Advanced Cosmology

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Contents

Part I

Introduction and recap

About these lecture notes

- These lecture notes contain the material presented on the board.
- They are designed to guide you through the lecture content.
- While I primarily follow Amendola's script, these notes are complementary because:
    1) Some arguments are restructured, and I have selectively chosen the topics.
    2) Most computations are detailed step-by-step, leaving little to the imagination.
    3) Mathematical terms involved in the computations are marked with arrows for clarity.
    4) Instead of lengthy explanations, I use concise "slogans" to emphasize key points.
    5) The detailed discussion will be provided during the lectures, where I will be very talkative.
- If you find any error, kindly let me know.

Relativistic cosmological model

What do we need?

Universe is neutral + only long-range forces are relevant

$\Rightarrow$ Relevant force is gravity: General Relativity

Einstein Fields equation in vacuum

$$S_H = \int R\sqrt{-g}\, d\Omega \qquad R = g^{ab} R_{ab} \qquad d\Omega = dx^0 dx^1 dx^2 dx^3$$

$$\delta S_H = \int \left(R_{\mu\nu} - R\frac{g_{\mu\nu}}{2}\right)\delta g^{\mu\nu}\sqrt{-g}\, d\Omega \overset{!}{=} 0 \qquad \boxed{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \equiv G_{\mu\nu} = 0}$$

Fields equation in vacuum with source

$$S = S_H + \alpha_M S_M$$

$$\delta S = \delta(S_H + \alpha_M S_M) = \int\left[G_{\mu\nu} + \alpha_M \frac{\delta(\mathcal{L}_M\sqrt{-g})}{\sqrt{-g}\,\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\, d\Omega \overset{!}{=} 0$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g^{\mu\nu}} \qquad T_{ab} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{ab}} \qquad \boxed{G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}} \qquad \text{with } \nabla_\mu G^\mu{}_\nu = 0 = \nabla_\mu T^\mu{}_\nu$$

Need for cosmological const. or Dark energy!

$$S = \int (R - 2\Lambda)\sqrt{-g}\, d\Omega + S_M \to R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad \text{or} \quad R_{ej} - \frac{1}{2}R g_{ej} = \frac{8\pi G}{c^4}\underbrace{\left(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}\right)}_{T^{tot}_{\mu\nu}}$$

$$T^{(\Lambda)}_{\mu\nu} \equiv -\frac{c^4\Lambda}{8\pi G}g_{\mu\nu} \quad \text{vacuum}$$

Field eq.s can also be written as

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) \quad \text{by exploiting} \quad g^{\mu\nu}G_{\mu\nu} = \kappa g^{\mu\nu}T_{\mu\nu} \qquad R = -\kappa T$$

Diffeomorphysm invariance of the theory and energy conservation

variation under diffeomorphysm invariance see GR notes

$$\delta S_M = \cdots = -2\int V_b \nabla_a\left(\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{ab}}\right)\sqrt{-g}\, dx^\mu = 0 \Rightarrow \nabla_a\Big(\underbrace{\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{ab}}}_{T^{ab}\,!}\Big) = 0 \Rightarrow \boxed{\nabla_a T^{ab} = 0}$$

$$\nabla_\nu T^{\mu\nu} \overset{*}{=} \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\,T^{\mu\nu}\right) + \Gamma^\mu_{\nu\gamma}T^{\gamma\nu} = 0 \qquad \text{energy-momentum conservation}$$

eg. This has an "effect" on photons, examples: Rees-Sciama effect $\to$ CMB
Sachs-Wolfe " $\to$ CMB
cosmological redshift

Homogeneous and isotropic space-time

- impose symmetries : cosmological principle

- $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\sigma^2\right)$ FLRW metric $a \overset{!}{=} 1$ today

  polar coordinates: $(x^1, x^2, x^3) = (r, \vartheta, \varphi)$ $\quad d\sigma^2 \equiv d\vartheta^2 + \sin^2\vartheta \, d\varphi^2$

  Curvature : $K = -1, 0, 1$ open, flat, closed $\left(K \to \frac{K}{|K|}, \; r \to \sqrt{|K|}\, r, \; a \to \frac{a}{\sqrt{|K|}}\right)$

  Conformal time: $\tau \equiv \int_0^t a^{-1} dt \quad ds^2 = a^2(-d\tau^2 + d\Omega)$; for $K=0$: $ds^2 = a^2(\tau)\, \eta_{\mu\nu} dx^\mu dx^\nu$

- $(T_{\mu\nu}) = \mathrm{diag}(\rho c^2, P, P, P)$ Source : perfect fluid $T_{\mu\nu} = (\rho + \frac{P}{c^2}) u_\mu u_\nu + P g_{\mu\nu}$

  isotropy + homogeneity
  at rest in comoving frame

  $\left[T^\mu{}_\mu \equiv T = -\rho c^2 + 3P \;\curvearrowright\; R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)\right]$

Dynamical eq. of motion : Friedmann eq.s

solve Einstein eq.s for the metric and source we defined above

$$(1) \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right)$$

$$(2) \quad H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} \qquad H \equiv \frac{\dot{a}}{a}$$

continuity eq. $\nabla_\mu T^{\mu 0} = \dot{\rho} c^2 + 3H(\rho c^2 + P) = 0 \qquad \frac{d(c^2 \rho a^3)}{dt} + P\frac{da^3}{dt} = 0$

1° law of thermodynamic: $dE + pdV = 0$
$E = \rho c^2 V \quad V = V_0 a^3$

Multi-component universe

$T^{TOT}_{\mu\nu} = \sum_i (T_{\mu\nu})_i = \sum_i \left[\left(\rho_i + \frac{P_i}{c^2}\right) u_\mu u_\nu + P_i g_{\mu\nu}\right] = u_\mu u_\nu \left(\sum_i \rho_i + \sum_i \frac{P_i}{c^2}\right) + g_{\mu\nu} \sum_i P_i$ i-th component

$\Rightarrow G_{\mu\nu} = \frac{8\pi G}{c^4} T^{TOT}_{\mu\nu}$ Friedmann eq. remains valid!

(matter
radiation
dark energy
...)

Equation of state parameters

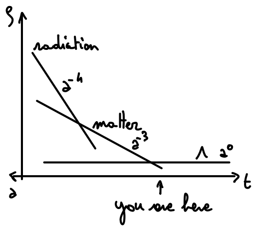
$p = w \rho c^2$ — from adiabatic condition $\Rightarrow$ $\rho = \rho_0 a^{-3(1+w)}$

$w = 0$ dust — $\rho_m = \rho_{m0} a^{-3}$ matter

$w = 1/3$ relativistic matter — $\rho_r = \rho_{r0} a^{-4}$ photons, neutrinos

$w = -1$ cosmo. const. — $\rho_\Lambda = \text{const}$ vacuum

$w \equiv P/\rho c^2$ dark energy a — ρ_{DE} can evolve a dynamical fluid



accelerated expansion, $\ddot{a} > 0$: $w < -1/3$ — No ghosts : $w \geq -1$

Λ has $a \propto \exp(Ht)$ — exponential expansion could last forever $\neq$ inflaton (inflation ended)

Hubble function (Friedmann 2)

$$H^2 = \frac{8\pi G}{3}\left(\rho_{m0} a^{-3} + \rho_{r0} a^{-4} + \rho_\Lambda\right) - K c^2 a^{-2} = H_0^2 \left(\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_\Lambda - \Omega_{K0} a^{-2}\right) = H_0^2 E(a)$$

$\rho_c \equiv \frac{3H^2}{8\pi G}$ critical density: density for which the universe is flat

$\Omega \equiv \rho/\rho_c$ density contrast

$\Omega_K \equiv -\frac{Kc^2}{a^2 H^2}$ density contrast of curvature

$\Rightarrow$ Friedmann (2): $1 = \Omega_{tot} + \Omega_K$

flat: $K = 0 \rightarrow \Omega_K = 0$, $\Omega_{tot} = 1$

Hubble stuff:

"0" = today

$H_0 = 100 \cdot h \text{ km s}^{-1} \text{ Mpc}^{-1}$ — Hubble parameter — $h = 0{,}72 \pm 0{,}08$ Hubble key project

$t_H \equiv H^{-1} = 9{,}78 \cdot 10^9 h^{-1}$ years — " time

$D_H \equiv c/H_0 = 2998 h^{-1}$ Mpc — " radius

$v \simeq H_0 r$ — " law

$\rho_c^{(0)} \equiv \frac{3H_0^2}{8\pi G} = 1{,}88 h^2 \times 10^{-29} \text{ g cm}^{-3}$ — average density

Earth $\rho_\oplus \sim 5 \text{ g/cm}^3$

Galaxy $\rho \sim 10^{-24} \text{ g/cm}^3$

Current observations: $H_0 \simeq 70$ $\Omega_{m0} \sim 0{,}3$ $\Omega_\Lambda \sim 0{,}7$ $\Omega_{K0} \sim 0$ flat

Cosmological distances

Distances depends on how they are performed (defined)

Convenient to know: $H \equiv \frac{\dot{a}}{a}$ $\quad \dot{a} = aH(a)$ $\quad \dot{a} = \frac{da}{dt}$ $\quad dt = \frac{da}{\dot{a}} = \frac{da}{aH(a)}$

- Proper distance: distance covered by a photon in dt

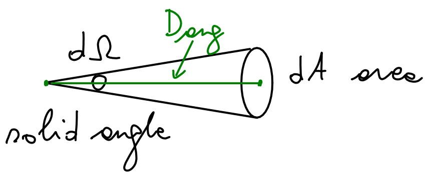
$$dD_p \equiv c\,dt = \frac{c\,da}{a\,H(a)} \qquad \boxed{D_p = \frac{c}{H_0}\int \frac{da}{aE(a)}}$$

- Comoving distance: distance between 2 hypersurfaces for t

$$ds^2 = -c^2dt^2 + a^2d\Omega^2 = 0 \qquad d\Omega = c\frac{dt}{a} = \frac{c\,da}{a^2H(a)} \qquad \boxed{D_c = \frac{c}{H_0}\int \frac{da}{a^2E(a)}}$$

- Angular diameter distance: distance obtained by measuring angles

$$dA \equiv d\Omega\, D_{ang}^2$$



$$D_{ang} = \left(\frac{dA}{d\Omega}\right)^{1/2}$$

solid angle → $\frac{d\Omega}{\cancel{4\pi}} = \frac{dA}{\cancel{4\pi}\,a^2D_c^2}$ ← area

solid angle of sphere ↗ ↖ area of sphere

$$\frac{dA}{d\Omega} = a^2D_c^2 \qquad \Rightarrow \qquad \boxed{D_{ang} = aD_c}$$

- Luminosity distance: distance obtained by measuring fluxes F

$$F \equiv \frac{L}{4\pi D_L^2} \quad \left[\frac{J}{s\,m^2}\right]$$

redshift: "stretching of λ" $\quad (a_1/a_2)$

spatial dilution: $\quad (a_1/a_2)^2 \quad \Rightarrow \propto \left(\frac{a_1}{a_2}\right)^4$ on F

delayed arrival time: $\quad (a_1/a_2)$

$$\boxed{D_L = \left(\frac{a_1}{a_2}\right)^2 D_{ang}(a_1, a_2)}$$

← D between a_1 and a_2

Redshift

$$z \equiv \frac{\lambda_0 - \lambda}{\lambda} = \frac{\nu}{\nu_0} - 1 \qquad ds^2 = 0 = -c^2dt^2 + a^2(t)R \qquad R = \text{const.} \qquad a(z) = (1+z)^{-1}$$

The "ingredients" of the universe: a closer look

"Matter" components (source in Einstein eq.s)

1) Relativistic particles : photons, neutrinos (if they have a small mass)
2) Non relativistic particles : baryons, dark-matter
3) Scalar fields : Inflaton, dark energy

In general: Phase space occupation in thermal equilibrium

$$f(\bar p) = \frac{1}{e^{(E-\mu)/kT} \pm 1}$$ (+) Fermi-Dirac (−) Bose-Einstein

$f(\bar p, \cancel{\bar x})$ homogeneity

μ = chemical potential , E energy of the cell

g_* internal degrees of freedom

$d^3x\, d^3p/(2\pi\hbar)^3$ # of phase space elements

$$\rho = g_* \int \frac{d^3p}{(2\pi\hbar)^3} E(p) f(p) \qquad P = g_* \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p v}{3} f(p)$$

$\frac{1}{3}$ from relation between momentum and pressure $p_K = \frac{1}{3}\frac{n_K K^2}{E}$

Consider equilibrium : $\mu = 0$ reaction rate $\Gamma \gg H$

non-degenerate particles ($T \gg \mu$)

(Riemann function $\zeta(m) \equiv \int_0^\infty (1-e^{-x}) x^m dx$)

	relativistic distributions: Bose-Einstein (spin 1 : γ)	Fermi-Dirac (spin ½ : e^+, e^-, n, p, ν)	matter $kT \ll mc^2$ non-relativistic Maxwell-Boltzmann both Bosons and Fermions
number density	$n_B = g_B \frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$	$n_F = \frac{3}{4}\frac{g_F}{g_B} n_B$	$n = g_*\left(\frac{kT}{2\pi\hbar}\right)^{3/2} e^{-kT/mc^2}$
energy density	$u_B = g_B \frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3}$!	$u_F = \frac{7}{8}\frac{g_F}{g_B} u_B$	$u = \frac{3}{2} n kT$
pressure	$(w = 1/3)$ $P_B = \frac{u_B}{3} = g_B \frac{\pi^2}{90}\frac{(kT)^4}{(\hbar c)^3}$!	$(w = 1/3)$ $P_F = \frac{u_F}{3} = \frac{7}{8}\frac{g_F}{g_B} P_B$!	$P = n kT = k \frac{\rho}{m} T \simeq 0$ $kT \ll mc^2$ ($\frac{u}{mc^2}$)
entropy	$s_B = g_B k \frac{2\pi^2}{45}\left(\frac{kT}{\hbar c}\right)^3$	$s_F = \frac{7}{8}\frac{g_F}{g_B} s_B$	continuous case $\varepsilon_\alpha = mc^2 + p^2/2m$ 1° order relat. limit

Thermal equilibrium despite universe is dynamic

$$\frac{d \ln N_s}{d \ln a} = \frac{\alpha}{H}\left(\frac{N_{sT}}{N_s} - 1\right)$$ Freeze-out equation / Boltzmann eq. it tells you how the reaction proceeds

1) Relativistic components

- **photons** (Bosons, relativistic, $g_* = 2$)

Bose-Einstein distribution $\left\{\begin{array}{l} w = \frac{1}{3},\quad \rho = \rho_0 a^{-3(1+w)} \Rightarrow u_\gamma \propto a^{-4} \\ u_\gamma \propto T^4 \end{array}\right\} \Rightarrow T \propto a^{-1} = 1+z$

COBE / Planck $T_{CMB} = 2{,}725$ K $\;\;$ $\rho_{\gamma,0}$: $4 = 1{,}279 \cdot 10^{-35}$ g cm^{-3} $\Rightarrow \rho_{\gamma,0} = 4{,}641 \cdot 10^{-34}$ g cm^{-3} (*) $\;\; \Omega_{\gamma,0} \equiv \frac{\rho_{\gamma,0}}{\rho_{c,0}} = 2{,}469 \cdot 10^{-5} h^{-2}$

- **neutrinos** (Fermions, relativistic, $g_* = 1$ spin)
  - $\mu = 0$, in standard model: ν_e electron-, ν_μ muion-, ν_τ tau-neutrinos, anti-neutrinos
  - 1 deg. of freedom per specie $\Rightarrow N_{eff} = 3$ effective number $\quad$ including anti-particles ×2

    presence of extra relativistic deg. of freedom of ν → affects BBN (Big Bang Nucleosynthesis, at ~ 1 MeV ?) D, He (, Li, Be)

    $\Rightarrow$ Bounds on $N_{eff} \simeq 3{,}04$

  - Reheating and neutrino background (freez-out)

    At ~0,1 MeV: $e^+ + e^- \to \gamma + \gamma \Rightarrow \nu$ decoupling from plasma (no more electro weak interaction via e^-)

