To calculate the level populations we must solve the equation of *statistical equilibrium*. For every level i we demand that the rate at which atoms/molecules are being (de-)excited out of level i is equal to the rate at which level i is being re-populated by (de-)excitation from other levels:

$$\sum_{j>i} n_j A_{ji} - \sum_{j<i} n_i A_{ij} + \sum_j \left[n_j C_{ji} - n_i C_{ij} \right] = 0 \tag{7.54}$$

This must be true for all levels i, and therefore Eq. (7.54) constitutes a coupled set of $N_{\rm lev}$ linear equations with $N_{\rm lev}$ unknowns (where $N_{\rm lev}$ is the number of levels of the atom/molecule), or in other words: a matrix equation. This can be solved using, for instance, an LU-decomposition and backsubstitution method (see e.g. the Numerical Recipes book).

The temperature and density-dependence is "hidden" in the C_{ij} coefficients via Eqs. (7.6, 7.8) and via the temperature-dependence of $K_{ij}(T)$. In particular, because of $C_{ij} = NK_{ij}(T)$ (Eq. 7.6), the C_{ij} is linear in N, the number density of the collision partner (typically e^- for hot atomic gases and H_2 and He for cold molecular gases).

For a given temperature, one can define the *critical number density* $N_{\rm crit}$ as the density above which the collisions are so frequent, that they keep the populations close to their LTE values, while below which there are substantial deviations from LTE. This depends typically also on the level at which you look: some of the lower states might be LTE while the higher states might display strong non-LTE behavior. The critical density is therefore a somewhat vaguely defined concept, but it can nevertheless be extremely useful for estimating from the value of the gas density and the gas temperature whether we risk having non-LTE effects or whether LTE is a good assumption. In addition to that, it allows us to use line intensities and line intensity ratios are a (admittedly somewhat indirect) probe of the density: If we see line ratios that are inconsistent with LTE populations, then this indicates that the density is lower than the critical density for that molecule and that pair of levels. This is an important capability, because normally in the optically thin regime we can only measure *column densities* (i.e. densities integrated along the line of sight).

7.8 Non-LTE: The full problem

7.8.1 The equations for full non-LTE line transfer

In addition to photon emission, photons can also be absorbed by lines. Given a radiation field $I_\nu(\mathbf{x},\mathbf{n})$ the number of photons absorbed by some transition $j \rightarrow i$ is:

$$\begin{aligned}
&\oint \alpha_{ij,\nu}(\mathbf{n}) \frac{I_\nu(\mathbf{n})}{h\nu} d\nu \\
&\simeq \frac{1}{h\nu_{ij}} \oint \alpha_{ij,\nu}(\mathbf{n}) I_\nu(\mathbf{n}) d\nu \\
&= \frac{1}{h\nu_{ij}} \oint \frac{h\nu_{ij}}{4\pi} (N_j B_{ji} - N_i B_{ij}) \phi_{ij}(\nu,\mathbf{x},\mathbf{n}) I_\nu(\mathbf{n}) d\nu \\
&= (N_j B_{ji} - N_i B_{ij}) \frac{1}{4\pi} \oint \phi_{ij}(\nu,\mathbf{x},\mathbf{n}) I_\nu(\mathbf{n}) d\nu \\
&= (N_j B_{ji} - N_i B_{ij}) J_{ij}
\end{aligned} \tag{7.55}$$

with J_{ij} is the *line profile integrated mean intensity* defined as

$$J_{ij} \equiv \frac{1}{4\pi} \oint \phi_{ij}(\nu,\mathbf{x},\mathbf{n}) I_\nu(\mathbf{n}) d\nu \tag{7.56}$$

where $\phi_{ij}(\nu,\mathbf{x},\mathbf{n})$ is

$$\phi_{ij}(\nu,\mathbf{x},\mathbf{n}) = \phi_{ij}(\nu,\mathbf{v}(\mathbf{x})) \tag{7.57}$$

with $\phi_{ij}(\nu, \mathbf{v}(\mathbf{x}))$ given by Eq. (7.20).

