Exam Radiative Transfer (SS 2012)
NOTE: This Exam has 5 problems!

1. **Moments of intensity and the Eddington approximation**

   (a) The intensity $I_\nu(\mathbf{n})$ is a function of direction $\mathbf{n}$. How are the first three angular moments of intensity defined?

   (b) Why are the moment equations of radiative transfer incomplete? Or in other words: why do we always need a "closure relation" of some kind to close the system of moment equations?

   (c) The Eddington approximation is one way of closing the moment equations, by assuming $K_{ij,\nu} = \frac{1}{3} J_\nu \delta_{ij}$, where δ_{ij} is the Kronecker delta (=1 for $i = j$, =0 for $i \neq j$). Explain where the factor $\frac{1}{3}$ comes from (i.e. why not choose for instance $K_{ij,\nu} = J_\nu \delta_{ij}$?). Hint: To simply things, you can derive this for the case of plane-parallel transfer with $\mu = \cos\theta$.

   (d) Under which conditions is this choice reasonable, and why?

   (e) Show that in the Eddington approximation the mean intensity J_ν obeys

   $$\frac{1}{\alpha_\nu} \nabla \cdot \left(\frac{\nabla J_\nu}{3\alpha_\nu} \right) = J_\nu - S_\nu \tag{1}$$

   where $S_\nu = j_\nu / \alpha_\nu$ is the source function.

2. **Line transfer in a spherical cloud**
   Suppose we have a spherical gas cloud with radius $R_{\mathrm{cloud}} = 10^{16}$ cm consisting mostly of H_2 molecules. Let us assume that the gas density in the cloud is constant with $\rho = 10^{-17}$ g/cm^3 so that the total mass of the cloud is $M = 4.2 \times 10^{31}$ g $= 0.021$ $M_\odot$. The cloud is entirely dust-free. The temperature of the cloud is also constant at $T = 40$ K. The velocity of the gas is everywhere zero and there is no turbulence.

   Let us now introduce a fictional two-level molecule X which has an abundance relative to H_2 of $N_{\mathrm{X}}/N_{\mathrm{H_2}} = 10^{-5}$ and a molecular weight of $20 m_p$ (where m_p is the proton mass). The ground level has energy $E_d = 0$ and statistical weight $g_d = 1$. The upper level has $E_u = 3 \times 10^{-3}$ eV and statistical weight $g_u = 3$. The spontaneous decay rate is $A_{ud} = 10^{-8}$ s^{-1}. The collisional rate per X molecule per H_2 molecule at the temperature of 40 K is $K_{ud} = 3 \times 10^{-12}$ cm^3/s.

   (a) What is the wavelength λ_0 of the line corresponding to the u→d transition?

   (b) Assuming Local Thermodynamic Equilibrium (LTE), what are the *fractional* level populations n_u and n_d of molecule X?

   (c) Assuming that the only line-broadening mechanism is thermal broadening, what is the optical depth from the center of the cloud to the edge of the cloud at line-center?

   (d) By comparing the spontaneous radiative decay rate A_{ud} to the collisional decay rate C_{ud}, answer the question whether LTE is a good assumption or not.

3. **An optically thin dust cloud**
Consider an optically thin dust cloud of any shape (you can choose the shape you like). The cloud is at a large distance d from the observer, i.e. it spans a small angle on the sky of the observer. The temperature of the cloud is T, the opacity at frequency ν is κ_ν (in units of cm^2/gram-of-dust).

Starting from the formal transfer equation, prove that the flux we observe on Earth is (again: choose a convenient shape if you like!):

$$F_\nu = \frac{M_{\mathrm{dust}}\kappa_\nu B_\nu(T)}{d^2} \tag{2}$$

4. **UX Orionis stars**
UX Orionis stars are stars that have occasional extinction events where the stellar light (in the *optical*) suddenly, on a time scale of a day or two, becomes dimmer. A few days later they regain their original brightness again. From infrared observations we know that these stars are surrounded by dusty material. It is believed that the cause of the dimming in the optical is the passage of one of these circumstellar dust clumps or filaments in front of the star.

   (a) As these stars become dimmer, their color becomes redder. Explain.

   (b) If the dimming is *very strong*, their color becomes blue, sometimes even bluer than the original unextincted starlight! Explain.

5. **Accelerated Lambda Iteration**
The Lambda Iteration Method can be written as the following formula:

$$\mathbf{S}^{m+1} = \epsilon\mathbf{B} + (1-\epsilon)\Lambda[\mathbf{S}^m] \tag{3}$$

   (a) Explain under which conditions this method requires (too) many iterations to converge, and why.

   (b) Explain how this convergence is improved using Accelerated Lambda Iteration (ALI). Please write down the relevant formulae, including the ALI-version of Eq. (3). Allow, in your explanation, for a general approximate operator.

   (c) Now simplify this to a local operator; why does this make ALI a lot simpler to implement into a program?