    (reheating) $\Rightarrow$ CνB $T_\nu = T_\gamma (4/11)^{1/3}$ from entropy conservation before/after decoupling

    link it to T_γ because T_ν has not beein measured yet $\;\; u_\nu = N_{eff} \frac{7\pi^2}{8 \cdot 30} \frac{(kT_\nu)^4}{(\hbar c)^3} = N_{eff} \frac{7\pi^2}{8\cdot 30} \frac{(kT_\gamma)^4}{(\hbar c)^3} \left(\frac{4}{11}\right)^{\frac{4}{3}}$

    $\Omega_{relat,0} = \frac{\rho_{\gamma,0} + \rho_{\nu,0}}{\rho_{c,0}} = \Omega_{\gamma,0}(1 + 0{,}2271\, N_{eff}) = 8{,}051 \cdot 10^{-5}$

(*) **Density of photons in grams ?!**

What is the meaning of mass? Well ... "it depends on the framework"

- **Newton's law**: $\bar{F} = m\bar{a} \equiv m\dot{\bar{v}} \equiv \dot{\bar{P}} \qquad P \equiv m\bar{v}$ $\qquad$ m = inertial mass (!) "resistence" to the accelerations when a force is applied

  $dW = \bar{F}\, d\bar{l} = m\bar{a}\, d\bar{l}$ $\quad$ associated to work (energy)

- **Relativistic particle**: $E^2 = p^2c^2 + m_0^2 c^4$

  meaning of $m_0 c^2$: sets the lower energy bound for particles

  c^2 is just needed because of our wierd unit of measures: meters, seconds

  set natural units i.e. $c = 1 \Rightarrow E^2 = p^2 + m^2 \qquad [m] = [E] =$ energy

  Now the meaning of m_0 is clearer: m_0 is the lowest energy that a particle can have

  for photons the lowest energy is $m_0 = 0$

  mass ⇔ energy

2) Non relativistic components ($kT \ll mc^2$) matter

$P = k\frac{T}{m}\rho \simeq 0$ i.e. $w \simeq 0$ pressureless fluid (dust)

$\Rightarrow \rho$ do not depend on T alone $\Rightarrow$ need other observations to constrain it in contrast to γ

- Baryons :

baryons of interest

– constrain on Ω_b from Big Bang Nucleosynthesis (BBN) : $p, n \to H, D, He$ (, Li, Be)

before and during this process: $n \to p^+ + e^- + \bar{\nu}_e$ (β-decay)

# of n changes because of decay and because of nuclear fusion

if ρ_B is larger, reactions are faster $\Rightarrow$ less time for n to decay (i.e. more available n) $\Rightarrow \rho_{He}\uparrow \rho_D \downarrow$

$\models\!\!\Rightarrow \rho_B$ and ρ_D tunes the amount of He produced

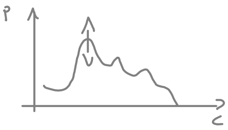
Abundance of D from spectroscopy of far quasars $\Rightarrow D/H = (3.0 \pm 0.4)\cdot 10^{-5}$ Burles+(2001)

$\Rightarrow \Omega_{B,0} = (0.020 \pm 0.002)h^{-2}$ (2σ)

– Constrain from CMB, galaxy clustering

if $\rho_B \uparrow \Rightarrow c_s$ of "CMB fluid" $\downarrow$ (i.e. smaller spring constant of harmonic oscillator)

$\Rightarrow$ amplitude of 1° acoustic peak $\uparrow$



$\Omega_{B,0} = \left(0.02267^{+0.00058}_{-0.0059}\right)h^{-2}$ WMAP5 (1σ)

$\models\!\!\Rightarrow$ Baryons are 4% ($h = 0.72$) NOT sufficient to explain cosmic structure formation galaxies, galaxy clusters, ... $\Rightarrow$ Need Dark Matter (DM)

- Dark Matter

Data say must be non-relativistic when it decoupled from photons : Cold (CDM)

$\Omega_{DM,0} = 0.1131 \pm 0.0034\, h^{-2}$ WMAP5 (structure formation)

1) Neutrinos? No because ν are relativistic (free streaming $\Rightarrow$ dump CDM)
2) Primordial black holes? maybe MACHOs not observed $\Rightarrow$ small mass
3) Fundamental particles? " eg. WIMPs, there is a zoo... pg 13
4) No DM at all? modified gravity ...

3) Scalar fields

– To explain current accelerated expansion of the universe

– Cosmological constant or dark-energy (DE) $\to$ scalar field / modified gravity

- Cosmological constant Λ, allowed by GR

- Needed to explain the accelerated expansion of the universe
- Allowed to add $\Lambda g_{\mu\nu}$ term because of metric compatibility condition $D^{\mu} g_{\mu\nu}=0$ (to respect energy cons.)
- We associate an energy-momentum tensor to Λ

$$R_{\mu\nu}-\frac{1}{2}R g_{\mu\nu}+\Lambda g_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu} \qquad R_{\mu\nu}-\frac{1}{2}R g_{\mu\nu}=\frac{8\pi G}{c^4}\left(T_{\mu\nu}+T^{\Lambda}_{\mu\nu}\right) \Rightarrow \boxed{T^{\Lambda}_{\mu\nu}\equiv-\frac{c^4\Lambda}{8\pi G}g_{\mu\nu}}$$

Most general, 2° order in $g_{\mu\nu}$ in 4D, respecting $D_{\mu}T^{\mu\nu}=0$, Lovelock theorem

Action: $S=\int(R-2\Lambda)\sqrt{-g}\,d\Omega+S_M$

- Friedmann eq.s with Λ

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\left(\rho+3\frac{P}{c^2}\right)+\frac{\Lambda}{3}$$

$\Lambda>0 \Rightarrow$ repulsive force at background level (hence the acceleration)

$$H^2=\frac{8\pi G}{3}\rho-\frac{Kc^2}{a^2}+\frac{\Lambda}{3}$$

- Λ as an effective fluid

not to have $\bar{u}$ dependency

$$T^{\Lambda}_{\mu\nu}=-\frac{c^4\Lambda}{8\pi G}g_{\mu\nu}\overset{!}{=}\underbrace{\left(\rho+\frac{P}{c^2}\right)}_{\overset{!}{=}0}u_{\mu}u_{\nu}+P g_{\mu\nu} \Rightarrow \boxed{P_{\Lambda}=-c^2\rho_{\Lambda}}$$

Eq. of state $\Rightarrow$ ($P=w\rho c^2$) $\boxed{w_{\Lambda}=-1}$

$$=P g_{\mu\nu}$$

$$=-\rho c^2 g_{\mu\nu} \Rightarrow \boxed{\rho_{\Lambda}=\frac{c^2\Lambda}{8\pi G}}$$

constant density!

- Interpretation of Λ

No dependency on $\bar{u}$ $\Rightarrow$ characteristic of empty space $\rightarrow$ vacuum energy $\qquad T^{\Lambda}_{\mu\nu}\equiv\frac{c^4\Lambda}{8\pi G}g_{\mu\nu}$

cosmological observations: $|\Lambda|<10^{-56}\,\text{cm}^{-2} \sim m_{\Lambda}<10^{-32}\,\text{eV}$

- The problems of Λ

(1) Fine tuning problem: why Λ so smaaal but not zero? Conflict with particle physics

(2) Coincidence problem: why $\Omega_{\Lambda,0}\sim\Omega_{m,0}$?

Problem (1)

From an elementary particle perspective: vacuum energy is $\sim 10^{121}$ times larger!

(a) from cosmological observations: $|\Lambda| < 10^{-56}\,cm^{-2}$

- Friedmann eq. $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$ => $\Lambda \sim H_0^2$ observations: $H_0 \simeq 70\,Km/s\,Mpc$

$$\rho_\Lambda \equiv \frac{c^2\Lambda}{8\pi G} = \frac{\cancel{c^2}\Lambda m_p^2}{8\pi\hbar \cancel{c}} = \frac{\Lambda m_p^2}{8\pi\hbar} \approx \underline{10^{-47}\,GeV^4} \approx 10^{-123}\,m_p^4$$

- As a reference, to see how small this is, consider a photon with the smallest possible λ i.e. the size of the universe: $\lambda \overset{!}{=} R_H = \frac{c}{H_0} \approx 3\,Gpc/h$ => $E_\gamma = \hbar\nu \sim 10^{-27}\,eV$

↳ fine tuning problem, why so "unnaturally small" but not zero?!

(b) From an elementary particle perspective: vacuum energy is $\sim 10^{121}$ times larger!

- vacuum energy $\langle\rho\rangle$ of empty space ($\hbar = 1 = c$)
- zero energy of a field of mass m with momentum k and frequency ω $E = \frac{\omega}{2} = \frac{1}{2}\sqrt{k^2+m^2}$
- sum zero-point energies of the field up to a cut-off k_{max} ($\gg m$)

$$\rho_{vac} = \int_0^{k_{max}} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} = \underbrace{\int_0^{k_{max}} \frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}}_{\text{on k shells (isotropy)}} \underbrace{\simeq}_{k\gg m} \int_0^{k_{max}} \frac{k^3 dk}{(2\pi)^2} = \frac{k_{max}^4}{16\pi^2} \simeq \underline{10^{74}\,GeV^4}$$

GR valid up to Planck scale => take k_{max} to l_p, m_p

Even using other energy scales the problem remains: QCD scale $k_{max} \approx 0{,}1\,GeV$ => $\rho_{vac} \approx 10^{-3}\,GeV^4$

- Possible solution

· $\underline{\Lambda = 0}$ => $\exists$ of some symmetry (supersymmetric theories)
even if symmetry is broken still possible to have $\Lambda = 0$

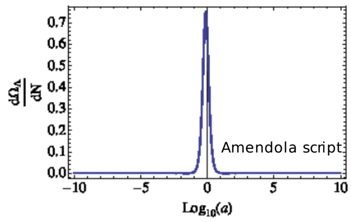
· Introduce Dark-Energy (DE) to explain acceleration
↳ not a constant, it can evolve with time: $w(a)$
↳ behaviour of DE must be very close to Λ: so far observations are compatible with Λ!

Problem (2) Coincidence problem

Λ and matter are not linked..., why $\Omega_{\Lambda,0} \sim \Omega_{m,0}$?!

Coincidence time z_c: $\Omega_\Lambda \overset{!}{=} \Omega_m = \Omega_{m,0}(1+z)^3$ $\qquad z_c = \left(\frac{\Omega_\Lambda}{\Omega_{m,0}}\right)^{1/3} - 1 = \left(\frac{\Omega_\Lambda}{\Omega_\Lambda - 1}\right)^{1/3} - 1 \simeq 0{,}3$ i.e. "today"...

flat cosmo

$\frac{d\Omega_\Lambda}{d\ln a} =$ 

i.e. Λ is visible today

DE is affected by problem (2) as well! because DE behaveiour must be very close to the one of Λ

– "solving" this problem for DE, 4 ways:

a) Tracker model: attractor solutions, ρ_{DE} responds to ρ_m

$\rho_{DM} \to \rho_m$ regardless it own initial conditions

better... but still... accelerated expansion is happening today, not justified here

b) Scaling attractors: postulate 2 components $\to$ matter that cluster (structure formation)

$\to$ DE does not cluster (very large c_s)

tune their eq. of state to have $\Omega_m \sim \Omega_{DE}$ at all times, so no coincidence

$w_{DE}(a) \to -1$ at the right time

difficult to explain all observations at once

— o —

c) Antropic explenation:

$\rho_{DE} \sim \rho_m$ is the largest ρ_{DE} value with sufficient structure formation to allow life

large ρ_{DE} are more likely but life could not form

$\Rightarrow$ our universe is the one likely to host us

d) Backreaction:

structure formation triggers the accelerated expansion through some cumulative non-linear effect

there is no real acceleration, we just used the wrong model...

eg. Lamaitre-Tolman-Bondi model strongly inhomogeneous model

Part II

Quintessence Dark energy

Alternatives to the cosmological constant

- $\Lambda \overset{!}{=} 0$: how to explain accelerated expansion $\ddot{a} > 0$?
  GR: $G_{\mu\nu} = \alpha T_{\mu\nu}$
- Two streams of thought: 1) Dark energy: a matter field with negative pressure
  or
  2) Modified gravity (e.g. extensions of GR)
- Not a fundamental distinction (like for Λ, move extension of geometry sector into the source)
  Total action $S = \int \left(R + \alpha \mathcal{L}_\phi\right)\sqrt{-g}\, d\Omega + S_M$ R = Ricci scalar S_M = standard matter term
  With GR, no way to distinguish the 2 approaches,
  Need a fundamental theory to distinguish the two... quantum field theory
- careful... need not to screw up local measures of gravity

Dark energy: Quintessence model

- The concept: scalar field ϕ with potential $V(\phi)$ $\Rightarrow$ $w(a)$ evolves (not const. like for Λ)
  No need for ρ_{DE} to be small
  Attempts to construct it based on particle physics models
  $\Rightarrow V(\phi)$ flat enough: slow-roll at late times, $\rho_{DE} \sim 10^{-123}\, m_{Pl}^4$, $m_\phi \lesssim 10^{-33}\,eV$ ||
- Self interacting scalar field ϕ (spin 0 particle like Higgs field)
  $T^\phi_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\,\mathcal{L}_\phi)}{\delta g^{\mu\nu}}$
  Perfect fluid with energy density ρ_ϕ, pressure P_ϕ and eq. of state $w_\phi = \frac{P_\phi}{\rho_\phi}$
  Lagrangian density
  $$\mathcal{L}_\phi = -\frac{1}{2}c^2 \partial_\alpha\phi\, \partial^\alpha\phi - V(\phi)$$
  energy-momentum
  $$T^\phi_{\mu\nu} = c^2 \partial_\mu\phi\, \partial_\nu\phi + \mathcal{L}_\phi g_{\mu\nu} = c^2\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}c^2\partial_\alpha\phi\,\partial^\alpha\phi\, g_{\mu\nu} - V(\phi)\, g_{\mu\nu}$$
- Isotropy: only diagonal components of T are $\neq 0$
  Homogeneity: $\partial_i\phi = 0$ * i.e. $\phi(t)$ only as well as isotropy (vector $\vec{\nabla}\phi$ = privilege direction)

$$\begin{cases} T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \cancel{\frac{1}{2}c^2(\vec{\nabla}\phi)^2} \overset{!}{=} \rho_\phi c^2 & \text{energy density} \\ T_{ii} = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \cancel{\frac{1}{6}c^2(\vec{\nabla}\phi)^2} \overset{!}{=} P_\phi & \text{pressure} \\ P = w\rho c^2 : \quad w = \frac{P}{\rho c^2} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} & \text{effective eq. of state} \end{cases}$$

ideal fluid $T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}$ "$\dot{\ }$" $= \frac{d}{c\,d\tau}$ conformal time

$\Rightarrow$ w not constant in general! see dynamics...

