix B.4 on Gaussian signals. We have thus expressed the coherence function of the wave in terms of the structure function of the phase! So our next goal is to find an expression for $D_\phi(\vec{r})$.

Since $D_\phi(\vec{r}) = 2B_\phi(0) - 2B_\phi(\vec{r})$ (Eq. B.9), we now compute $B_\phi(\vec{r})$:

$$\begin{aligned}
B_\phi(\vec{r}) &= \langle \phi(\vec{x})\phi(\vec{x} + \vec{r}) \rangle & (2.28)\\
&= k^2 \int_{z_0}^{z_1} \int_{z_0}^{z_1} \langle n(\vec{x}, z') n(\vec{x} + \vec{r}, z'') \rangle dz' dz'' & (2.29)\\
&= k^2 \int_{z_0}^{z_1} \int_{z_0 - z'}^{z_1 - z'} \langle n(\vec{x}, z') n(\vec{x} + \vec{r}, z' + z) \rangle dz' dz & (2.30)
\end{aligned}$$

where k is the wave number of the radiation field and n the refractive index. We take here $\vec{x}$ and $\vec{r}$ to be pointing in the horizontal 2-D plane. We now assume that $z_1 - z_0$ is much larger than the correlation length of the refractive index, so that we can effectively assume the integrals in z to be infinite and the other integral to yield simply the length $z_1 - z_0$

$$B_\phi(\vec{r}) = k^2(z_1 - z_0) \int_{-\infty}^{+\infty} B_n(\vec{r}, z) dz \tag{2.31}$$

Now use, as already announced above, Eq.(B.9) to find an expression for $D_\phi(\vec{r})$:

$$\begin{aligned}
D_\phi(\vec{r}) &= 2[B_\phi(0) - B_\phi(\vec{r})] & (2.32)\\
&= 2k^2(z_1 - z_0) \int_{-\infty}^{+\infty} [B_n(0, z) - B_n(\vec{r}, z)] dz & (2.33)\\
&= 2k^2(z_1 - z_0) \int_{-\infty}^{+\infty} [B_n(0, 0) - B_n(\vec{r}, z) - B_n(0, 0) + B_n(0, z)] dz & (2.34)\\
&= k^2(z_1 - z_0) \int_{-\infty}^{+\infty} [D_n(\vec{r}, \vec{z}) - D_n(0, z)] dz & (2.35)\\
&= k^2(z_1 - z_0) C_n^2 \int_{-\infty}^{+\infty} [(r^2 + z^2)^{1/3} - z^{2/3}] dz & (2.36)\\
&= 2.914\, k^2(z_1 - z_0) C_n^2 r^{5/3} & (2.37)
\end{aligned}$$

where in the last step you can perform a numerical integral to find the constant 2.914, though it must be said that this value is only obtained when $(z_1 - z_0)$ is really thousands of correlation lengths large. In general this number will be anywhere between 1 and 2.914 for less extreme limits.

Now that we know $D_\phi(\vec{r})$ we can insert this back into Eq. (2.27) to obtain

$$B_\psi(\vec{r}) = \exp\left[-\frac{1}{2}\left(2.914\, k^2 (z_1 - z_0) C_n^2\, r^{5/3}\right)\right] \tag{2.38}$$

This is still only for a single layer. Integrating over the entire atmosphere, and taking now also the zenith-angle ζ into account, we obtain

$$B_\psi(\vec{r}) = \exp\left[-\frac{1}{2}\left(2.914\, k^2\, r^{5/3} \sec\zeta \int_0^\infty C_n^2(z) dz\right)\right] \tag{2.39}$$

where we define z such that the observatory is at $z = 0$.