If we include this rate into the statistical equilibrium equation Eq. (7.54) then we obtain

$$\begin{aligned}&\sum_{j>i}\Big[n_j A_{ji} + (n_j B_{ji} - n_i B_{ij})J_{ji}\Big]\\ &- \sum_{j<i}\Big[n_i A_{ij} + (n_i B_{ij} - n_j B_{ji})J_{ij}\Big]\\ &+ \sum_{j\neq i}\left[n_j C_{ji} - n_i C_{ij}\right] = 0\end{aligned} \tag{7.58}$$

which is the full form of the statistical equilibrium equation for non-LTE line transfer. This is a local equation (to be solved at each location separately), but it has a global character due to the dependency on J_{ij} and J_{ji} which can only be calculated using full radiative transfer.

As usual, we can write the J_{ij} and J_{ji} in Eq. (7.58) as a Lambda-operator acting on the source function:

$$J_{ij} = \Lambda_{ij}[S_{ij}] \tag{7.59}$$

where the source function for the line $i \rightarrow j$ is

$$S_{ij} = \frac{j_\nu(\mathbf{n})}{\alpha_\nu(\mathbf{n})} = \frac{n_i A_{ij}}{n_j B_{ji} - n_i B_{ij}} \tag{7.60}$$

Note that the source function S_{ij} is independent of ν, at least over the small frequency range across the line $i \rightarrow j$. It has a single value for each line. For LTE line transfer this becomes:

$$S_{ij}\Big|_{\rm LTE} = B_{\nu_{ij}}(T) \tag{7.61}$$

The Lambda Operator Λ_{ij} can written as

$$\Lambda_{ij}[.] = \frac{1}{4\pi}\int d\nu \oint d\Omega \phi_{ij}(\nu, \mathbf{x}, \mathbf{n})\Lambda_{\nu,\mathbf{n}}[.] \tag{7.62}$$

where $\Lambda_{\nu,\mathbf{n}}[.]$ is the angle-dependent Lambda Operator (which gives the intensity $I_\nu(\mathbf{n})$ for given source function $S_\nu(\mathbf{x})$.

Eq. (7.58) then becomes

$$\begin{aligned}&\sum_{j>i}\Big[n_j A_{ji} + (n_j B_{ji} - n_i B_{ij})\Lambda_{ji}[S_{ji}]\Big]\\ &- \sum_{j<i}\Big[n_i A_{ij} + (n_i B_{ij} - n_j B_{ji})\Lambda_{ij}[S_{ij}]\Big]\\ &+ \sum_{j\neq i}\left[n_j C_{ji} - n_i C_{ij}\right] = 0\end{aligned} \tag{7.63}$$

This is now the formulation of the full non-LTE line radiative transfer problem in terms of a Lambda Operator.

7.8.2 Assumptions underlying the equations

In writing Eq. (7.59) and Eq. (7.63) we made an important assumption: that J_{ij} only depends on S_{ij}. But Eq. (7.62) contains an integral over $S_\nu(\mathbf{x})$ whereever $\phi_{ij}(\nu, \mathbf{n}) \neq 0$. If there is another line $k \rightarrow l$ with frequency ν_{kl} very close to ν_{ij}, then this may also contribute to $S_\nu(\mathbf{x})$ and thus to Eq. (7.62). We would then have Λ_{ij} not only depend on $S_{ij} = S(\nu_{ij})$ but also on S_{kl}. In writing Eq. (7.59) we made the implicit assumption of *non-overlapping lines*, i.e. that none of the line interfere radiatively with any of the other lines. In principle the treatment of overlapping lines is straightfoward, at least in

the simplest of methods such as Lambda Iteration (Section 7.9 below). But for more sophisticated methods it can become tricky. The extreme case in which many lines overlap to form a kind of pseudo-continuum is called *line blanketing*.