How does ϕ evolve? Eq. of motion (flat geometry)

∘ ϕ satisfies continuity eq. ($\nabla_\mu T^{\mu 0}=0$): $\dot{\varrho}_\phi c^2 + 3H(\varrho_\phi c^2 + P_\phi) = 0$ $\qquad \frac{1}{\cancel{2}}\cancel{2}\,\dot\phi\,\ddot\phi + \frac{\delta V}{\delta\phi}\dot\phi + 3H\left(\frac{1}{2}\dot\phi^2 + \cancel{V(\phi)} + \frac{1}{2}\dot\phi^2 - \cancel{V(\phi)}\right) = 0$

$$\Rightarrow \boxed{\underbrace{\ddot\phi}_{\text{acceleration}} + \underbrace{3H\dot\phi}_{\text{friction}} + \underbrace{\frac{\delta V}{\delta \phi}}_{\text{force (source)}} = 0} \quad (1)$$

Klein-Gordon eq. in an expanding universe (eq. of motion)

- scalar field mass $m^2 = \frac{d^2V}{d\phi^2}$ in general is a func. of time
- Dynamic depends on V, choose the model of your like
  ↳ get inspiration from supersymmetry, extra dimensions, ...
- As ϕ evolves with time, so does w!

to convince yourself use: $(\Box + m^2)\phi(t,x) = 0$ with $m^2 = \frac{\delta V}{\delta\phi}$ KG eq. $\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$ $\quad g_{\mu\nu} = a^2\eta_{\mu\nu}$, $\quad$ "$'$" $= \frac{d}{c\,d\tau}$ $\quad \tau$ = conformal time

∘ Need evolution of H for universe with ϕ and 'matter'

Friedmann eq.s:

(2) $H^2 = \frac{8\pi G}{3}\varrho - \underbrace{\frac{Kc^2}{a^2}}_{\overset{!}{=}0} + \underbrace{\cancel{\frac{\Lambda}{3}}}_{\overset{!}{=}0} \quad \rightarrow \quad \boxed{H^2 = \frac{8\pi G}{3c^2}\left(\varrho_M c^2 + \frac{1}{2}\dot\phi^2 + V(\phi)\right)}$ Friedmann (2)

(3) $3H^2 + 2\dot H = -\frac{8\pi G}{c^2}P - \underbrace{\cancel{\frac{Kc^2}{a^2}}}_{\overset{!}{=}0} \quad \rightarrow \quad \boxed{\dot H = -\frac{4\pi G}{c^2}\left(\varrho_M c^2 + P_M + \dot\phi^2\right)}$ Friedmann (1)

useful

$\models\!\!\Rightarrow$ Eq. of motion: solve for $\begin{cases} \ddot\phi + 3H\dot\phi + \frac{dV}{d\phi} = 0 \\ H^2 = \frac{8\pi G}{3c^2}\left(\varrho_M c^2 + \frac{1}{2}\dot\phi^2 + V(\phi)\right) \end{cases}$

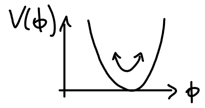
Same model, different physical scenarios

Dynamic is given by KG eq. $\ddot{\phi} + 3H\dot{\phi} + \frac{\delta V}{\delta \phi} = 0$

Still very generic model, ϕ could be anything: a inflaton, matter, photons, Λ, DE

A) Case of steep potential $\frac{\dot{\phi}}{2} \gg V(\phi) \Rightarrow w_\phi \simeq 1$, $\rho_\phi \propto a^{-6}$, i.e. it "dies" faster than the rest

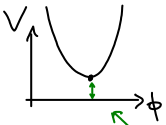
B) If ϕ oscillates harmonically around a zero-energy potential



$\Rightarrow$ kinetic energy = potential energy $\left(\frac{1}{2}\dot{\phi}^2 = V\right)$ when averaging over several cycles $\Rightarrow P = 0$ i.e. $w=0$

$\Rightarrow$ Dust!

C) Static field ($\dot{\phi} = 0$) at non-zero energy minimum of V

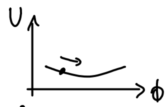


$\neq 0$ or $\rho c^2 = V = 0$ = nothing

$\Rightarrow \boxed{w = -1} = \frac{P}{\rho c^2} = \frac{-V}{V} \Rightarrow$ field represents the cosmological const. Λ!

state in which the field has non zero vacuum energy

D) Slow roll limit $\dot{\phi}^2 \ll V(\phi)$:



- dynamical field ($\dot{\phi} \neq 0$), slow evolution, no oscillations!
- slow roll limit is ensured by conditions $\epsilon_s \equiv \frac{1}{2\kappa^2}\left(\frac{V_\phi}{V}\right)^2 \ll 1$ $\quad |\eta_s| \equiv \left|\frac{V_{\phi\phi}}{\kappa^2 V}\right| \ll 1$ slow roll parameters
- $w_\phi = \frac{P}{\rho c^2} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \simeq \frac{-\cancel{V}\left(1 - \frac{1}{2}\frac{\dot{\phi}^2}{V}\right)}{\cancel{V}\left(1 + \frac{1}{2}\frac{\dot{\phi}^2}{V}\right)} \simeq -\left(1 - \frac{1}{2}\frac{\dot{\phi}^2}{V}\right)^2 \simeq -1 + \frac{\dot{\phi}^2}{V}$ with $\frac{\dot{\phi}}{V} \ll 1$ (slow roll)
- in agreement with observations: $w \sim -1$ and nearly const.

  but as field evolves $\dot{\phi} \uparrow \Rightarrow w_\phi \uparrow \Rightarrow$ necessary dynamic! Key difference from Λ Dynamical DE!
- Constrains on w_ϕ : for $V > 0 \Rightarrow w_\phi > -1$ no ghosts (theories are unstable)

  $w_\phi < -1/3$ to have acceleration $\ddot{a} > 0 \Rightarrow \dot{\phi}^2 < V$ $\quad w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} < -\frac{1}{3}$ $\quad \dot{\phi}^2 - 2V < -\frac{1}{3}(\dot{\phi}^2 + 2V)$

  tune $V(\phi)$ to have $\rho_\phi \sim \rho_M$ at late times (with or without tracking)

Therefore: we need case (D)!

DE in the Friedmann equations

Condition on w to have acceleration

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \overset{\overset{!}{=}0}{\cancel{\frac{\Lambda}{3}}} \overset{!}{>} 0 \quad \Rightarrow \quad \left(\rho + \frac{3P}{c^2}\right) < 0 \quad \Rightarrow \quad P > -\frac{1}{3}\overset{w_{max}}{\rho c^2} \quad \Rightarrow \quad w < -\frac{1}{3} \quad \text{and} \quad \dot{\phi}^2 < V(\phi)$$

DE $\boxed{-1 \leq w \leq -\frac{1}{3}}$ for $w < -1$ most fields are unstable : ghosts

Density ρ_ϕ evolution

- Adiabatic condition $\dot{\rho}c^2 + 3H(\rho c^2 + \overset{\overset{P = w\rho c^2}{\downarrow}}{P}) = \dot{\rho}\cancel{c^2} + 3H\rho\cancel{c^2}(1 + w(a)) = 0$

$$\Rightarrow \frac{d\rho}{\rho} = -3\cancel{\rho}(1 + w(a))\frac{da}{a} \qquad \ln\frac{\rho}{\rho_0} = -3\int_1^a (1 + w(a))\frac{da}{a} \qquad \boxed{\rho = \rho_0 e^{-3\int_1^a (1+w(a))\frac{da}{a}}}$$

Chevelier-Polarski-Linder parametrization

There are many DE models and $w(a)$ evolves slowly over a large range of cosmic time

$\Rightarrow$ Assume linear model for w_{DE}: $\boxed{w_{DE} = w_0 + w_a \cdot (1 - a)} = w_0 + w_a \frac{z}{1+z}$

$w_0 = w_{DE}(1)$ today

i.e. as Taylor expansion of $w(a)$ at 1° order

$$\Rightarrow d\rho = -3(1 + \overset{\downarrow}{w})\frac{da}{a} = -3(1 + w_0 + w_a(1-a))\frac{da}{a} \quad \text{integrate}$$

$$\int \frac{d\rho}{\rho} = -3(1 + w_0 + w_a)\int \frac{da}{a} + 3\int w_a\, da \qquad \ln\rho'\Big|_\rho^{\rho_0} = -3(1 + w_0 + w_a)\ln a'\Big|_a^1 + 3w_a \cdot a'\Big|_a^1$$

$$\ln\rho^0 - \ln\rho = +3(1 + w_0 + w_a)\ln a + 3w_a \cdot (1 - a) \qquad \Rightarrow \qquad \boxed{\rho_{DE} = \rho_0\, e^{-3(1+w_0+w_a)a} \cdot e^{3w_a(a-1)}}$$

Friedmann eq.

$$\boxed{H^2(a) = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\phi e^{-3\int_1^a (1+w(a))\frac{\dot{a}}{a}}\right]}$$

$$H^2(a) = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\phi e^{-3(1+w_0+w_a)a} \cdot e^{3w_a(a-1)}\right]$$

Appendix

Friedmann (1):

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) \;:\quad \ddot a = d_t \dot a = d_t(aH) = \dot a H + a\dot H = aH^2 + a\dot H$$

$$\frac{4\pi G}{3}\rho = \frac{H^2}{2} - \frac{Kc^2}{2a^2} \quad \text{(Friedmann (2))}$$

$$H^2 + \dot H = -\frac{H^2}{2} - \frac{4\pi G}{c^2}P - \frac{Kc^2}{2a^2} \quad (\times 2) \qquad 3H^2 + 2\dot H = -\frac{8\pi G}{c^2}P - \frac{Kc^2}{a^2} \quad ✓$$

Friedmann (1) with a scalar field:

$$3H^2 + 2\dot H = -\frac{8\pi G}{c^2}P - \frac{\overset{!}{=}0}{\cancel{K}}{a^2}\;:\quad 3H^2 + 2\dot H = \cancel{3}\frac{8\pi G}{\cancel{3}c^2}\left(\rho_M c^2 + \frac{1}{2}\dot\phi^2 + \cancel{V(\phi)}\right) + \cancel{2}\dot H \overset{!}{=} -\frac{8\pi G}{c^2}P = -\frac{8\pi G}{c^2}\left(P_M + \frac{1}{2}\dot\phi^2 - \cancel{V(\phi)}\right)$$

$$\dot H = -\frac{4\pi G}{c^2}\left(\rho_M c^2 + P_M + \dot\phi^2\right) \quad ✓$$

Dark energy as a self interacting fluid

signature $(-1,1,1,1)$

$$\mathcal{L} = -\frac{1}{2}c^2\partial_\alpha\phi\,\partial^\alpha\phi - U(\phi) \qquad T_{\mu\nu} = c^2\partial_\mu\phi\,\partial_\nu\phi + \mathcal{L}\,g_{\mu\nu} = c^2\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}c^2\partial_\alpha\phi\,\partial^\alpha\phi\,g_{\mu\nu} - U(\phi)\,g_{\mu\nu}$$

$$T_{00} = c^2(\partial_0\phi)^2 + \frac{1}{2}c^2\left[-(\partial_0\phi)^2 + (\vec\nabla\phi)^2\right](-1) - U(\phi)(-1) = \dot\phi^2 - \frac{1}{2}\dot\phi^2 + \frac{1}{2}c^2(\vec\nabla\phi)^2 + U(\phi) = \frac{1}{2}\dot\phi^2 + U(\phi) + \frac{1}{2}c^2(\vec\nabla\phi)^2$$

$$T_{ii} = c^2(\partial_i\phi)^2 - \frac{1}{2}c^2\left[-(\partial_0\phi)^2 + (\vec\nabla\phi)^2\right]\cdot 1 - U(\phi)\cdot 1 = c^2(\partial_i\phi)^2 + \frac{1}{2}\dot\phi^2 - \frac{1}{2}c^2(\vec\nabla\phi)^2 - U(\phi) = \frac{1}{2}\dot\phi^2 - U(\phi) - c^2\frac{1}{6}(\vec\nabla\phi)^2$$

$$* \; (\vec\nabla\phi)^2 = \vec\nabla\phi\cdot\vec\nabla\phi = (\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 = 3(\partial_i\phi)^2$$

here we defined "$\dot{\ }$" $\equiv \frac{d}{c\,d\tau}$ dot "$\cdot$" derivative with respect to conformal time $\qquad \frac{d}{dx^0} = \frac{d}{c\,dt} \equiv \frac{d}{a\,c\,d\tau}$

Recall: $ds^2 = a^2(t)\left[-c^2d\tau^2 + d\chi^2 + f_K^2(\chi)\left(d\theta^2 + \sin\theta\,d\phi^2\right)\right] \quad d\tau \equiv \frac{dt}{a(t)} \quad \tau = $ conformal time

Investigating the properties of dark energy models

- Schenario

Flat geometry + Scalar field + background fluid with $w_M = 0, 1/3$ i.e. matter or radiation

eq. of motion given by thes two coupled eq.s

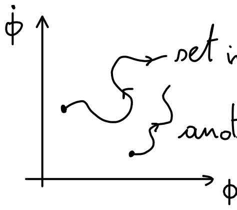
$$\begin{cases} \ddot\phi + 3H\dot\phi + V_{,\phi} = 0 \\ H^2 = \frac{8\pi G}{3}(\rho_M + \rho_\phi) \end{cases}$$

- ρ_M known behavejour: $\rho_M = \rho_{M,0}\, a^{-m}$ $m = 3, 4$ $\rho_{M,0}$ fixed parameter
- V is given your DE model $\Rightarrow$ known
- The only deg.s of freedom are init. conditions of ϕ, $\dot\phi$
- in general, no analytical solutions, only numerical

- Phase-Space approach

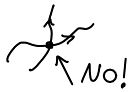
We will not find solutions in terms of $\phi(t)$, $\dot\phi(t)$ as usual (dynamical system)

instead we will describe the system in terms of curves in phase-space $\dot\phi(\phi)$: a phase space approach



set init. conditions $(\phi_i, \dot\phi_i) \Rightarrow$ system evolves somehow

another init. condition $\Rightarrow$ different path in the phase space



we do not know when something happens but we know what happen

describe the system without solving equations

describe not individual solutions but the entire phase-space: a "qualitative" behavejour of all solutions

in our case, convenient to define other dynamical variables $(\phi, \dot\phi) \to (X_1, X_2)$

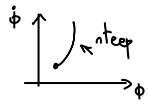
- Trivial examples of the power of this approach

- Trajectories can not cross: crossings would imply that the system could evolve in different ways given one init. cond.

  see theorems of uniqueness for differential eq.s (no singularities, regularity)

$\Rightarrow$ if you have for example a solution like  you know that solutions in region (1) are confined there

- e.g. $w = \dfrac{\frac{1}{2}\dot\phi^2 - V}{\frac{1}{2}\dot\phi^2 + V} \simeq -1$ i.e. $\dot\phi^2 \ll V$

  if $\dot\phi(\phi)$ grows a lot, you get no Λ like behavejour 

  $\Rightarrow$ you can exclude solutions with steep $\dot\phi(\phi)$ trajectories

- moreover: $\exists$ radiation domination?

  $\exists$ matter domination?

  is there the onset of accelerated expansion?

  Can we transition from one domination to another?

All of this and more can be understood... :)

How we do proceed in practice

(1) Make your life simpler: Define more convenient phase-space variables: $(\phi,\dot\phi)\to(x_1,x_2)$

• Friedmann (2): $H^2=\frac{8\pi G}{3c^2}\left(\rho_M c^2+\overbrace{\frac{1}{2}\dot\phi^2+V(\phi)}^{\rho_\phi c^2}\right)$ $c\overset{!}{=}1,\ \kappa^2\equiv 8\pi G$

recall: $\Omega_{tot}=1$ (flat)

$$\boxed{\Omega_M}\equiv\frac{\kappa^2\rho_M}{3H^2}=1-\frac{\kappa^2\dot\phi^2}{6H^2}-\frac{\kappa^2V(\phi)}{3H^2}\equiv 1-\underbrace{x_1^2-x_2^2}\overset{\downarrow}{=}1-\Omega_\phi \qquad x_1\equiv\frac{\kappa\dot\phi}{\sqrt6 H}\quad x_2\equiv\frac{\kappa\sqrt V}{\sqrt3 H}\ \text{dimentionless}$$

$$\boxed{\Omega_\phi}\equiv\frac{\kappa^2\rho_\phi}{3H^2}=\frac{\kappa^2}{3H^2}\left(\frac12\dot\phi^2+V\right)=x_1^2+x_2^2 \qquad x_1^2=\Omega_{\dot\phi}\ \ x_2^2=\Omega_V$$

kinetic potential

$$\boxed{w_\phi}=\frac{P_\phi}{\rho_\phi}=\frac{\frac12\dot\phi^2-V}{\frac12\dot\phi^2+V}=\frac{x_1^2-x_2^2}{x_1^2+x_2^2}$$

$$\boxed{w_{eff}}\equiv\frac{P_{tot}}{\rho_{tot}}=\frac{\sum_i P_i}{\rho_{tot}}=\sum_i\frac{w_i\rho_i}{\rho_{tot}}=\sum_i w_i\Omega_i=w_M\Omega_M+w_\phi\Omega_\phi=w_M(1-x_1^2-x_2^2)+\frac{x_1^2-x_2^2}{\cancel{x_1^2+x_2^2}}(\cancel{x_1^2+x_2^2})$$

ρ_{tot} ← = ρ_{cr} because of flat geom.

$$=w_M+x_1^2(1-w_M)-x_2^2(1+w_M)$$

$P=w\rho$

$$\boxed{\dot H}=-\frac{\kappa^2}{2}\left(\dot\phi^2+\rho_M+P_M\right)=-\frac{\kappa^2}{2}\left[\dot\phi^2+\rho_M(1+w_i)\right] \quad \div H^2 \qquad \text{Friedmann (1)}$$

$$\frac{\dot H}{H^2}=-\frac{\kappa^2\dot\phi^2}{2H^2}-\frac{\kappa^2\rho_M}{2H^2}(1+w_M)=-3x_1^2-\frac32\Omega_M(1+w_M)=-3x_1^2-\frac32(1-x_1^2-x_2^2)(1+w_M)$$

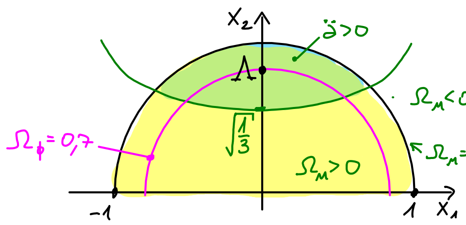
$$=-\frac32\left(\cancel{2}x_1^2+1+w_M-\cancel{x_1^2}-w_Mx_1^2-x_2^2-w_Mx_2^2\right)=-\frac32\left(1+w_M+x_1^2(1-w_M)-x_1^2(1+w_M)\right)$$

$$=-\frac32(1+w_{eff}) \qquad \text{dynamic is given by } w_{eff}$$

Solution in terms of (x_1,x_2) give you everything: $w_\phi,\Omega_\phi,\Omega_M,w_{eff}$!