We now arrived where we wanted to be: we have expressed the autocorrelation (or autocovariance) of the phase function $\psi(\vec{x}, t)$ in terms of an integral of the turbulence structure constant C_n^2 over height in the atmosphere. To get from here to a full understanding of how turbulence affects imaging is still more work. But it turns out (without proof) that many of the effects of turbulence on observations can be expressed in terms of *moments of turbulence*:

$$\mu_m = \int_0^\infty C_n^2(z) z^m dz \tag{2.40}$$

where m can be any real number. For our expression of $B_\psi(\vec{r})$ we use the 0-th moment ($m = 0$). But the moment of $m = 5/3$ is used for anisoplanatic and temporal errors, $m = 2$ for tilt errors and $m = 5/6$ for scintillation (see below).

2.5 Fried parameter r_0 and "seeing"

The zeroth moment of turbulence is usually given in terms of the *Fried parameter* r_0, which is defined as

$$r_0 \equiv \left(0.423\, k^2 \sec\zeta \int_0^\infty C_n^2(z) dz\right)^{-3/5} \tag{2.41}$$

It has the dimension of length, so it defines an important length scale of the theory of "seeing": the scale length over which phase errors in a wave front are of the order of 1 radian. We can write:

$$B_\psi(\vec{r}) = \exp\left[-3.44\left(\frac{r}{r_0}\right)^{5/3}\right] \tag{2.42}$$

and

$$D_\phi(\vec{r}) = 6.88\left(\frac{r}{r_0}\right)^{5/3} \tag{2.43}$$

If we make a long exposure of some object on the sky with seeing conditions characterized by Fried parameter r_0, then we obtain an image with the same quality (angular resolution) as a telescope of diameter r_0. The phase variance over an aperture of diameter r_0 is approximately 1 rad^2. The Fried parameter depends on the turbulence profile $C_n^2(z)$, the zenith angle ζ and the wavelength λ. The wavelength dependence is:

$$r_0 \propto \lambda^{6/5} \tag{2.44}$$

which leads to an image size ("seeing disc" or "seeing PSF") of

$$\alpha_{\rm seeing} \propto \frac{\lambda}{r_0} \propto \lambda^{-1/5} \tag{2.45}$$

Note that if we have a telescope of diameter D, we have a PSF size proportional to λ. That means that, relative to the diffraction limited PSF, the seeing-PSF becomes less and less important for longer wavelengths.

Typical values for r_0 at 500 nm are 10 to 20 cm, corresponding to PSF sizes of 0.5" to 1". But r_0 changes strongly from day to night, and varies on all time scales during the night. In fact, one can encounter brief moments where it is very large, i.e. the seeing is very small. If one can make a brief exposure during this moment, your image looks pretty sharp. For bright sources, which allow brief exposures, this allows for an observing strategy called "lucky imaging", which means that of a large number of brief exposures you only take the "best" ones, which (after correcting for possible shifts) are then co-added to result in a fairly sharp image.

But mostly one does not have the luxury of bright sources, and one must expose your image including the seeing effects. The *Strehl ratio* S is defined as the peak intensity of a point source divided by the peak intensity of a diffraction-limited image (no seeing). For phase errors $\lesssim 2$ radian, $S \simeq \exp(-\sigma_\phi^2)$.

2.6 Image motion, speckles

The effects of seeing for small telescopes is different from that of large telescopes. Important is the ratio D/r_0. If this ratio is smaller than 1, then the Airy PSF is broader than the seeing disc. For $D \lesssim 0.5\, r_0$ this effect is so strong, that the seeing smearing is unimportant compared to the diffraction limit of the telescope. For $0.5\, r_0 \lesssim D \lesssim 3$, however, the seeing can still shift the centroid of the Airy PSF in a way as to degrade the image beyond the diffraction limit. This "image motion" effect of the seeing simply moves the diffraction-limited PSF around (by roughly λ/r_0) in a random fashion. If the exposure time is much less than $\tau_0 \simeq 20$ milliseconds, one captures this motion as a still snapshot. Longer exposures mean that this motion simply smears out a star image over the seeing disc. For telescopes of this intermediate size ($0.5\, r_0 \lesssim D \lesssim 3$) one can obtain substantially improved images if one can adjust the position of the image using *tip-tilt adaptive optics*: a real-time adjustment of one of the mirrors such that the image motion is compensated. This is the simplest form of adaptive optics.