Another assumption we tacitly made is that of *complete redistribution*. The assumption is that the velocity of any atom or molecule will be completely randomized between successive photon absorption or emission events. In other words: if at some instant in time an atom moves (through its thermal motions) with some velocity v toward the observer, its line is doppler shifted to the blue, and it may thus preferentially absorb photons that are blueshifted. The complete redistribution assumption says that before that atom/molecule absorbs or emits another photon, the thermal collisions have already moved the atom into another direction. This assumption is very important to keep the problem of line transfer tractable. If this were not the case, then we would have to solve an equation such as Eq. (7.63) not just for each location on the grid, but also for each velocity vector. Instead of a 3-D problem in $\mathbf{x}$ we would then have a 6-D problem in $\mathbf{x}, \mathbf{v}$. Unfortunately, complete redistribution is not always guaranteed. In particular in the interstellar medium, if we use sub-grid-scale turbulence as a simple broadening factor in the line profile (see Section 7.5.2), the time it takes for a typical turbulent eddy to "turn over" (i.e. randomize its velocity) is much larger than the typical $1/A_{ij}$ for the line transitions. Nevertheless the complete redistribution assumption is still used for such problems, simply because the full treatment of non-LTE line transfer with *partial redistribution* would be computationally unfeasible in full 3-D.

7.9 Lambda Iteration for line transfer

The most straightfoward method of solution of this non-LTE line transfer problem is Lambda Iteration (see Section 4.4). It is simply the iteration between solving the set of equations Eq. (7.58) for given J_{ij}, and computing J_{ij} for given n_i. Or, starting from Eq. (7.63), we can write Lambda Iteration as:

$$\begin{aligned}
&\sum_{j>i}\left[n_j^{m+1}A_{ji} + (n_j^{m+1}B_{ji} - n_i^{m+1}B_{ij})\Lambda_{ji}[S_{ji}^m]\right] \\
&- \sum_{j<i}\left[n_i^{m+1}A_{ij} + (n_i^{m+1}B_{ij} - n_j^{m+1}B_{ji})\Lambda_{ij}[S_{ij}^m]\right] \\
&+ \sum_{j\neq i}\left[n_j^{m+1}C_{ji} - n_i^{m+1}C_{ij}\right] = 0
\end{aligned} \tag{7.64}$$

where the m-index is the iteration counter. After each iteration you solve the coupled set of linear equations Eq. (7.64) to obtain a new set of populations $\{n_1^{m+1}, \cdots, n_{N_{\rm levels}}^{m+1}\}$.

In practice the frequency integration of Eq. (7.62) is done over small frequency windows covering each line:

$$\Lambda_{ij}[.] = \frac{1}{4\pi}\int_{\nu_{ij}-\Delta\nu}^{\nu_{ij}+\Delta\nu} d\nu \oint d\Omega \phi_{ij}(\nu, \mathbf{x}, \mathbf{n})\Lambda_{\nu,\mathbf{n}}[.] \tag{7.65}$$

where $\Delta\nu$ must be chosen large enough that the line always stays fully within the window, even when doppler shifted as a result of gas motion within the model. This means that you will have to find out what the largest velocity is inside the model box, and assure that the shifted line, including its broadened wings, fits well inside the window. Note that formally each line is infinitely wide since there is no ν where the line profile is *exactly* zero. But typically one can find a distance from the line where the line profile function is sufficiently small that it no longer contributes.

It is also useful to choose $\Delta\nu$ not too large, because the frequency integral in Eq. (7.65) is, in the computer, a discrete sum over a frequency grid. If $\Delta\nu$ is very large, then that would require a lot of frequency points to sum over. The spacing of the frequency

points must be small enough that they nicely resolve the intrinsic line profile $\phi_{ij}(\nu)$. Taking, for instance, a window $\Delta\nu$ that is 100× the intrinsic line width, requires about 400 frequency points at the very least, if not more. This also means doing the full formal transfer for 400 frequencies *per line* for each iteration of the Lambda Iteration scheme. This is hugely numerically costly. Therefore, a wise (not too small, not too big) choice of $\Delta\nu$ is crucial to keep the problem doable.

This also shows us that non-LTE line transfer problems in which the gas experiences large differential velocities (i.e. large velocity differences between different locations in the model) will be very numerically costly to solve using this Lambda Iteration method, even for the moderately optically thick case.

7.10 ALI for lines: The MALI method of Rybicki & Hummer

In Section 4.4 we learned that the Lambda Iteration method can be rather slow in convergence, and that a better method exists: the Accelerated Lambda Iteration method.