Flat geometry

• Basic structure of the phase-space:



- $x_1\equiv\frac{\kappa\dot\phi}{\sqrt6 H}=0 \to \dot\phi=0$ all is in potential
- $x_2\equiv\frac{\kappa\sqrt V}{\sqrt3 H}=0\to V=0$ all is in kinetic term
- $V\geq0 \Rightarrow x_2\geq0$
- $\Omega_M=1-x_1^2-x_2^2\geq0 \Rightarrow x_1^2+x_2^2\leq1$

physical solutions can not escape this region: $\dot\rho_M+3H\rho_M=0$, at border $\rho_M=0\to\dot\rho_M=0$, ρ_M get stuck there

↳ not because of a theorem, if your theory "escapes" ⇒ indication that it is wrong

- limit for acceleration $\ddot a>0$ $(w_M=0)$: $w_{eff}=\overset{=0}{w_M}+x_1^2(1-\overset{=0}{w_M})-x_2^2(1+\overset{=0}{w_M})=x_1^2-x_2^2\leq-1/3$ $x_2^2\geq\frac13+x_1^2$
- $\Omega_{\phi,0}=x_1^2+x_2^2\simeq0{,}7$ $x_2^2=0{,}7-x_1^2$
- $w_\phi=\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\overset{!}{=}-1$ $x_1=0$ cosmological const. Λ
- realistic case: $0\leq w_M<1$ ($w_M=0$ dust, $w_M=1/3$ relativistic)

(2) Derivatives with respect of e-foldings number

$$N \equiv \log a \qquad H \equiv \frac{\dot a}{a} = \frac{1}{a}\frac{da}{d\tau} = \frac{d\ln a}{d\tau} \equiv \frac{dN}{d\tau} \qquad \overset{\text{"}\cdot\text{"}}{\frac{d}{d\tau}} = H \overset{\text{"}\prime\text{"}}{\frac{d}{dN}}$$

(3) Use (x_1, x_2) to move from one 2° order diff. eq. to two 1° order diff. eq.s $\quad (\kappa \equiv 8\pi G)$

- $\ddot\phi + 3H\dot\phi + V_{,\phi} = 0 \qquad H^2\phi'' + 3H^2\phi' + V_{,\phi} = 0 \qquad \phi'' + 3\phi' + \frac{V_{,\phi}}{H^2} = 0 \qquad \phi'' = -3\phi' - \frac{V_{,\phi}}{H^2} = -3\frac{\sqrt6}{\kappa}x_1 - \frac{3V_{,\phi}}{\kappa^2 V}x_2^2$

- $\dot x_1 = \frac{\kappa}{\sqrt6}\left(\frac{\ddot\phi H - \dot\phi\dot H}{H^2}\right) = \frac{\kappa}{\sqrt6}\left[\frac{\ddot\phi}{H} + \dot\phi\left(3x_1^2 + \frac{3}{2}(1-x_1^2-x_2^2)(1+w_M)\right)\right]$

$$\cancel{H}\boxed{x_1'} = \frac{\kappa}{\sqrt6}\left[\phi'' + \phi'\left(3x_1^2 + \frac{3}{2}(1-x_1^2-x_2^2)(1+w_M)\right)\right]\cancel{H} \qquad x_1 \equiv \frac{\kappa\dot\phi}{\sqrt6 H} = \frac{\kappa\phi'}{\sqrt6}$$

$$= -3x_1 - \frac{3}{\sqrt6}\frac{V_{,\phi}}{\kappa V}x_2^2 + \frac{3}{2}x_1\left(2x_1^2 + (1-x_1^2-x_2^2)(1+w_M)\right)$$

$$= -3x_1 + \lambda x_2^2 + \frac{3}{2}x_1\left(\cancel{2}x_1^2 + 1 + w_M - \cancel{x_1^2} - x_1^2 w_M - x_2^2 - x_2^2 w_M\right)$$

$$= \quad '' \quad \frac{3}{2}x_1\left(1 + w_M + x_1^2(1-w_M) - x_2^2(1+w_M)\right)$$

$$= \quad '' \quad \frac{3}{2}x_1\left(x_1^2(1-w_M) - (1+x_2^2)(1+w_M)\right)$$

$$\boxed{x_2'} = -\frac{\sqrt6}{2}\lambda x_1 x_2 + \frac{3}{2}x_2\left[(1-w_M)x_1^2 + (1+w_M)(1-x_2^2)\right]$$ same derivation as for x_1'

$$\boxed{\lambda} \equiv -\frac{V_{,\phi}}{\kappa V} = -\frac{1}{\kappa}\frac{d\log V}{d\phi}$$ logaritmic slope of the potential

- In general $\lambda(\phi)$ can evolve $\Rightarrow$ need a third equation to close the system

$$\boxed{\lambda'} = -\frac{V_{,\phi\phi}\phi'\kappa V - \kappa V_{,\phi}^2\phi'}{\kappa^2V^2} = -\underbrace{\frac{V_{,\phi}^2}{\kappa^2V^2}}_{=\lambda^2}\left(\underbrace{\frac{V V_{,\phi\phi}}{V_{,\phi}^2}}_{\equiv\Gamma} - 1\right)\kappa\underbrace{\phi'}_{\phi' = \frac{\sqrt6}{\kappa}x_1} = -\sqrt6\lambda^2(\Gamma-1)x_1$$ recall: $V(\phi(N))$

$$\boxed{\Gamma} \equiv \frac{VV_{,\phi\phi}}{V_{,\phi}^2}$$ characterizes the curvature of the potential

- Summary of key equations

$$\begin{cases} \dfrac{dx_1}{dN} = -3x_1 + \dfrac{\sqrt6}{2}\lambda x_2^2 + \dfrac{3}{2}x_1\left[(1-w_M)x_1^2 + (1+w_M)(1-x_2^2)\right]\,, \\ \dfrac{dx_2}{dN} = -\dfrac{\sqrt6}{2}\lambda x_1 x_2 + \dfrac{3}{2}x_2\left[(1-w_M)x_1^2 + (1+w_M)(1-x_2^2)\right]\,, \\ \dfrac{d\lambda}{dN} = -\sqrt6\lambda^2(\Gamma-1)x_1 \qquad \lambda \equiv -\dfrac{V_{,\phi}}{\kappa V} \qquad \Gamma \equiv \dfrac{VV_{,\phi\phi}}{V_{,\phi}^2} \end{cases}$$

solve numerically and/or study phase space

If λ = const. $= \lambda$ is a fixed parameter and these two equations (x_1', x_2') are sufficient

λ and Γ come from the potential you have choose

(4) Further characterize the phase-space: critical points

- Critical points

The solutions of $x_1' = 0 = x_2'$ are called critical solutions $(\hat{x}_1, \hat{x}_2)$
they are one of the possible solutions (trajectories in phase-space)
these "trajectories" just happen to be points in the phase-space
Positions $(\hat{x}_1, \hat{x}_2)$ are called critical points, they can be more than one
If you start from $(\hat{x}_1, \hat{x}_2)$ $\Rightarrow$ the system will remain there forever (clearly $x_1'=0$, $x_2'=0$)
There are different types of critical points, classified in base of their stability properties

- Stability of the critical points

What happens in the surroundings of critical points? To see, use a perturbative approach at 1° order

a) Add small perturbation, δx_i, about a given critical point $(\hat{x}_1, \hat{x}_2)$

$$\begin{cases} x_1 = \hat{x}_1 + \delta x_1 \\ x_2 = \hat{x}_2 + \delta x_2 \end{cases} \quad |\hat{x}_i| \ll |\delta x_i| \; ; \quad \text{plug in} \quad \begin{aligned} x_1' &= f_1(x_1, x_2) \simeq \cancel{f_1(\hat{x}_1, \hat{x}_2)}^{=0} + \delta f_1(\delta x_1, \delta x_2) = 0 \\ x_2' &= f_2(x_1, x_2) \simeq \cancel{f_2(\hat{x}_1, \hat{x}_2)} + \delta f_2(\delta x_1, \delta x_2) = 0 \end{aligned}$$

keep 1° order terms only
i.e. linearize eq.s

$$\Rightarrow \frac{d}{dN}\begin{pmatrix}\delta x_1' \\ \delta x_2'\end{pmatrix} = A \begin{pmatrix}\delta x_1 \\ \delta x_2\end{pmatrix} \quad \text{1° order linear diff. eq with} \quad A(x_1, x_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(\mathbb{R}, 2)$$

b) Find eigenvalues of A (i.e. rotate axis)

$$\begin{aligned} \delta y_1 &= a\,\delta x_1 + b\,\delta x_2 \\ \delta y_2 &= c\,\delta x_1 + d\,\delta x_2 \end{aligned} \Rightarrow \frac{d}{dN}\begin{pmatrix}\delta y_1 \\ \delta y_2\end{pmatrix} = \begin{pmatrix}\mu_1 & 0 \\ 0 & \mu_2\end{pmatrix}\begin{pmatrix}\delta y_1 \\ \delta y_2\end{pmatrix} \quad \text{solve} \quad \overset{i=1,2}{\frac{\delta y_i'}{\delta y_i}} = \mu_i \quad \frac{dy_i}{y_i} = \mu_i dN \quad \delta y_i = \alpha_i e^{\mu_i N} \quad i=1,2$$

$$\mu_{1,2} = \frac{1}{2}\left(-Tr(A) \pm \sqrt{Tr(A)^2 - 4\det(A)}\right) = \frac{1}{2}\left[-(a_{11}+a_{22}) \pm \sqrt{D}\right] \qquad D \equiv (a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$$

c) Solution of linearized eq. $x_i = \hat{x}_i + \alpha_{i1} e^{\mu_1 N} + \alpha_{i2} e^{\mu_2 N}$ (Back to original coordinates)

eigenvalues determine behaviour of solution around fixed points

$\mu_i = r + is \Rightarrow e^{\mu_i N} = e^{rN} \cdot e^{isN} \qquad r, s \in \mathbb{R}$

decaying ($r<0$) or growing ($r>0$) — (stability) (instability)

periodic term ($s \neq 0$) (spiraling) — sign(s) gives clockwise or anti-clockwise

Type of stability is given μ_1, μ_2

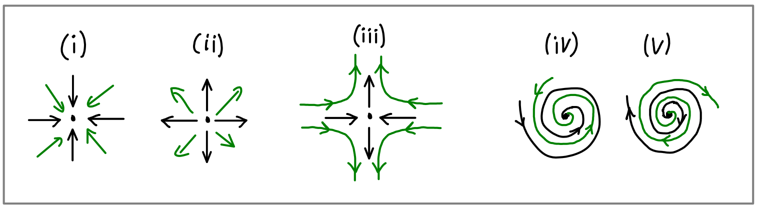
(i) Stable node: $\mathcal{D} > 0$ and $\mu_1 < 0$, $\mu_2 < 0$.

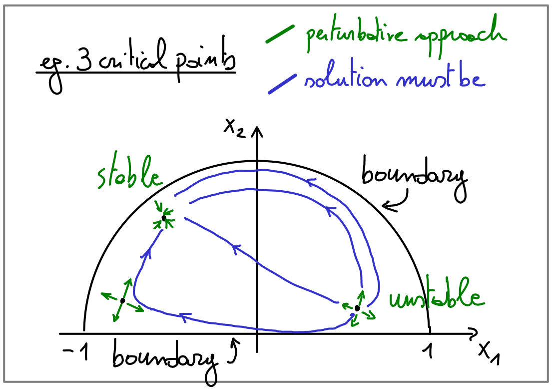
(ii) Unstable node: $\mathcal{D} > 0$ and $\mu_1 > 0$, $\mu_2 > 0$.

(iii) Saddle point: $\mathcal{D} > 0$ and $\mu_1 < 0$, $\mu_2 > 0$ (or $\mu_1 > 0$ and $\mu_2 < 0$). stable for one parameter, unstable in the other

(iv) Stable spiral: $\mathcal{D} < 0$ and the real parts of μ_1 and μ_2 are negative
(v) Unstable spiral: $\mathcal{D} < 0$ and the real parts of μ_1 and μ_2 are positive } μ_i has complex part

(vi) if $\mathcal{D}=0$, $\mathcal{A}$ is singular $\Rightarrow$ system is 1-dimensional around fixed point (one parameter is redundant at 1° order)





Just the concept regarding critical points

(i) Solutions are goes toward (i) very fast
(ii) " go away from (ii) " "
(iii) { " can go toward (iii) along one direction, eg. $\mu_1 < 0$
" can spend there some time
" and go away along the other direction, eg. $\mu_2 > 0$ }

How do you work with that?

is the critical point real? i.e. $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$, $\hat{x}_i$ complex has no physical meaning
check constrains to see if they are in physical region
look what they represent (w_ϕ, w_{eff}, ...). Matter era? Accelerated phase? Else...