For large telescopes, with $D/r_0 \gtrsim 10$, the seeing is not just an image motion, but a diffraction pattern of interfering wave fronts. If we make a "short exposure" (much shorter than τ_0) then this interference pattern originating from the distorted wave function $\psi(\vec{x})$ entering the aperture yields a semi random set of dots called *speckles*, each having a width of about the telescope diffraction limit. The speckles are distributed over an area the size of the seeing disc. Simple tip-tilt adaptive optics will thus not work to improve long exposures. For such large telescopes true adaptive optics, in which the entire wave front is corrected, is mandatory.

The time scale τ_0 of the "seeing" is often not so much related to the time scales of the turbulent motions themselves, but by the passage of a turbulent layer over the observatory by a macroscopic wind. Suppose that the time scales of turbulent eddies is of the order of seconds or (much) more, but an entire layer of turbulent atmosphere at an altitude of, say, 4 km moves with, say, 10 m/s in one direction. The turbulence scale is $r_0 = 10$ cm. This means that a single line of sight to a star passes through 100 uncorrelated turbulent cells of size r_0 per second. This thus sets the time scale τ_0 at 10 milliseconds. In such situations, where wind motions are the dominant source of time scales, one can regard turbulence as *frozen-in*.

2.7 From $B_\psi(\vec{r})$ to image distortion

In Section 2.4 we ended with an expression of the autocovariance $B_\psi(\vec{r})$ of the phase function, and we then qualitatively looked at how seeing affects images. But we can also be more mathematically rigorous. Here we appeal to what we learned in Chapter 1. We learned that, for a plane wave entering an aperture, the PSF on the image plane is the Fourier transform of the aperture function. Let us write the aperture function as $P(\vec{u})$, where $\vec{u} = \vec{x}/\lambda$ in the aperture plane, and let it be 1 where radiation is let through, and 0 where it is blocked. The image on the image plane is then the Fourier transform:

$$A(\vec{\alpha}) = \mathcal{F}[P(\vec{u})] \tag{2.46}$$

where $\vec{\alpha}$ is the location on the image plane measured in angle from the aperture to the image plane. This is for a plane wave.

Now if we have a distorted wave $\psi(\vec{u})$ then we would obtain

$$A(\vec{\alpha}) \propto \mathcal{F}[\psi(\vec{u})P(\vec{u})] \tag{2.47}$$

The flux measured on the image plane is then

$$S(\vec{\alpha}) = AA^* \propto |\mathcal{F}[\psi(\vec{u})P(\vec{u})]|^2 \tag{2.48}$$

We can now use the *Wiener-Khinchin theorem* (see Appendix B.1, Eq. B.10) to write this as

$$\begin{aligned} S(\vec{\alpha}) &\propto |\mathcal{F}[\psi(\vec{u})P(\vec{u})]|^2 && (2.49) \\ &= \mathcal{F}\left[\int \psi(\vec{u})\psi^*(\vec{u}+\vec{f})P(\vec{u})P^*(\vec{u}+\vec{f})d\vec{u}\right] && (2.50) \end{aligned}$$

Or, if we look at the expectation value of the Fourier transform of S:

$$\begin{aligned} \langle S(\vec{f})\rangle &= \langle \int \psi(\vec{u})\psi^*(\vec{u}+\vec{f})P(\vec{u})P^*(\vec{u}+\vec{f})d\vec{u}\rangle && (2.51) \\ &= \int \langle\psi(\vec{u})\psi^*(\vec{u}+\vec{f})\rangle P(\vec{u})P^*(\vec{u}+\vec{f})d\vec{u} && (2.52) \\ &= B_\psi(\vec{f})T(\vec{f}) && (2.53) \end{aligned}$$

where we have defined the *telescope transfer function*

$$T(\vec{f}) \equiv \int P(\vec{u})P^*(\vec{u}+\vec{f})d\vec{u} \tag{2.54}$$

and $B_\psi(\vec{f})$ is the autocorrelation function of the wave at the aperture plane, given by Eq.(2.42) with $\vec{f} = \vec{r}/\lambda$:

$$B_\psi(f) = \exp\left[-3.44\left(\frac{\lambda f}{r_0}\right)^{5/3}\right] \tag{2.55}$$

If we have no turbulence, then $B_\psi(f) = 1$.