To generate an ALI method out of Eq. (7.63) we introduce the usual splitting of the Λ-operator:

$$\Lambda_{ij} = \Lambda^*_{ij} + (\Lambda_{ij} - \Lambda^*_{ij}) \tag{7.66}$$

such that (cf. Eq. 7.62):

$$\Lambda^*_{ij}[.] = \frac{1}{4\pi}\int d\nu \oint d\Omega \phi_{ij}(\nu, \mathbf{x}, \mathbf{n})\Lambda^*_{\nu,\mathbf{n}}[.] \tag{7.67}$$

where $\Lambda^*_{\nu,\mathbf{n}}$ is, for instance, the diagonal part of the Lambda Operator or its tri-diagonal part.

Let us for convenience assume $\Lambda^*_{\nu,\mathbf{n}}$ to be the diagonal of the full Lambda Operator, so that Λ^*_{ij} is just a scalar instead of a non-local operator. We can then work out the following *local* expression:

$$\Lambda^*_{ij}[S_{ij}] = \Lambda^*_{ij}S_{ij} = \Lambda^*_{ij}\frac{n_i A_{ij}}{n_j B_{ji} - n_i B_{ij}} \tag{7.68}$$

Since in Eq. (7.63) this is then multiplied with $(n_i B_{ij} - n_j B_{ji})$ let us work out:

$$(n_i B_{ij} - n_j B_{ji})\Lambda^*_{ij}[S_{ij}] = (n_i B_{ij} - n_j B_{ji})\Lambda^*_{ij}\frac{n_i A_{ij}}{n_j B_{ji} - n_i B_{ij}} = -\Lambda^*_{ij} n_i A_{ij} \tag{7.69}$$

Therefore, if we insert Eq. (7.66) into Eq. (7.63) ans we use Eq. (7.69), we obtain:

$$\begin{aligned}
&\sum_{j>i}\left[n_j A_{ji}(1-\Lambda^*_{ji}) + (n_j B_{ji} - n_i B_{ij})(\Lambda_{ji} - \Lambda^*_{ji})[S_{ji}]\right] \\
&- \sum_{j<i}\left[n_i A_{ij}(1-\Lambda^*_{ij}) + (n_i B_{ij} - n_j B_{ji})(\Lambda_{ij} - \Lambda^*_{ij})[S_{ij}]\right] \\
&+ \sum_{j\neq i}\left[n_j C_{ji} - n_i C_{ij}\right] = 0
\end{aligned} \tag{7.70}$$

This is almost identical to Eq. (7.63), only with the following substitions:

$$A_{ij} \quad \rightarrow \quad A_{ij}(1-\Lambda^*_{ij}) \tag{7.71}$$

$$\Lambda_{ij} \quad \rightarrow \quad (\Lambda_{ij} - \Lambda^*_{ij}) \tag{7.72}$$

And so the ALI iteration scheme, for a local operator, becomes:

$$\begin{aligned}
&\sum_{j>i}\left[n_j^{m+1} A_{ji}(1-\Lambda^*_{ji}) + (n_j^{m+1} B_{ji} - n_i^{m+1} B_{ij})(\Lambda_{ji} - \Lambda^*_{ji})[S^m_{ji}]\right] \\
&- \sum_{j<i}\left[n_i^{m+1} A_{ij}(1-\Lambda^*_{ij}) + (n_i^{m+1} B_{ij} - n_j^{m+1} B_{ji})(\Lambda_{ij} - \Lambda^*_{ij})[S^m_{ij}]\right] \\
&+ \sum_{j\neq i}\left[n_j^{m+1} C_{ji} - n_i^{m+1} C_{ij}\right] = 0
\end{aligned} \tag{7.73}$$

This is the Multilevel Accelerated Lambda Iteration (MALI) scheme of Rybicki & Hummer (1991, Astronomy & Astrophysics 245, 171) for non-overlapping lines, for a local operator and without a background continuum. One can relatively easily include Ng-acceleration (Section 4.4.7) on the level populations to speed things up more.