Our universe

- if universe is in a stable node or stable spiral it will be "trapped" in that state (w_{eff}, Ω_ϕ, Ω_M)
- to transition to a different epoch we need saddle points (attracted there for some time and then repulsed away)
- we need:

radiation era	stable for a while but not forever	saddle
matter era	" " " " " " "	saddle
current acceleration	stable for a while or forever	stable/saddle

↑ to avoid coincidence problem

- Do you get that? This is the 1° test for a theory/model
- For the full picture, we need to consider ρ_m and ρ_r simultaneously
  $\Rightarrow$ you get a third axis in the phase space

$\boxed{\text{Case } \lambda = \text{const}}$

- $\frac{d\lambda}{dN} = -\cancel{\sqrt{6}}\,\cancel{\lambda^2}(\Gamma - 1)\cancel{x_1} \overset{!}{=} 0 \quad \Rightarrow \quad \boxed{\Gamma = 1}$

λ is the exponent

$\lambda \equiv -\frac{V_{,\phi}}{\kappa V} = \frac{dV}{d\phi}\frac{1}{\kappa V} \qquad \frac{dV}{V} = -\kappa\lambda\, d\phi \quad \Rightarrow \quad \boxed{V(\phi) = V_0 e^{-\kappa\lambda\phi}}$ exponential potential

	Fixed points	Cosmology			Eigenvalues		"stability"
	$(\hat{x}_1, \hat{x}_2)$	Ω_ϕ	w_{eff}	w_ϕ	μ_1	μ_2	
(a)	$(0,0)$	0	w_M	undefined	$-\frac{3}{2}(1-w_M)$	$\frac{3}{2}(1+w_M)$	saddle
(b_1)	$(1,0)$	1	1	1	$3-\frac{\sqrt{6}}{2}\lambda$	$3(1-w_M)$	unstable $\lambda<\sqrt{6}$, saddle $\lambda>\sqrt{6}$
(b_2)	$(-1,0)$	1	1	1	$3+\frac{\sqrt{6}}{2}\lambda$	$3(1-w_M)$	" $\lambda>-\sqrt{6}$, " $\lambda<-\sqrt{6}$
(c)	$\left(\frac{\lambda}{\sqrt{6}}, \sqrt{1-\frac{\lambda^2}{6}}\right)$	1	$-1+\frac{\lambda^2}{3}$	$-1+\frac{\lambda^2}{3}$	$\frac{1}{2}(\lambda^2-6)$	$\lambda^2-3(1+w_M)$	stable $\lambda^2<3(1+w_M)$, " $3(1+w_M)<\lambda^2<6$
(d)	$\left(\sqrt{\frac{3}{2}}\frac{(1+w_M)}{\lambda}, \sqrt{\frac{3(1-w_M^2)}{2\lambda^2}}\right)$	$\frac{3(1+w_M)}{\lambda^2}$	w_M	w_M	$\frac{3}{4}(1+w_M)\left(1\pm\sqrt{1-\frac{8(1+w_M)[\lambda^2-3(1+w_M)]}{\lambda^2(1-w_M)}}\right)$		$\begin{cases}\text{saddle } \lambda^2<3(1+w_M), \text{ stable } 3(1+w_M)<\lambda^2<\gamma \\ \text{stable spiral } \gamma<\lambda^2 \quad \gamma=\frac{24(1+w_M)^2}{7+9w_M}\end{cases}$

- a: $\Omega_\phi = 0 \Rightarrow \Omega_M = 1$, i.e. Matter domination, saddle $\Rightarrow$ "attracts" and than "repels", acceleration not possible
- b: $\Omega_\phi = 1 \Rightarrow \Omega_M = 0$, i.e Field ", $w=1$: $\rho_\phi \propto a^{-6}$, dies very fast; unstable: last for a "blink"; saddle: last for a wile and ends
- c, d : can have stable solutions $\forall\lambda$

$w_{eff} = -1 + \frac{\lambda^2}{3} < -\frac{1}{3}$

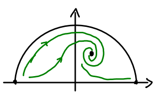
- c : $\forall$ for $\lambda^2 < 6$ scalar field domination

  the only one giving stable attractor, accelerated $\begin{cases}\text{for } \lambda^2<2 \Rightarrow w_{eff} < -1/3, \text{ i.e. acceleration!} \\ \text{for } \lambda \to 0 \Rightarrow V(\phi) \to V_0,\ w_{eff} = w_\phi = -1+\frac{\lambda^2}{3}\end{cases}$ similar to Λ but not the identical

- d : $\Omega_\phi = \frac{3(1+w_M)}{\lambda^2}$ $\begin{cases} w_M = 0 & \Omega_\phi = 3/\lambda^2 \leq 1 \Rightarrow \lambda^2 \geq 3 \\ w_M = 1/3 & \Omega_\phi = 4/\lambda^2 \leq 1 \Rightarrow \lambda^2 \geq 2\end{cases}$ or you can not have $\Omega_\phi + \Omega_M = 1$

  $w_{eff} = w_\phi = w_M = 0, 1/3 \Rightarrow$ no acceleration unless $w_M < -1/3$ ... too wierd

if d is in valid area



Example on how to interpret scenarios

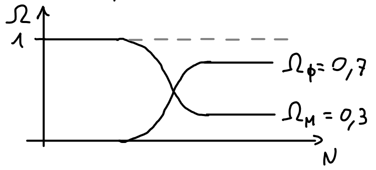
- for $\lambda^2 > 3(1+w_M)$

$\left.\begin{array}{l}\text{(c) is a saddle}\\ \text{(d) is stable}\end{array}\right\} \Rightarrow$ solutions must end in (d) ... and the way around for $\lambda < 3(1+w_M)$

- start from : matter domination : $w_M = 0$, $\Omega_\phi = 0$, $\Omega_M = 1$, $\mu_1 = -\frac{3}{2}$ $\mu_2 = \frac{3}{2}$ saddle $\Rightarrow$ stay and escape
- move to (d) : field " : $w_M = 0$, $\Omega_\phi = \frac{3}{\lambda^2} = \text{const}$, $\Omega_M = 1 - \Omega_\phi = 1 - \frac{3}{\lambda^2} = \text{const}$

$\Rightarrow \Omega_\phi / \Omega_M = \text{const}$. this is called a "scaling model", solves the coincidence problem

Observations: $\Omega_{\phi,0} = \frac{3}{\lambda^2} = 0{,}7 \Rightarrow \lambda^2 \sim 3/0{,}7 = 4{,}2$: $\hat{x}_1 = \hat{x}_2 = \frac{1}{\lambda}\sqrt{\frac{3}{2}} \sim 0{,}28$ is in physical region $\Omega_M < 1$



They reach an attractor

but $w_\phi = w_M = 0 > -\frac{1}{3}$ $\Rightarrow$ no acceleration

solution remains in scaling era $\Omega_\phi = \text{const} \Rightarrow$ can not shift to accelerated era

- example (a) $\rightarrow$ (c)

for $\lambda \sim 0$ solution starting from (a) approaching (c) with $w_\phi = -1 + \frac{\lambda^2}{3}$ (similar to Λ scenario)

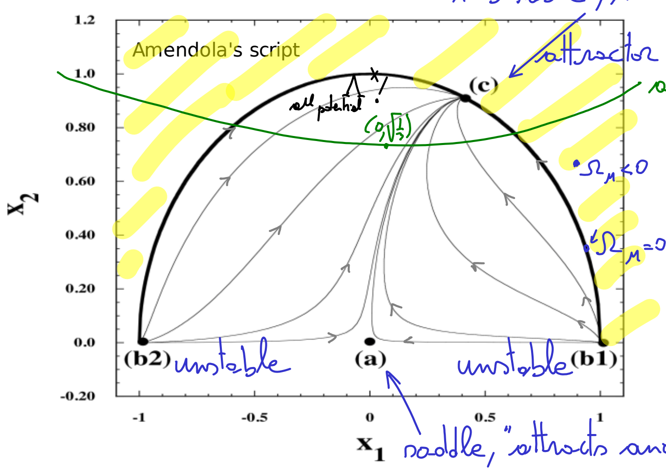
(d) not valuable ($\Omega_M > 1$)

attractor solution you must end up there

- Example of full solutions for : $w_M = 0$, $\lambda = \text{const} = 1$ (exponential potential)

- only as an illustration, in (c) $w_\phi = -1 + \frac{\lambda^2}{3} = -0{,}66$, against observations
- Point (d) is not valuable $(\hat{x}_1, \hat{x}_2)_d = (1.22, 1.22)$

$\lambda < 3$ stable, $\lambda < 2$ acceleration



$w_M = 0$ acceleration $w_{eff} = x_1^2 - x_2^2 < -\frac{1}{3}$ $x_2^2 = x_1^2 + \frac{1}{3}$

saddle, "attracts and repels", matter domination

Different potentials : $\lambda \neq$ const.

- $V(\phi)$ defines the characteristics of DE
- For $\lambda =$ const. => exponential potential
- For $\lambda \neq$ const. => large variety of models (2 main families)

1) Freezing models:

- V rolling along ϕ in past
- slows down when accelerated expansion starts:

a) $V(\phi) = M^{4+n}\,\phi^{-n}$ $\quad n > 0$ $\quad$ M mass scale (Look in lagrangian, V has unit of mass)

no minimum => ϕ rolls down V towards ϕ infinity
↳ like exponential case => same qualitative behaviour
Fermion condensate as a dynamical supersymmetry breaking

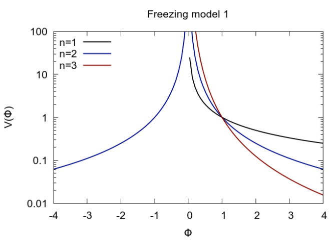
b) $V(\phi) = M^{4+n}\,\phi^{-n}\exp(\alpha\phi^2/m_{pe}^2)$ $\quad$ has minimum => ϕ is trapped by the exponential $w_\phi = -1$
supersymmetry framework

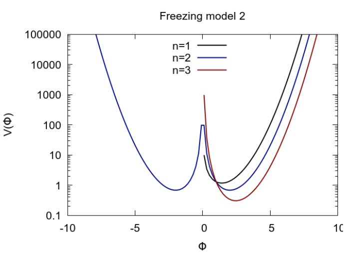
2) Thawing models: $\qquad$ Thawin ~ de-freezing

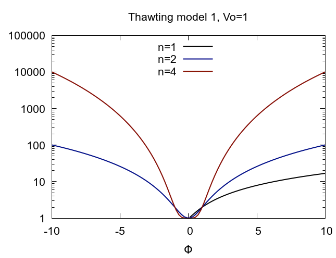
- Potential start flat
- field frozen by Hubble friction, $3H\dot{\phi}$ in KG eq., until late times
- Then it evolves when H drops below $m_\phi \equiv \sqrt{\frac{\delta^2 V}{\delta\phi^2}}$
- $w_\phi \sim -1$ at early times than it grows

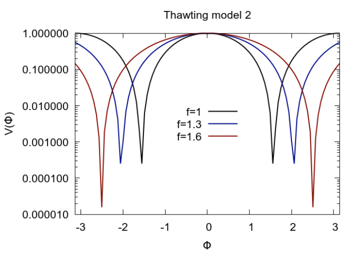
a) $V(\phi) = V_0 + M^{4-n}\,\phi^{n}$ $\quad n > 0$ $\quad$ $n = 2, 3$ similar to chaotic inflation ($V_0 = 0$) but M is very different
$n = 1$ slowly varying field

b) $V(\phi) = M^4 \cos^2(\phi/f)$ $\quad$ ϕ is nearly frozen at maximum of V when $m_\phi < H$
then rolls down at late times when $m_\phi \simeq H_0$
Pseudo-Nambu-Goldstone-Boson framework









Constraints on the field values/properties

- Case with $V(\phi) = M^{4+m}\phi^{-m}$ $(m>0)$

- $\Gamma \equiv \frac{V V_{,\phi\phi}}{V_{,\phi}^2} = \frac{m+1}{m} > 1$ tracking condition is automatically satisfied
- $\lambda \equiv -\frac{V_{,\phi}}{\kappa V} = -\frac{m\phi^{-1}}{\kappa}$ $\leftarrow V_{,\phi} = -M^{4+m}\, m\, \phi^{-(m+1)}$
- $\lambda^2 = \left(\frac{m\phi^{-1}}{\kappa}\right)^2 < 2$ acceleration condition $\Rightarrow$ $\phi > \frac{m}{4\sqrt{\pi}} m_{pl}$ $\left(\phi^2 > \frac{m^2}{2\kappa^2} \quad \kappa^2 \equiv 8\pi G \quad m_{pl} = \sqrt{\frac{\hbar c}{G}} \quad \hbar = 1 \quad c = 1\right)$

  no dependency on mass scale M

  ϕ at start of acceleration $\phi = \frac{m}{4\sqrt{\pi}} m_{pl} \sim O(1)$

- Normalization M, set by observations: $H_0, \Omega_{m,0}$

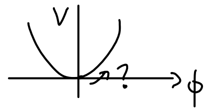
  Friedmann $H^2 = \frac{\kappa^2}{3}\left[\underbrace{\cancel{\tfrac{1}{2}\dot\phi^2}}_{\text{slow roll}} + V(\phi) + \rho_M\right] = \frac{8\pi}{3m_{pl}^2} M^{4+m}\phi^{-m}$ $\Rightarrow$ $M^{4+m} = H_0^2 \frac{3m_{pl}^2}{8\pi}\phi_0^m$

  $\phi_0 \sim m_{pl}$ $H_0 \simeq 10^{-42}\,\text{GeV}$ $\Rightarrow$ $\left.\begin{array}{ll} m=2 & M \approx 10^{-1}\,\text{GeV} \\ m=4 & M \approx 10^{4}\,\text{GeV}\end{array}\right\}$ ok for particle physics :)

- Consider power law $V(\phi) = M^{4+m}\phi^{+m}$ $m>0$

  Thawing model, ϕ frozen in the past, started to move recently

- $\Gamma \equiv \frac{V V_{,\phi\phi}}{V_{,\phi}^2} = \frac{m-1}{m} < 1$ tracking condition never satisfied
- $\lambda \equiv -\frac{V_{,\phi}}{\kappa V} = \frac{m\phi^{-1}}{\kappa}$ because $V_{,\phi} = M^{4+m}\, m\, \phi^{m-1}$



- $|\lambda|^2 = \frac{m^2 m_{pl}^2}{{}_4\cancel{8}\pi|\phi|^2} < \cancel{2}^{1}$ acceleration condition $\Rightarrow$ $|\phi| > \frac{m}{4\sqrt{\pi}} m_{pl} \sim m_{pl}$

  when ϕ has this value, acceleration starts $\Rightarrow$ ϕ_{init} and m_ϕ about potential minimum are crucial to set the right time of acceleration

  $|\phi_i| < m_{pl}$ and $m_\phi \lesssim H_0$ $\Rightarrow$ temporary phase of accelerated expansion

  acceleration stops when $|\phi|$ drops down to to the order of m_{pl}

- Case $V(\phi) = M^4 \cos^2\left(\frac{\phi}{f}\right)$ similar expansion history

- Case $V(\phi) = V_0 + M^{4-m}\phi^m$ $V_0 > 0$

  $|\lambda| \to 0$ eventually $\Rightarrow$ when $\phi = 0$ you get Λ behaviour but $w_\phi > -1$ by a bit

Behaviour of solutions with non constant λ

- $\lambda \equiv -\frac{V_{,\phi}}{\kappa V}$ $V(\phi)$ not exponential $\Rightarrow \lambda(\phi)$ can evolve

  $\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x_1$ (*) give λ evolution together with $x_1'(x_1, x_2)$ and $x_2'(x_1, x_2)$

- Get fixed points for $\underline{\lambda}$ (eq.*) and $\underline{H}$ at a given time ("instantaneous" values)

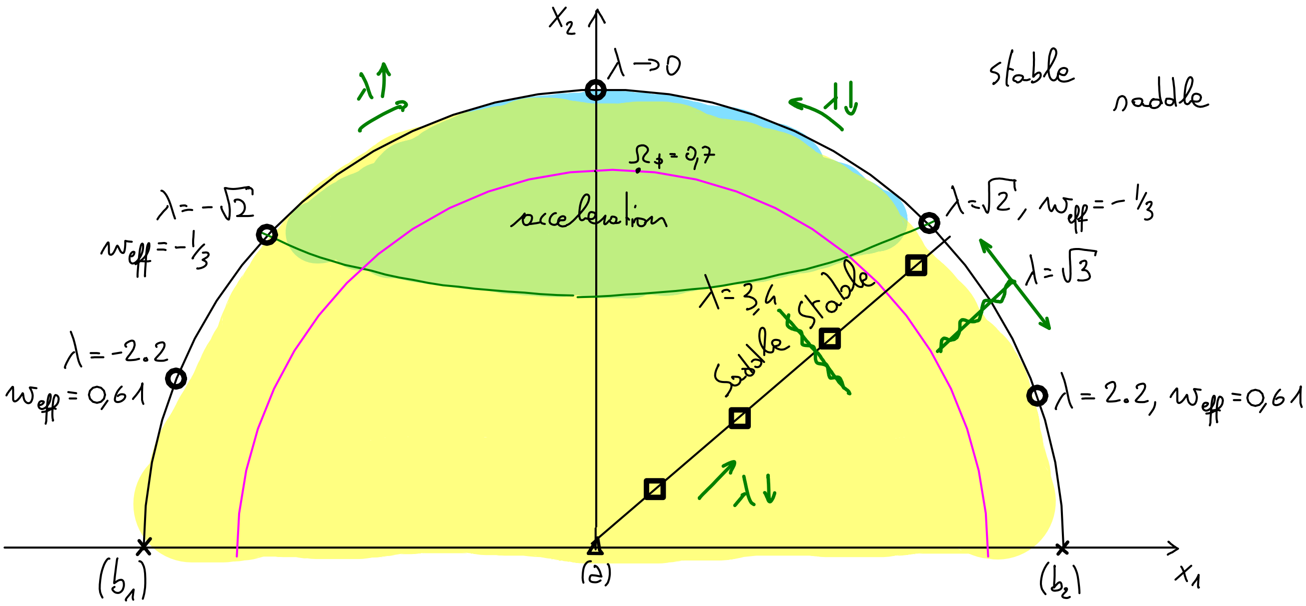
  possible if (time scale variation of λ) $\ll (H^{-1})$