The *resolving power* R is defined as

$$R \equiv \int B(\vec{f})T(\vec{f})d\vec{f} \tag{2.56}$$

The resolving power without turbulence, for a circular aperture, is

$$R_{\rm diffract} = \frac{\pi}{4}\left(\frac{D}{\lambda}\right)^2 \tag{2.57}$$

For strong turbulence, $T = 1$ in the region where $B_\psi(f)$ is non-zero, and we obtain

$$R_{\rm turb} = \frac{\pi}{4}\left(\frac{r_0}{\lambda}\right)^2 \tag{2.58}$$

2.8 Angular anisoplanatism

As mentioned above, short-term exposures yield speckle patterns that, for long term exposures, average out to a seeing disc. The question that is addressed now is, will two stars that are very close together, yield the same instantaneous speckle pattern (of course, slightly shifted with respect to each other)? Let us look at a single layer of turbulence at a hight h above the telescope. A stellar separation of θ radian means that the optical paths from the stars to the telescopes pass through the layer at a distance of $r = \theta h \sec\zeta$, where ζ is again the zenith angle. Now evaluate

$$D_\phi(r) = \langle|\phi(0) - \phi(r)|^2\rangle = 2.914\, k^2 \sec\zeta\; \delta h C_n^2 r^{5/3} \tag{2.59}$$

If we now insert $r = \theta h \sec\zeta$ and integrate over all layers we obtain

$$\begin{aligned}
\langle\sigma_\phi^2\rangle &= \langle|\phi(0) - \phi(r)|^2\rangle && (2.60)\\
&= 2.914\, k^2 (\sec\zeta)^{8/3}\theta^{5/3}\int_0^\infty C_n^2 z^{5/3} dz && (2.61)\\
&= 2.914\, k^2 (\sec\zeta)^{8/3}\theta^{5/3}\mu_{5/3} && (2.62)
\end{aligned}$$

Because of the 5/3-moment, angular anisoplanatism is dominated by high layers in the atmosphere. If we define θ_0 as

$$\theta_0 \equiv \left[2.914\, k^2 (\sec\zeta)^{8/3}\mu_{5/3}\right]^{-3/5} \tag{2.63}$$

then we can write

$$\langle\sigma_\phi^2\rangle = \left(\frac{\theta}{\theta_0}\right)^{5/3} \tag{2.64}$$

A pair of stars separated by more than θ_0 have different PSFs at short exposures, even though their PSFs at long exposures are the same.

2.9 Diffraction and scintillation

Most of the above analysis follows the light along rays: the geometric optics approximation. This is valid if the turbulent layers are not higher than the *Fresnel length* d_F belonging to the correlation length r_0. This is the length for which the Fresnel number F is unity (see Section 1.1, Eq.1.13).

$$d_F = \frac{r_0^2}{\lambda} \tag{2.65}$$

For $r_0 = 10$ cm and $\lambda = 500$ nm we have $d_F = 20$ km. This means that we are mostly in good shape with geometric optics. But diffraction can play a role, especially at short wavelengths (recall that $r_0 \propto \lambda^{6/5}$), large zenith angles and poor observing sites.

One effect of diffraction is *scintillation*, the stochastic variation in brightness of stars. This is given by

$$\sigma_{\ln I}^2 = 2.24\, k^{7/6} (\sec\zeta)^{11/6} \mu_{5/6} \tag{2.66}$$

This shows that scintillation is dominated by turbulence at high altitude.