Eq. (7.73) is a coupled set of linear equations that has to be solved at each location for each iteration. We were in fact quite lucky that Eq. (7.73) remained linear in the populations! The reason why this is not trivial is that in line transfer the opacity $\alpha_\nu(\mathbf{x},\mathbf{n})$ changes from one iteration to the next. In Section 4.4 we worked out the ALI method under the assumption that the $\alpha_\nu(\mathbf{x})$ stays constant with iteration, in which case the linearity of the problem is evident. With $\alpha_\nu(\mathbf{x},\mathbf{n})$ changing in line transfer this is not evident at all. But because of the miraculous cancellation of the $(n_i B_{ij} - n_j B_{ji})$ factor in Eq. (7.69) all the non-linearity fortunately got cancelled out. It turns out that if you include complexities such as a background continuum by the dust that this cancellation does no longer take place. Please refer to the paper by Rybicki & Hummer for how the MALI method solves this problem.

7.11 The two-level atom problem: Line "scattering"

People often talk about "scattering" when they are talking about non-LTE line transfer. This is not real scattering that we know from scattering off dust particles. It is the process of excitation and subsequent de-excitation of a level pair that has the netto effect of redirecting a photon into a different direction. Mathematically this process is similar to true isotropic scattering (see Section 4.1), and it is for this reason that the word "scattering" is used. However, this often leads to confusion. In fact, the mathematical correspondence is only valid in the case of a two-level system. If we have multiple system, the excitation to a higher level could be followed by a two-step de-excitation, meaning that a high-energy photon is absorbed while two lower energy photons are being emitted. This is no longer similar to scattering.

Let us work out a simple example, assuming that the medium is at rest (i.e. $\mathbf{v}(\mathbf{x}) = 0$). We have just two levels: "u" (up) and "d" (down), so that $n_d = 1 - n_u$. Let us first assume that the collision rates $C_{ud} = C_{du} \simeq 0$ for simplicity. Then the statistical equilibrium equation becomes

$$n_u A_{ud} + (n_u B_{ud} - n_d B_{du}) J_{ud} = 0 \tag{7.74}$$

which is just *one* equation (rather than a coupled set). The emissivity and extinction coefficients are (Eqs. 7.11, 7.13):

$$j_{ud,\nu} = \frac{h\nu_{ud}}{4\pi} N n_u A_{ud} \phi(\nu) \tag{7.75}$$

$$\alpha_{ud,\nu} = \frac{h\nu_{ud}}{4\pi} N (n_d B_{du} - n_u B_{ud}) \phi(\nu) \tag{7.76}$$

where N is the total number density of the two-level atom. With Eq. (7.74) we can then write

$$j_{ud,\nu} = \alpha_{ud,\nu} J_{ud} \tag{7.77}$$

which is the line equivalent of the isotropic scattering formula Eq. (4.2).

The main difference is now that J_{ud} is a frequency-integrated mean intensity, integrated over a line profile. This means that photons can hop from one frequency to the other within the line profile upon each scattering event. This is reminiscent to the "scattering"-nature of the absorption+re-emission events in dust thermal radiative transfer (see Section 5.4.4), but now only over a tiny frequency domain just around the line. It means that, like with the absorption+re-emission, the scattering "random walk" can take spatial steps of different distance, depending on which frequency (and thus which opacity) the photon happens to be at a given moment.

Another difference with the isotropic scattering problem of Section 4.1 is that in the two-level atom problem the opacity $\alpha_{ud,\nu}$ will be dependent on J_{ud}: if we have a very large J_{ud}, then most of the atoms will be in the "up" state, meaning that there are fewer "down" state atoms to excite. This reduces $\alpha_{ud,\nu}$. However, as long as J_{ud} is small enough that $n_u \ll 1$, then $\alpha_{ud,\nu}$ can be considered to be constant.