  $\Rightarrow$ we can exploit previous results (critical points)

  careful: critical points stability/position evolve because of $\lambda' \neq 0$

For $w_M = 0$ Critical point (c) ○ $\Omega_\phi = 1$, $w_{eff} = w_\phi = -1 + \frac{\lambda^2}{3}$

" " (d) □ $\Omega_\phi = 3/\lambda^2$ $w_{eff} = w_\phi = w_M$



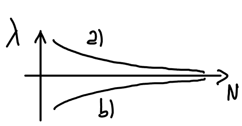
Scaling ($\Gamma = 1$) and Tracking conditions ($\Gamma > 1$), and Tracker solutions

- Example, freezing model

a) if $V_{,\phi} < 0$: $\lambda > 0 \Rightarrow x_1 > 0$

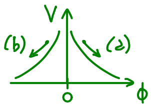
b) if $V_{,\phi} > 0$: $\lambda < 0 \Rightarrow x_1 < 0$

$$\Rightarrow \frac{d\lambda}{dN} = -\sqrt{6}\lambda^2 \underbrace{(\Gamma - 1)}_{>0} x_1 \quad \begin{matrix} <0 \\ >0 \end{matrix}$$



(!) $\lambda \to 0$ regardless sign$(V_{,\phi})$

recall $V>0$



e.g.(a): $\phi \uparrow$ i.e. $\dot\phi > 0$ $\quad x_1 \propto \dot\phi > 0$

i.e. eventually solutions move to accelerated point (c) even if initially $\lambda^2 > 2$

1) acceleration: when λ reaches $\lambda < 2$

2) tracking behevejour: starting with $\Omega_\phi \ll 1$ (Matter domination): $w_\phi < w_M \Rightarrow \rho_\phi$ catches up ρ_M

in fact: $w_\phi < w_M \Rightarrow$ since $\rho = \rho_0 a^{-3(1+w)}$ slowlier evolution of ρ_ϕ with respect to ρ_M

- Origin of tracking condition

- $w_\phi = \frac{P}{\rho} = \frac{\frac{1}{2}\dot\phi^2 - V}{\frac{1}{2}\dot\phi^2 + V} = \frac{\cancel{V}\left(\frac{\dot\phi^2}{2V} - 1\right)}{\cancel{V}\left(\frac{\dot\phi^2}{2V} - 1\right)} = \frac{x-1}{x+1}$ $\qquad x \equiv \frac{\dot\phi^2}{2V} = \frac{1+w_\phi}{1-w_\phi}$ $\qquad y' \equiv \frac{d\ln x}{dN}$

- $-\lambda = \frac{V_{,\phi}}{\kappa V} = \pm\sqrt{\frac{3(1+w_\phi)}{\Omega_\phi}}\left(1 + \frac{1}{6}\frac{d\ln x}{dN}\right)$ (*) $\quad (\pm : \dot\phi \lessgtr 0)$

from $\frac{dx}{dN}$ and $\Omega_\phi = x_1^2 + x_2^2$

∂_ϕ

- $$\Gamma = 1 + \frac{3(1-\overset{\sim 0}{\Omega_\phi})(w_M - w_\phi)}{(1+w_\phi)(6+\underset{\sim 0}{y'})} - \frac{\overset{\sim 0}{y'}}{(1+w_\phi)(6+y')(1+x)} - \frac{\overset{\sim 0}{2y''}}{(1+w_\phi)(6+y')^2} \qquad y' \equiv \frac{d\ln x}{dN}$$

$$\simeq 1 + \frac{\cancel{3}^{\,1}(w_M - w_\phi)}{(1+w_\phi)\cancel{6}_{\,2}}$$

Assuming Matter (m or r) domination ($\Omega_\phi \ll 1$) and if Γ varies slowly $\left|\frac{d(\Gamma-1)}{dN}\right| \ll |\Gamma - 1|$ because of slow rolling (y', y'' negligible)

$\Downarrow$

$$\boxed{w_\phi \simeq \frac{\overset{\text{const}}{w_M} - 2(\Gamma - 1)}{1 + 2(\Gamma - 1)} \sim \text{const.}}$$

if $\Gamma > 1$ $\quad w_\phi < w_M$ $\quad$ tracking: Ω_ϕ catches up Ω_M

$\Gamma = 1$ $\quad w_\phi = w_M$ $\quad$ scaling: $\frac{\Omega_\phi}{\Omega_M} = \text{const.}$ $\quad \rho = \rho_0 a^{-3(1+w)}$

- For $V(\phi) = M^{4-n}\phi^{-n}$ $(n>0)$: $\Gamma = \frac{n+1}{n} > 1$ tracking condition is automatically satisfied

$$w_\phi \simeq \frac{n w_M - 2}{n+2}$$

- Tracker solution

- Using (*) : $\quad \frac{1}{6}\frac{d\ln x}{dN} = \lambda\sqrt{\frac{\Omega_\phi}{3(1+w_\phi)}} - 1 \equiv \Delta(t) - 1 \qquad$ (a)

- Using $x \equiv \frac{1+w_\phi}{1-w_\phi}$: $\quad \frac{d\ln x}{dN} = \frac{1}{x}\frac{dx}{dN} = \frac{1-w_\phi}{1+w_\phi}\,\frac{w_\phi'(1-w_\phi) + (1+w_\phi)w_\phi'}{(1-w_\phi)^{\cancel{2}}} = w_\phi'\frac{1-\cancel{w_\phi}+1+\cancel{w_\phi}}{1-w_\phi^2} = \frac{2}{1-w_\phi^2}\frac{dw_\phi}{dN}$ $\qquad$ (b)

$\Rightarrow$ (a) = (b) : $\frac{dw_\phi}{dN} = 3(\Delta(t) - 1)(1 - w_\phi^2) \longrightarrow \boxed{\frac{dw_\phi}{dN} \gtrless 0 \text{ for } \Delta \gtrless 1}$ $\quad (w_\phi \leq 1)$

$\Rightarrow$ (a) $\simeq 0$: $\boxed{\Omega_\phi \simeq \frac{3(1+w_\phi)}{\lambda^2}}$ $\qquad \boxed{\Delta(t) \sim 1}$ tracker solution $w_\phi \sim$ const. for

- Tracker solution leads to stable fixed points
  - Ω_ϕ as in scaling solution of fixed point (d): $\Omega_\phi = \frac{3(1+w_M)}{\lambda^2}$, $w_\phi = w_M$ (scaling), stable for $\lambda^2 > 3(1+w_M)$
  - as λ decreases, (d) becomes a saddle when $\lambda^2 < 3(1+w_M)$

  $\Rightarrow$ solution can move away and reach fixed point (c): $\Omega_\phi = 1$, $w_\phi = -1 + \frac{1}{3}\lambda^2$, stable for $\lambda^2 < 3(1+w_M)$
- we say: tracker solution is a stable attractor (goes to stable points)
- Key points: This happens "regardless" the initial conditions
  for λ=const. stable solution in (d) does NOT exit to reach (c)
  now for decreasing λ can go to accelerated (c) through the tracking solution
- Concept: Special trajectories attracting other trajectories
  have w_ϕ, Ω_ϕ nearly constant
  several initial conditions converge to the same tracker

Two "Matter" (M) contributions: dust (m) and radiation (r)

$\rho_M = \rho_m + \rho_r \qquad P_M = 0 + \frac{1}{3}\rho_r$ matter (dust) and radiation simultaneously (!)

$\Omega_\phi = x_1^2 + x_2^2 \qquad \Omega_r \equiv \frac{\kappa^2 \rho_r}{3H^2} \equiv x_3^2 \qquad \Omega_m = 1 - x_1^2 - x_2^2 - x_3^2$ where we introduced $x_3 \equiv \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}$ $\quad \kappa = 8\pi G$

because of flatness

$w_{eff} = -1 - \frac{2}{3}\left(\frac{\dot H}{H^2}\right)^* = x_1^2 - x_2^2 + \frac{1}{3}x_3^2$ * as we did earlier but for 2 fluids solution 4.96

$$\frac{dx_1}{dN} = -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{1}{2}x_1(3 + 3x_1^2 - 3x_2^2 + x_3^2)\,,$$
$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + 3x_1^2 - 3x_2^2 + x_3^2)\,,$$
$$\frac{dx_3}{dN} = -2x_3 + \frac{1}{2}x_3(3 + 3x_1^2 - 3x_2^2 + x_3^2)\,.$$
$$\frac{d\lambda}{dN} = -\sqrt{6}\frac{\lambda^2}{n}x_1 \quad \text{because } \Gamma = \frac{n+1}{n}$$

(as before but with $w_M = 0$, radiation is in x_3)

autonomous eq.s to be solved numerically

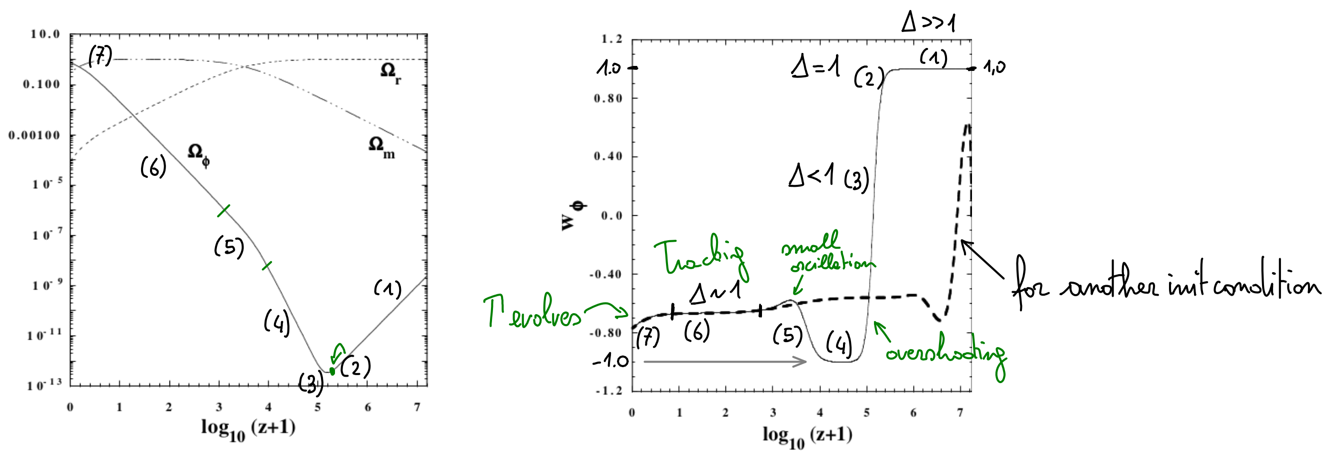
now 3D phase-space: (x_1, x_2, x_3)

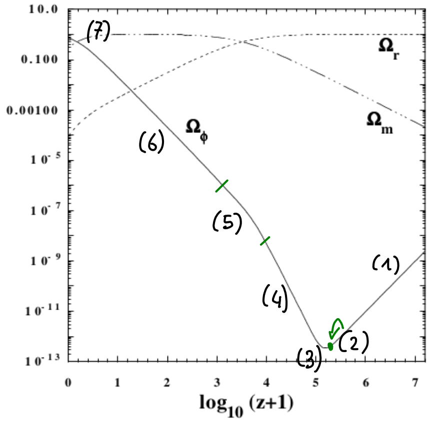
- Universe evolution
  - interested in case Ω_ϕ not negligible during radiation era (not like Λ)
  - as $\lambda \downarrow$ with time, we can have init. condition with $\Delta(t_i) \gg 1$ (during radiation era)
  - than universe converges to tracking solution.

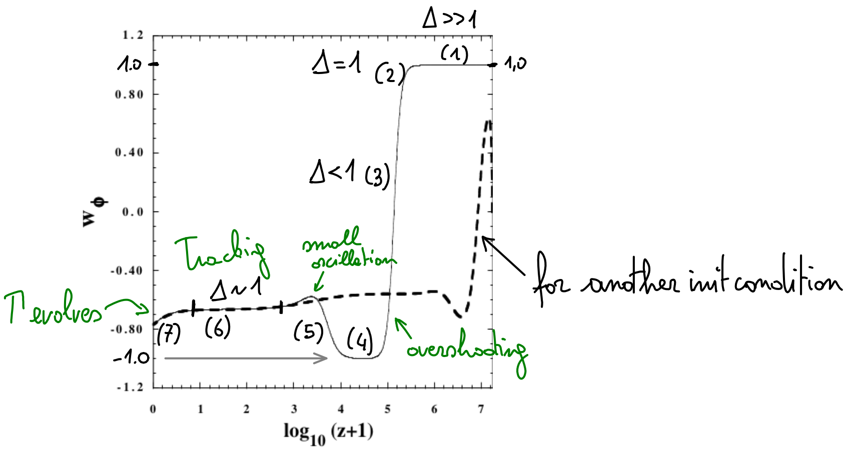
Example: $V(\phi) = M^5\phi^{-1} \quad (n=1) \qquad \Omega_{\phi,0} = 0{,}72$

initial conditions: $(x_1, x_2, x_3)_{init} = (5\cdot10^{-5}, 10^{-8}, 0.999) \quad \lambda_{i\,t} = 10^9 \quad \log_{10}(z_{init}+1) = 7{,}21$

Only numerical solution: see plot next page







Recall: $\frac{dw_\phi}{dN} = 3(\Delta(t) - 1)(1 - w_\phi^2) \rightarrow \begin{cases} \boxed{\frac{dw_\phi}{dN} \gtrless 0 \text{ for } \Delta \gtrless 1} \quad (w_\phi \le 1) \\ \boxed{\Delta(t) \sim 1} \text{ tracker solution } w_\phi \sim \text{const.} \end{cases}$

$\boxed{\Omega_\phi \simeq \frac{3(1+w_\phi)}{\lambda^2}}$ $\boxed{w_\phi \simeq \frac{m w_M - 2}{m+2}}$

(0) at high z Ω_ϕ not negligible

(1) $\Delta \gg 1$: $\frac{dw_\phi}{dN} > 0$ $w_\phi \uparrow$

$\frac{dw_\phi}{dN} = 3(\Delta(t) - 1)(1 - w_\phi^2) \simeq 3\Delta(t)(1 - w_\phi^2) \overset{!}{=} 0 \Rightarrow w_{\phi\,min} = -1 \quad w_{\phi\,max} = 1$

i.e. $w_\phi = 1 \Rightarrow \rho_\phi \propto a^{-6}$ in fact ϕ is dominant fast rolling phase $w_\phi = \frac{\frac{1}{2}\dot\phi^2 - V}{\frac{1}{2}\dot\phi^2 + V} \simeq 1$

(2) $\Delta = 1$: condition for tracker solution $w_\phi' = 0$

but $\dot\phi$ very large $\Rightarrow$ overshoot tracker solution (still fast rolling)

(3) $\Delta < 1$: $\frac{dw_\phi}{dN} < 0$ w_ϕ start to decrease rapidly $w_\phi \rightarrow w_{\phi\,min} = -1$

(4) w_ϕ reaches minimum $w_\phi \sim -1$ and $\frac{dw_\phi}{dN} \sim 0 \Rightarrow \rho_\phi \propto a^{3(1+w_\phi)} = \text{const}$, i.e. field is frozen

$\Omega_\phi \equiv \frac{\rho_\phi}{\rho_c}$ start $\uparrow$ (because $\rho_c \downarrow$)

ϕ frozen $\Rightarrow \lambda = \frac{V_\phi}{\kappa V} \rightarrow 0$ because $\dot\phi \sim 0$ i.e. $V(\phi) = \text{const.}$

