Solutions Klausur
Radiative Transfer (SS 2012)

1. **Moments of intensity and the Eddington approximation**

   (a)

$$J_\nu = \frac{1}{4\pi}\oint I_\nu(\mathbf{n})d\Omega \tag{1}$$
$$H_{i,\nu} = \frac{1}{4\pi}\oint I_\nu(\mathbf{n})n_i d\Omega \tag{2}$$
$$K_{ij,\nu} = \frac{1}{4\pi}\oint I_\nu(\mathbf{n})n_i n_j d\Omega \tag{3}$$

   (b) First two equations:

$$\nabla\cdot\mathbf{H}_\nu(\mathbf{x}) = j_\nu(\mathbf{x}) - \alpha_\nu(\mathbf{x})J_\nu(\mathbf{x}) \tag{4}$$
$$\nabla\cdot\mathcal{K}_\nu(\mathbf{x}) = -\alpha_\nu(\mathbf{x})\mathbf{H}_\nu(\mathbf{x}) \tag{5}$$

   That means: there is always one more moment than equations.

   (c) If we assume an isotropic radiation field, then

$$\begin{aligned} K_{ij,\nu} &= \frac{1}{4\pi}\oint I_\nu n_i n_j d\Omega \\ &= I_\nu\frac{1}{4\pi}\oint n_i n_j d\Omega \\ &= I_\nu\frac{\delta_{ij}}{4\pi}\oint n_1 n_1 d\Omega \\ &= I_\nu\frac{\delta_{ij}}{2}\int_{-1}^{+1}\mu^2 d\mu \\ &= I_\nu\frac{\delta_{ij}}{2}\int_{-1}^{+1}\mu^2 d\mu \\ &= I_\nu\frac{\delta_{ij}}{6}[\mu^3]_{-1}^{+1} \\ &= I_\nu\frac{\delta_{ij}}{3} \end{aligned} \tag{6}$$

   (d) The radiation is only approximately isotropic if the optical depth is high.

   (e) The first moment equation is then:

$$\nabla_i H_{i,\nu} = j_\nu - \alpha_\nu J_\nu \tag{7}$$

   The second moment equation is then:

$$\frac{1}{3}\nabla J_\nu = -\alpha_\nu H_{k,\nu} \tag{8}$$

Dividing the latter by α_ν and taking the ∇_k:

$$\nabla_k \left(\frac{1}{3\alpha_\nu} \nabla_k J_\nu \right) = -\nabla_k H_{k,\nu} \tag{9}$$

Inserting this into the first equation yields

$$\nabla_k \left(\frac{1}{3\alpha_\nu} \nabla_k J_\nu \right) = \alpha_\nu J_\nu - j_\nu \tag{10}$$

2. **Line transfer in a spherical cloud**

   (a) $\lambda_0 = 413\ \mu$m.

   (b) $e^{-\Delta E/kT} = 0.42$, $Z = g_d + g_u e^{-\Delta E/kT} = 2.26$, $n_d = 1/Z = 0.44$, $n_u = (3/Z)e^{-\Delta E/kT} = 0.56$.

   (c) $\phi_0 = c/(a_{th}\nu\sqrt{\pi}) = 1.28 \times 10^{-6}$, $B_{ud} = 1.78 \times 10^3$, $B_{du} = 5.35 \times 10^3$, $N_x = 30$. So $\alpha = 2.02 \times 10^{-17}$. This gives $\tau = 0.2$.

   (d) $C_{ud} = N_{H_2} K_{ud} = 9 \times 10^{-6}$, so $C_{ud}/A_{ud} = 900$. So LTE is fine.

3. **An optically thin dust cloud**
   Take a square shape with Δx in the direction of the observer, and Δy and Δz perpendicular. We have $M_{\rm dust} = \rho_{\rm dust}\Delta x \Delta y \Delta z$. We also have

$$I_\nu = j_\nu \Delta x = \alpha_\nu B_\nu(T)\Delta x = \rho_{\rm dust}\kappa_\nu B_\nu(T)\Delta x \tag{11}$$

   The $\Delta\Omega = \Delta y \Delta z / d^2$. The flux is

$$F_\nu = I_\nu \Delta\Omega = \rho_{\rm dust}\kappa_\nu B_\nu(T)\frac{\Delta x \Delta y \Delta z}{d^2} = \kappa_\nu B_\nu(T)\frac{M_{\rm dust}}{d^2} \tag{12}$$

4. **UX Orionis stars**

   (a) Dust extinction (if the grains are small) is stronger at shorter wavelengths, hence the reddening.

   (b) If the star is fully extincted, then the only optical light you see is scattered light, which is also stronger at shorter wavelength. Hence the bluing.

5. **Accelerated Lambda Iteration**

   (a) If the optical depth is large and ϵ is small, then a photon can pingpong many times before it escapes or gets destroyed. Since each Lambda Iteration step accounts for one scattering, this means many iterations are required.

   (b) We split

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*) \tag{13}$$

   Inserting yields

$$[1 - (1-\epsilon)\Lambda^*]\mathbf{S}^{m+1} = \epsilon\mathbf{B} + (1-\epsilon)(\Lambda - \Lambda^*)[\mathbf{S}^m] \tag{14}$$

   This then means:

$$\mathbf{S}^{m+1} = [1 - (1-\epsilon)\Lambda^*]^{\rm invl}\left(\epsilon\mathbf{B} + (1-\epsilon)(\Lambda - \Lambda^*)[\mathbf{S}^m]\right) \tag{15}$$

(c) Local operator means Λ^* is a scalar, meaning that $[1-(1-\epsilon)\Lambda^*]$ is a scalar. You can then write:

$$\mathbf{S}^{m+1} = \frac{\epsilon\mathbf{B} + (1-\epsilon)(\Lambda - \Lambda^*)[\mathbf{S}^m]}{[1-(1-\epsilon)\Lambda^*]} \tag{16}$$