There are examples in Nature which behave nicely like a two-level atom. For instance, the Ly-α line that is emitted after recombination. If an electron recombines with a proton to form a hydrogen atom, it typically is first in an excited state. Through radiative decay the atom decays to ever lower levels, until eventually reaching the 1s ground state. If it reaches the 2p state before that, that state can only decay by sending out a Ly-α photon. However, this Ly-α photon can quickly excite any ground-state hydrogen atom nearby. In fact, in many cases the optical mean free path for a Ly-α photon is extremely small compared to the typical scales of the system. It will therefore almost immediately be reabsorbed, exciting the electron in that atom from 1s to 2p. That state will live for a short time and radiatively decay again, sending out, again, a Ly-α photon, and the entire procedure starts all over again. De-facto the Ly-α photon will scatter many times before either escaping from the system or being destroyed by a collisional de-excitation or two-photon de-excitation. Since the optical depth in the Ly-α line is often so high that the chance of escape is much lower than the chance of collisional or two-photon decay, it is often reasonable to use the *on-the-spot* approximation: assuming that the 2p state simply does not decay via Ly-α. But the validity of the on-the-spot approximation requires verification on a case-by-case basis.

7.12 Photon escape probability and the escape probability method

Even though the concept of "scattering" is strictly speaking only valid for a two-level atom, it can be very useful for multi-level atoms as well, to get a feeling for the problem. In line transfer energy (or photons) can be transported via the lines. But if the optical depth is very large in all the lines that are appreciably emitting light, then the transport of the energy is hampered substantially. It is then useful to look at the various ways by which radiation can nevertheless escape. One way is simply by multiple scattering. But another way is the escape in the line wings: if, occasionally, a photon happens to be emitted at a frequency far enough away from the line center that the optical depth of the cloud at that frequency is smaller than unity, then the photon can escape. To be more precise: the probability of a photon to escape from a cloud with optical depth τ_ν is:

$$p_{\mathrm{esc},\nu} = \frac{1 - e^{-\tau_\nu}}{\tau_\nu} \tag{7.78}$$

This is called the *monochromatic escape probabilty*. If we integrate this over the full line profile we obtain the *escape probability* for that line:

$$p_{\mathrm{esc}} = \int \frac{1 - e^{-\tau_\nu}}{\tau_\nu} \phi(\nu) d\nu \tag{7.79}$$

with of course $0 \leq p_{\mathrm{esc}} \leq 1$.

This insight can be used to contruct a very simple (though of course very approximate) method of non-LTE line transfer: we simply assume that

$$\Lambda_{ij} = (1 - p_{\mathrm{esc},ij}) \tag{7.80}$$

i.e. that, for each allowed pair (i, j) the Lambda Operator Λ_{ij} is a diagonal operator with $p_{\mathrm{esc},ij}$ on the diagonal. Inserting this into the statistical equilibrium equation

Eq. (7.63),

$$\begin{aligned}&\sum_{j>i}\Big[n_jA_{ji}+(n_jB_{ji}-n_iB_{ij})(1-p_{\mathrm{esc},ji})S_{ji}\Big]\\&-\sum_{j<i}\Big[n_iA_{ij}+(n_iB_{ij}-n_jB_{ji})(1-p_{\mathrm{esc},ij})S_{ij}\Big]\\&+\sum_{j\neq i}\left[n_jC_{ji}-n_iC_{ij}\right]=0\end{aligned}\tag{7.81}$$

and using Eq. (7.60) then leads to

$$\sum_{j>i}n_jA_{ji}p_{\mathrm{esc},ji}-\sum_{j<i}n_iA_{ij}p_{\mathrm{esc},ij}+\sum_{j\neq i}\left[n_jC_{ji}-n_iC_{ij}\right]=0\tag{7.82}$$

Thus, if we have a good way of computing the escape probabilities $p_{\mathrm{esc},ij}$, then we can solve Eq. (7.82) and thus find the level populations. This is called the *escape probability method*.