$\Delta \equiv \lambda \sqrt{\frac{\Omega_\phi}{3(1+w_\phi)}} \uparrow$ because $\Omega_\phi \uparrow$ and $(1+w_\phi) \sim 0$

$w_\phi \sim -1$ untill $\Delta > 1$

(5) $\Delta > 1$ again $\Rightarrow w_\phi \uparrow$, at $\Delta \sim 1$ oscillation

(6) $\Delta \sim 1$: tracking regime at $z \lesssim 10^3$, i.e. during matter domination, $w_M = 0 \Rightarrow w_\phi = -2/3 = -0{,}6\bar{6}$

(7) Ω_ϕ not negligible $\Rightarrow \Gamma$ evolves : $w_\phi \downarrow$

$$\Gamma = 1 + \frac{3(1 - \underbrace{\Omega_\phi}_{\text{not negligible and growing}})(w_M - w_\phi)}{(1 + w_\phi)(6 + \underbrace{y'}_{\sim 0})} - \frac{\overset{\sim 0}{y''}}{(1 + w_\phi)(6 + y')(\cdots}$$

Quintessence: Summary

Dark energy as a scalar field

$V>0$ or $\rho<0$ is possible (ghosts)

- $\boxed{\mathcal{L}_\phi = -\frac{1}{2}c^2\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)}$ $\quad\boxed{T^\phi_{\mu\nu} = c^2\partial_\mu\phi\,\partial_\nu\phi + \mathcal{L}_\phi g_{\mu\nu}}$ $\quad\boxed{\begin{aligned}T_{00} &= \tfrac{1}{2}\dot\phi^2 + V = \rho_\phi c^2\\ T_{ii} &= \tfrac{1}{2}\dot\phi^2 - V = P_\phi\end{aligned}}$ $\quad\boxed{w = \frac{P}{\rho c^2} = \frac{\frac{1}{2}\dot\phi^2 - V}{\frac{1}{2}\dot\phi^2 + V}}$

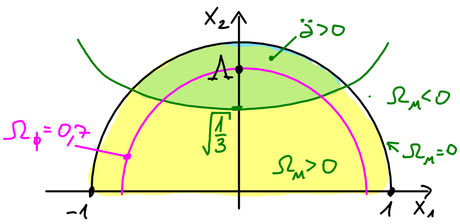
- $\boxed{\ddot\phi + 3H\dot\phi + \frac{\delta V}{\delta\phi} = 0}$

(dynamic) Klein-Gordon

A) $\frac{\dot\phi}{2} \gg V(\phi)$:	$w_\phi \simeq 1$	$\rho_\phi \propto a^{-6}$	
B) $\frac{1}{2}\dot\phi^2 = V$:	$w = 0$	$\rho \propto a^{-3}$	dust
C) $\dot\phi = 0$:	$w = -1$	$\rho = \text{const}$	Λ
D) $\dot\phi^2 \ll V(\phi)$:	$w_\phi \simeq -1 + \frac{\dot\phi^2}{V}$	$\rho \simeq \text{const}$	dark-energy

- $\boxed{H^2} = \frac{8\pi G}{3c^2}\left(\rho_M c^2 + \frac{1}{2}\dot\phi^2 + V(\phi)\right) = \boxed{\kappa^2\left(\rho_M + \frac{1}{2}\dot\phi^2 + V(\phi)\right)}$ $\qquad c \overset{!}{=} 1,\quad \kappa^2 \equiv 8\pi G$ $\qquad$ Friedmann (2)

Phase-space analysis



$$\boxed{\begin{aligned}
&\Omega_M = 1 - x_1^2 - x_2^2 = 1 - \Omega_\phi \qquad x_1 \equiv \frac{\kappa\dot\phi}{\sqrt{6}H} \quad x_2 \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}\\
&\Omega_\phi = x_1^2 + x_2^2\\
&w_\phi = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}\\
&w_{\text{eff}} = w_M + x_1^2(1-w_M) - x_2^2(1+w_M) \qquad \ddot a > 0 \text{ for } w_{\text{eff}} < -\tfrac{1}{3}
\end{aligned}}$$

- K-G $\rightarrow$ $N \equiv \log a$ $\quad$ "$\frac{d}{d\tau} = H\frac{d}{dN}$" $\quad$: $\phi'' + 3\phi' + \frac{V_\phi}{H^2} = 0$ $\qquad \phi'' = -3\phi' - \frac{V_\phi}{H^2} = -3\frac{\sqrt{6}}{\kappa}x_1 - \frac{3V_\phi}{\kappa^2 V}x_2^2$

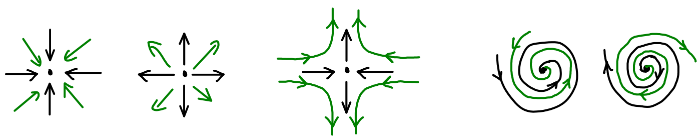
$$\begin{cases}
x_1' = -3x_1 + \lambda x_2^2 + \frac{3}{2}x_1\left(x_1^2(1-w_M) - (1+x_2^2)(1+w_M)\right)\\
x_2' = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{3}{2}x_2\left[(1-w_M)x_1^2 + (1+w_M)(1-x_2^2)\right]\\
\lambda' = -\sqrt{6}\lambda^2(\Gamma - 1)x_1
\end{cases}
\qquad
\begin{aligned}
\lambda &\equiv -\frac{V_{,\phi}}{\kappa V} = -\frac{1}{\kappa}\frac{d\log V}{d\phi}\\
\Gamma &\equiv \frac{V V_{,\phi\phi}}{V_{,\phi}^2}
\end{aligned}$$

- Critical solutions $\quad \begin{aligned}x_1'(x_1,x_2) &= 0\\ x_2'(x_1,x_2) &= 0\end{aligned} \Rightarrow (\hat x_1, \hat x_2)$ for $\lambda = \text{const}$ i.e. $V(\phi) = V_0 e^{-\kappa\lambda\phi}$

- Stability near critical points :

$$\begin{cases}x_1 = \hat x_1 + \delta x_1\\ x_2 = \hat x_2 + \delta x_2\end{cases} \quad |\hat x_i| \ll |\delta x_i| \Rightarrow \frac{d}{dN}\begin{pmatrix}\delta x_1'\\ \delta x_2'\end{pmatrix} = \mathcal{A}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix} \Rightarrow \begin{aligned}&x_i = \hat x_i + \alpha_{i1}e^{\mu_1 N} + \alpha_{i2}e^{\mu_2 N}\\ &\mu_{1,2} = \tfrac{1}{2}\left(-\mathrm{Tr}(\mathcal{A}) \pm \sqrt{\mathrm{Tr}(\mathcal{A})^2 - 4\det(\mathcal{A})}\right)\end{aligned}$$

	$(\hat x_1, \hat x_2)$	Ω_ϕ	w_{eff}	w_ϕ
(a)	$(0,0)$	0	w_M	undefined
(b$_1$)	$(1,0)$	1	1	1
(b$_2$)	$(-1,0)$	1	1	1
(c)	$\left(\frac{\lambda}{\sqrt{6}}, \sqrt{1-\frac{\lambda^2}{6}}\right)$	1	$-1+\frac{\lambda^2}{3}$	$-1+\frac{\lambda^2}{3}$
(d)	$\left(\sqrt{\frac{3}{2}}\frac{(1+w_M)}{\lambda}, \sqrt{\frac{3(1-w_M^2)}{2\lambda^2}}\right)$	$\frac{3(1+w_M)}{\lambda^2}$	w_M	w_M



- $\underline{\lambda'(N) \neq 0}$ $\Rightarrow$ $V(\phi)$ not exponential eg.:

$$\circledast \Rightarrow \quad \Gamma \equiv \frac{VV_{,\phi\phi}}{V_{,\phi}^2} = \frac{m+1}{m} > 1 \qquad \lambda \equiv -\frac{V_{,\phi}}{\kappa V} = -\frac{m\phi^{-1}}{\kappa}$$

$$\left.\begin{array}{l} V(\phi) = M^{4+m}\phi^{-m} \quad m>0 \ \circledast \\ V(\phi) = M^{4+m}\phi^{-m}\exp(\alpha\phi^2/m_{pe}^2) \end{array}\right\} \text{Freezing models}$$

$$\left.\begin{array}{l} V(\phi) = V_0 + M^{4-m}\phi^m \\ V(\phi) = M^4\cos^2(\phi/f) \end{array}\right\} \text{Thawing models}$$

- $\underline{\text{Scaling / Tracking}}$

$$w_\phi = \frac{\left(\frac{\dot\phi^2}{2V}-1\right)}{\left(\frac{\dot\phi^2}{2V}+1\right)} = \frac{x-1}{x+1} \qquad x \equiv \frac{\dot\phi^2}{2V} = \frac{1+w_\phi}{1-w_\phi} \quad \xRightarrow{dx/dN} \quad -\lambda = \frac{V_{,\phi}}{\kappa V} = \pm\sqrt{\frac{3(1+w_\phi)}{\Omega_\phi}}\left(1+\frac{1}{6}\frac{\mathrm{d}\ln x}{\mathrm{d}N}\right) \qquad (a)$$

$\Omega_\phi = x_1^2 + x_2^2,\ \delta_\phi$

$$\Gamma \simeq 1 + \frac{(w_m - w_\phi)}{(1+w_\phi)\,2} \quad \text{for } \Gamma \text{ nearly const}$$

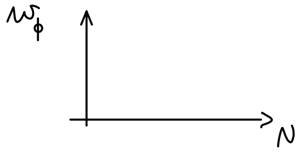
$$w_\phi \simeq \frac{\overset{\text{const}}{w_M} - 2(\Gamma-1)}{1+2(\Gamma-1)} \sim \text{const.}$$

$= -\frac{2}{3} = -0{,}66$ for $w_M = 0$, potential $\circledast$ $m=1$ $\Gamma = \frac{m+1}{m} = 2$

$\Gamma = 1$: $w_\phi = w_M$ scaling: $\frac{\Omega_\phi}{\Omega_M} = \text{const.}$ $\rho = \rho_0 a^{-3(1+w)}$

$\Gamma > 1$: $w_\phi < w_M$ tracking: Ω_ϕ catches up Ω_M $a^{-3(1+w)}$

- $\underline{\text{Tracker solutions}}$



$$(a)\quad \frac{1}{6}\frac{d\ln x}{dN} = \lambda\sqrt{\frac{\Omega_\phi}{3(1+w_\phi)}} - 1 \equiv \Delta(t) - 1 \qquad (b)\quad \frac{d\ln x}{dN} = \frac{2}{1-w_\phi^2}\frac{dw_\phi}{dN}$$

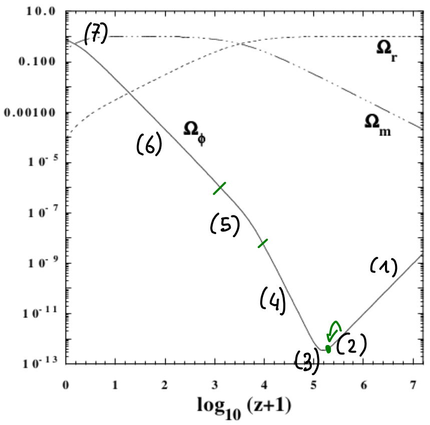
$$\Rightarrow (a)=(b):\ \frac{dw_\phi}{dN} = 3(\Delta(t)-1)(1-w_\phi^2) \longrightarrow \boxed{\frac{dw_\phi}{dN} \gtrless 0 \text{ for } \Delta \gtrless 1} \quad (w_\phi \le 1)$$

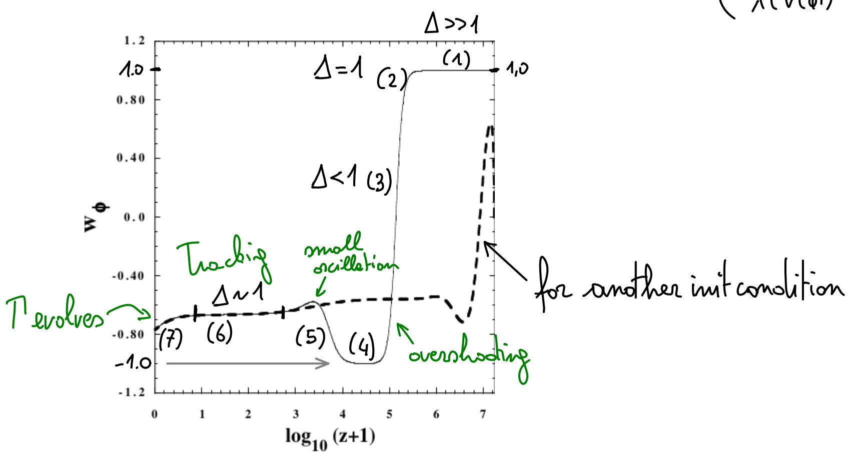
$\boxed{\Delta(t) \sim 1}$ $\underline{\text{tracker solution}}$ $w_\phi \sim$ const.

$$\Rightarrow (a) \simeq 0 : \quad \boxed{\Omega_\phi \simeq \frac{3(1+w_\phi)}{\lambda^2}} \qquad \text{like critical point (d)}$$

- $\underline{\text{Full solution for } \rho_m,\ \rho_r,\ \phi}$: 3D phase-space

$$\Omega_m = 1 - \underbrace{x_1^2 - x_2^2}_{\Omega_\phi} - \underbrace{x_3^2}_{\Omega_r} \qquad (\Omega_{\dot\phi}\ \Omega_V) \qquad x_3 \equiv \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \qquad \left\{\begin{array}{l} x_1'(x_1,x_2,x_3) \\ x_2'(x_1,x_2,x_3) \\ x_3'(x_1,x_2,x_3) \\ \lambda'(V(\phi)) \end{array}\right.$$





Part III

Coupled dark energy

Coupled Quintessence

- $\Omega_{\phi,0} \sim \Omega_{m,0}$ ⇒ there might be some physical relationship between matter and ϕ
  e.g. coupling between quintessence ϕ and dark matter

- Different ways to include such coupling
  - $f(R)$ gravity
  - dilaton gravity
  - scalar tensor theories (eg. Brans-Dicke)
    coupling between a scalar field ϕ and Ricci scalar R
    results in a coupling Q between ϕ and non-relativistic matter in Einstein frame
  - Fluid description of coupled DE (now)
    Scalar field (like Higgs field) → can couple to any other matter component (not only self interaction)
    introduce interaction $\Gamma \rho_m$ in the continuity eq. and $\Gamma = \delta \cdot H$ (Γ is related to Hubble parameter)
    └ dimensionless coupling $\in \mathbb{R}$
  - coupled DE indistinguishible from modified gravity

- Coupled quintessence with exponential potential

$$S = \int \sqrt{-g}\, d\Omega \left[R + \mathcal{L}_\phi(\phi) + \mathcal{L}_M(\phi) \right] : \quad T_{\mu\nu}(\mathcal{L}) \Rightarrow \nabla_\mu T^{\mu\nu}_{tot} = \nabla_\mu T^{\mu\nu}_{(r)} + \nabla_\mu T^{\mu\nu}_{(m)} + \nabla_\mu T^{\mu\nu}_{(\phi)} = 0 \models \boxed{\begin{array}{l} \nabla_\mu T^{\mu\nu}_{(m)} = J^\nu \\ \nabla_\mu T^{\mu\nu}_{(\phi)} = -J^\nu \end{array}}$$

interaction

$$\nabla_\mu T^{\mu\nu}_{(r)} = 0$$

- Include interaction term in Lagrangian
- J^ν expresses the energy-momentum exchange between (m) and (ϕ)
- Example: $\boxed{J_\nu = Q\, T_m \nabla_\nu \phi}$ (Brans-Dicke theory) ideal, homogeneous, isotropic fluid: $T^{(M)}_{\mu\nu}$, trace $T_M = -\rho_M + 3P_M$
- in general Q_{DM} dark-matter $\neq Q_b$ baryons
- radiation has $T^{(r)}_{\mu\nu}$ traceless ⇒ no interaction
- the specie with $Q \neq 0$ do not follow geodesics (extra force, not free particles): The 5th force!
- Later on we will distinguish dark matter and baryons

meaning	$\nabla_\mu T^{\mu\nu} = 0$	conserved ⇒ no energy-momentum exchange
	$\nabla_\mu T^{\mu\nu} = J^\nu$	$T_{\mu\nu}$ not conserved ⇒ energy-momentum exchange
	$\nabla_\mu T^{\mu\nu}_{(tot)} = \nabla_\mu \sum_i T^{\mu\nu}_{(i)} = -\cancel{Q T_M \nabla_\nu \phi} + \cancel{Q T_M \nabla_\nu \phi} + 0 = 0$	total energy conservation