The catch is, however, that Eq. (7.78) is only an approximation. One should in fact integrate Eq. (7.78) over all directions:

$$p_{\mathrm{esc},\nu}=\frac{1}{4\pi}\oint\frac{1-e^{-\tau_\nu(\mathbf{n})}}{\tau_\nu(\mathbf{n})}d\Omega\tag{7.83}$$

which involves an integral along a ray from the point of interest to infinity for each direction $\mathbf{n}$:

$$\tau_\nu(\mathbf{n})=\int_0^\infty\alpha_\nu(\mathbf{x}+s\mathbf{n},\mathbf{n})ds\tag{7.84}$$

Furthermore, the assumption that one can replace Λ_{ij} with $(1-p_{\mathrm{esc},ij})$ is also a very approximate one. The escape probability method completely ignores the fact that radiative energy can be transported from one region of the cloud to the next. Nevertheless, the escape probability method turns out to be quite helpful and usually gives a reasonable approximate answer. And because it is much faster computationally than the full MALI iteration, it is quite often used.

7.13 The Large Velocity Gradient method of Sobolev

Very closely related to the escape probability method described above is the *Large Velocity Gradient (LVG) method*, also called the *Sobolev method*. The idea here is that a photon can escape from the line due to differences in velocity between adjacent regions. Suppose a photon is emitted at position $\mathbf{x}$ from a gas parcel with velocity $\mathbf{v}$. Now it travels to a neighboring position $\mathbf{x}'$ where the gas velocity is $\mathbf{v}'$. Let $\mathbf{n}$ be the unit vector pointing from $\mathbf{x}$ into the direction $\mathbf{x}'$. The doppler shift between these two points is now $\Delta\nu_{ij}=\nu_{ij}\mathbf{n}(\mathbf{v}'-\mathbf{v})/c$. If this doppler shift is larger than the local line width, then the photon suddenly finds itself "free" because the opacity is then suddenly very low as a result of the line having been doppler shifted away from the frequency of the photon. A velocity gradient can thus lead to photon escape.

The LVG method is then identical to the escape probability method, but with $p_{\mathrm{esc},ij}$ caused by the velocity gradient. If we use Eqs. (7.83,7.84) then this is essentially automatically included in Eq. (7.84).

If we, however, do not want to explicitly integrate Eq. (7.84), then we can also approximate this, for the case of large enough velocity gradients, as:

$$\tau_{ij,\mathrm{LVG}}=\frac{A_{ij}c^3}{8\pi\nu_{ij}^3}\frac{N}{|dv/ds|}\left(\frac{g_i}{g_j}n_j-n_i\right)\tag{7.85}$$

where $|dv/ds|$ is the velocity gradient. In principle this has to be computed in all directions and averaged.

7.14 Some common line radiative transfer phenomena

7.14.1 Radiative pumping

Normally the very high levels of an atom or molecule are not populated because the thermal collisions cannot populate them. However, a common phenomenon in line transfer of cool gases in the presence of a hot light source (e.g. a star) is that the hot photons from the light source can radiatively excite the higher levels of the atom or molecule. Subsequently, these radiatively (or collisionally) decay and thus populate ever lower levels. In other words: they *cascade down*. This may thus lead to line radiation that is not expected on just the basis of thermal considerations of the gas in question. This process is called *radiative pumping*.

7.14.2 Recombination lines

If atoms get ionized by a hot light source (e.g. an O-star), then recombination can take place to the higher levels of the atom; levels that do not get thermally excited usually, because they are so high up. In that case you also get cascading, and thus line emission from rather high levels, even for relatively low-temperature gas. For hydrogen this may even lead to *radio recombination lines*, because the energy differences between the levels at principal quantum numbers close to 100 are at radio wavelengths.

7.14.3 Masers/lasers

If two levels lie very close together in frequency, and if the higher levels get somehow pumped, then the cascading may lead to populations of these two levels that are far from LTE. It may even lead to *population inversion* in which

$$n_j B_{ji} - n_i B_{ij} < 0 \qquad \text{for} \qquad j > i \tag{7.86}$$

which leads to *negative opacity*. As a result you get runaway emission, i.e. laser emission. For millimeter wavelength radiation this is called *maser emission*. Formally this could easily lead to absurdly strong emission, if you simply integrate the formal transfer equation. However, energy is conserved. This means that the process is self-limiting: if the emitted radiation becomes too strong, the de-population of the upper level is so strong that it reduces the masering effect. This is a highly non-linear process, which makes it very hard to model numerically. Most line radiative transfer codes cannot properly handle masers/lasers; special methods have to be used.