Dynamic of the Universe

- We assume:
  - Components: ϕ, ρ_m, ρ_r
  - For simplicity: 1 coupled matter fluid with $Q_\mu = \text{const.}$ (universal: no time/space dependency)
    such $Q = \text{const.}$ arises from Brans-Dike theory after a conformal transf in Einstein frame
  - To give a concrete example: $V(\phi) = V_0 e^{-\lambda\phi} \Rightarrow \lambda = \text{const}$; take $\lambda > 0$ (no loss of generality)
  - $J_i = 0$ isotropy
  - set: $k^2 = \frac{8\pi G}{c^4} \equiv 1$

- Eq. of motion

For $\nu = 0$ (energy "conservation") in FLRW background and conformal time τ

$$\begin{cases} \nabla_\mu T^\mu{}_{0(\phi)} = -Q T_\mu \nabla_\nu \phi \\ \nabla_\mu T^\mu{}_{0(m)} = +Q T_m \nabla_\nu \phi \\ \nabla_\mu T^\mu{}_{0(r)} = 0 \\ \text{Friedmann eq.} \end{cases} \Rightarrow \begin{cases} \dot\rho_\phi + 3H(\rho_\phi + P_\phi) = -Q\rho_m\dot\phi & (2a) \\ \dot\rho_m + 3H(\rho_m + 0) = Q\rho_m\dot\phi \\ \dot\rho_r + 3H(\rho_r + \frac{1}{3}\rho_r) = \dot\rho_r + 4H\rho_r = 0 & (**) \\ H^2 = \frac{1}{3}(\rho_\phi + \rho_m + \rho_r) & (*) \end{cases}$$

$$\begin{cases} \rho_\phi = \frac{1}{2}\dot\phi^2 + V \\ P_\phi = \frac{1}{2}\dot\phi^2 - V \end{cases} \Rightarrow (2a) \quad \cancel{\tfrac{1}{2}}\cancel{2}\cancel{\dot\phi}\ddot\phi + V_{,\phi}\cancel{\dot\phi} + 3H\dot\phi^{\cancel{2}} = -Q\rho_m\cancel{\dot\phi} \qquad \boxed{\ddot\phi + 3H\dot\phi + V_{,\phi} = -Q\rho_m}$$

source term

- Study dynamic of system as usual ($\lambda = \text{const.}$)
  - $x_1 \equiv \frac{\dot\phi}{\sqrt{6}H}$ ← ! $\quad x_2 \equiv \frac{\sqrt{V}}{\sqrt{3}H} \quad x_3 \equiv \frac{\sqrt{\rho_r}}{\sqrt{3}H} \quad (**)$
  - Friedmann: $\Omega_m = 1 - x_1^2 - x_2^2 - x_3^2 \qquad \Omega_\phi = x_1^2 + x_2^2 \qquad \Omega_r = x_3^2 \qquad \Omega_m = 1 - \Omega_\phi - \Omega_r$

$$w_\phi = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \qquad w_{eff} = x_1^2 - x_2^2 + \frac{1}{3}x_3^2$$

  - Look for fixed points (same method as before)

$$\begin{cases} \frac{dx_1}{dN} = -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 - x_1\frac{1}{H}\frac{dH}{dN} - \underbrace{\frac{\sqrt{6}}{2}Q\left(1 - x_1^2 - x_2^2 - x_3^2\right)}_{(!)} \\ \frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 - x_2\frac{1}{H}\frac{dH}{dN}, \\ \frac{dx_3}{dN} = -2x_3 - x_3\frac{1}{H}\frac{dH}{dN}. \end{cases}$$

$\frac{d}{dN}$(Friedmann): $\frac{1}{H}\frac{dH}{dN} = -\frac{3}{2}\left(1 + x_1^2 - x_2^2 + \frac{1}{3}x_3^2\right)$

$$\boxed{\text{KG: } \ddot\phi = -(3H\dot\phi^2 + V_{,\phi} + \underbrace{Q\rho_m}_{(!)}) \quad \rho_m = \Omega_m \qquad \dot x_1 = \frac{k}{\sqrt{6}}\left(\frac{\ddot\phi H - \dot\phi\dot H}{H^2}\right)}$$

$\frac{dx_i}{dN} \stackrel{!}{=} 0 \Rightarrow$ Critical points

• We will focus on relevant eras: radiation → matter → acceleration

Name	x_1	x_2	x_3	Ω_ϕ	Ω_r	w_ϕ	$w_{\rm eff}$	
! (a)	$-\frac{\sqrt{6}Q}{3}$	0	0	$\frac{2Q^2}{3}$	0	1	$\frac{2Q^2}{3}$	matter (ϕDM era) saddle for $Q(Q+\lambda)>-\frac{3}{2}$
(b1)	1	0	0	1	0	1	1	
(b2)	-1	0	0	1	0	1	1	
! (c)	$\frac{\lambda}{\sqrt{6}}$	$(1-\frac{\lambda^2}{6})^{1/2}$	0	1	0	$-1+\frac{\lambda^2}{3}$	$-1+\frac{\lambda^2}{3}$	acceleration
! (d)	$\frac{\sqrt{6}}{2(Q+\lambda)}$	$[\frac{2Q(Q+\lambda)+3}{2(Q+\lambda)^2}]^{1/2}$	0	$\frac{Q(Q+\lambda)+3}{(Q+\lambda)^2}$	0	$\frac{-Q(Q+\lambda)}{Q(Q+\lambda)+3}$	$\frac{-Q}{Q+\lambda}$	matter acceleration
! (e)	0	0	1	0	1	–	$\frac{1}{3}$	radiation era
(f)	$-\frac{1}{\sqrt{6}Q}$	0	$(1-\frac{1}{2Q^2})^{1/2}$	$\frac{1}{6Q^2}$	$1-\frac{1}{2Q^2}$	1	$\frac{1}{3}$	
(g)	$\frac{2\sqrt{6}}{3\lambda}$	$\frac{2\sqrt{3}}{3\lambda}$	$(1-\frac{4}{\lambda^2})^{1/2}$	$\frac{4}{\lambda^2}$	$1-\frac{4}{\lambda^2}$	$\frac{1}{3}$	$\frac{1}{3}$	

• Radiation era : e, f, g where $w_{\rm eff}=1/3$

BBN constrains
- (f) $Q^2>3.7$ not compatible with presence of matter domination era
- (g) $\lambda^2>88.9$ λ is too large to allow for acceleration (see later why)

$\Rightarrow$ excluded

$\Rightarrow$ (e) is the only possible radiation domination with $\mu_i=-1,1,2$: saddle to escape toward matter domination

• Matter era : a, d both scaling solutions with $\Omega_\phi=\text{const}$, $\Omega_r=0$ $\Rightarrow$ $\Omega_m=1-\text{const}$. $\frac{\Omega_\phi}{\Omega_m}=\text{const}'$

a: for $Q\ll 1$
d: for $|\lambda|\gg|Q|$
$\Big\}$ $w_{\rm eff}\approx 0$, $\Omega_\phi\approx 0$, $\Omega_r=0$, $\Omega_m=1$ (flat)

a: $\mu_i=\underbrace{\frac{3}{2}+Q(Q+\lambda)}_{>0},\ \underbrace{-\frac{3}{2}+\overset{\sim 0}{Q^2}}_{<0},\ \underbrace{-\frac{1}{2}+\overset{\sim 0}{Q^2}}_{<0}$ (!!) (called ϕ-matter dominated epoch ϕMDE) (not negligible)

(careful: $\mu_1<1$ if $Q<0$ and $\lambda\gg 1$ even if $Q\ll 1$)

saddle for $Q(Q+\lambda)>-\frac{3}{2}$ needed to escape toward late time acceleration ☺

Scaling! $w_\phi=\frac{2Q}{3}$ $\Rightarrow$ $\rho_\phi=\rho_{\phi 0}\,a^{-3(1+w_\phi)}=\rho_{\phi 0}\,a^{-3-2Q}\sim\rho_{\phi,0}\,a^{-3}$ $\quad\frac{\Omega_\phi}{\Omega_m}\sim\text{const.}$

d: $\mu=-\frac{4Q+\lambda}{2(Q+\lambda)},\ -\frac{3(2Q+\lambda)}{4(Q+\lambda)}\left[1\pm\sqrt{1+\frac{8[3-\lambda(Q+\lambda)][3+2Q(Q+\lambda)]}{3(2Q+\lambda)^2}}\right]>0$

stable node or stable spiral : can not escape toward accelerated phase ☹

• Accelerated era : c, d

c: $\Omega_\phi=1$ $\Omega_r=0=\Omega_m$

$\mu_i=\underbrace{\frac{1}{2}(\lambda^2-4)}_{<0},\ \underbrace{\frac{1}{2}(\lambda^2-6)}_{<0},\ \underbrace{\lambda(Q+\lambda)-3}_{<0\ (*)}$ accelerated for $\lambda^2<2$, (*) stable for $\lambda(Q+\lambda)<3$

d: acceleration for $\lambda<2Q$ or $\lambda>-Q$ (recall, $\lambda>0$) stable for $\lambda(Q+\lambda)>3$ (see eigenvalue μ_3)

• scaling solution allows global accelerated attractor $\Omega_\phi\simeq 0.7$

◦ $\lambda(Q+\lambda)\gtrless 3$ decides if we go to c or d (!)

- Summary of valuable critical points

	Name	x_1	x_2	x_3	Ω_ϕ	Ω_r	w_ϕ	w_{eff}	Constraints
Radiation	(e)	0	0	1	0	1	–	$\frac{1}{3}$	—
↓ Matter	(a)	$-\frac{\sqrt{6}Q}{3}$	0	0	$\frac{2Q^2}{3}$	0	1	$\frac{2Q^2}{3}$	matter domin: $\lvert Q\rvert \ll 1$, $Q(Q+\lambda) > -\frac{3}{2}$, saddle
↓ Acceleration	(c)	$\frac{\lambda}{\sqrt{6}}$	$(1-\frac{\lambda^2}{6})^{1/2}$	0	1	0	$-1+\frac{\lambda^2}{3}$	$-1+\frac{\lambda^2}{3}$	for acceleration $\lambda^2 < 2$
Or	(d)	$\frac{\sqrt{6}}{2(Q+\lambda)}$	$[\frac{2Q(Q+\lambda)+3}{2(Q+\lambda)^2}]^{1/2}$	0	$\frac{Q(Q+\lambda)+3}{(Q+\lambda)^2}$	0	$\frac{-Q(Q+\lambda)}{Q(Q+\lambda)+3}$	$\frac{-Q}{Q+\lambda}$	$\lambda < 2Q$ or $\lambda > -Q$

$\lambda(Q+\lambda) < 3 \Rightarrow$ (c)
$\lambda(Q+\lambda) > 3 \Rightarrow$ (d)
for stability

(c) is more likely:

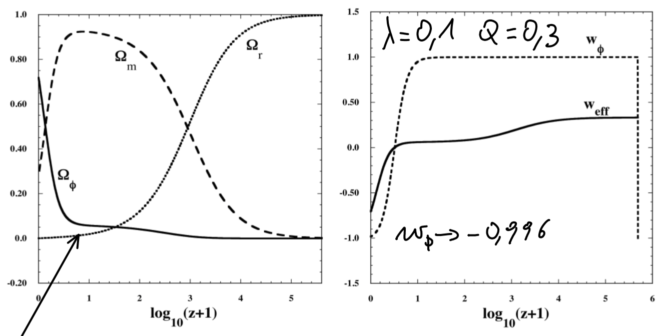
conflicting conditions for $\underset{\text{matter}}{(a)} \rightarrow \underset{\text{accelerat.}}{(d)}$ because (a) requires $Q^2 \ll 1$
(d) requires Q large to have sufficient acceleration

- Overall:

Valuable path: (e) $\longrightarrow$ (a) $\longrightarrow$ (c) (only possibility)

(a): $(Q \ll 1,\ Q(Q+\lambda) > -3/2)$ — $\Omega_m = 1$, $w_\phi = 0$, saddle

(c): $(\lambda^2 < 2,\ \lambda(Q+\lambda) < 3)$ — $w_{\text{eff}} < -1/3$, stable



ϕMDE: $\Omega_\phi = w_{\text{eff}} = \frac{2}{3}Q^2 \simeq 0{,}06$

- Constraints on observations

  - ϕMDE $\Rightarrow$ Q affects Hubble expansion
    this increases the distance to the last scattering surface
    c_s at decoupling is smaller, (7% for $Q = 0{,}1$) large but compensated by
    CMB data says that Q can not be larger than $0{,}1$

  - Q affects evolution of perturbations $\dot{\rho}_m + 3H(\rho_m + 0) = \underline{Q\rho_m\dot{\phi}}$ $\Rightarrow$ effect on growth factor
    $\rightarrow$ galaxy clustering, BAOs
    $\rightarrow$ galaxy clusters counts
    * See box next page

  - Local gravity constraints + Careful with baryons coupling

  - There is a degeneracy with the mass of neutrinos

*

$Q \neq 0$ and cosmic structure formation

- New force is unscreened ⇒ extend to large distances

⇒ it appears as a modification of gravity

$$\boxed{\nabla^2\phi = -4\pi G\, f(a,r)\, a^2 \hat{\rho}\,\delta}$$ modified Poisson eq.

$f(a,r)$: some function of time and space

- coupling, i.e. the 5th force, affects cosmic structure formation

growth eq. $\delta_m'' + \frac{1}{2}(1-3w_{eff})\delta_m' - \frac{3}{2} f(a,r)\,\Omega_m \delta_m = 0$ (we will look into that later...)

⇒ could be characterized by observations (e.g. galaxy clustering, CMB, BAOs, Clusters, ...)

more later...

- Particles with an evolving mass?!

$$\frac{\dot\rho_m + 3H\cancel{\rho_m} = Q\cancel{\rho_m}\dot\phi}{\rho_m} \qquad \frac{\dot\rho_m}{\rho_m} = Q\dot\phi - 3\frac{\dot a}{a} \qquad \log\rho_m\Big|_0^{\rho_m} = Q(\phi-\phi_0) - 3\log a\Big|_0^1 \qquad \rho_m = \rho_{m0}\, a^{-3} e^{Q(\phi-\phi_0)}$$

$Q = \text{const}!$

(assumptions: matter composed by only one particle with a given mass; number conserved)

$$= n_{m0}\cdot \underline{m}\, a^{-3}\, \underline{e^{Q(\phi-\phi_0)}} = n_{m0}\, m(\phi)\, a^{-3}$$

$$\boxed{m(\phi) = m_0 \exp\left(\int_{\phi_0}^{\phi} Q(\phi')\,d\phi'\right)}$$ variation of the particle mass because of coupling $\phi(t,\vec x)$ in general!

in newtonian limit: $\ddot{\vec r} = \frac{Gm}{r^3}\vec r$ can be seen as varying G; current cosmological constraints: $|\dot G/G| \leq 10^{-11}\,\text{yrs}^{-1}$

"trick" used in numerical simulations

- An issue with baryons

if baryons would couple ($Q_b \neq 0$) $\Rightarrow$ m_e evolves and it might even depend on position
$\rightarrow$ but many observations implicitly assume $m = \text{const.}$

e.g. ionization energy of H ($E = 13{,}6$ eV) depends on Rydberg constant $R_\infty = \frac{m_e e^4}{8\varepsilon_0^2 h^3 c}$

R_∞ evolves because of $m_e(\phi,\vec x)$
$\downarrow$
this affects spectral emission/absorbtion (redshift measures!)
$\downarrow$
need to reinterpret everything!
$\downarrow$
keep $\nabla_\mu T^{\mu\nu}_{(b)} = 0$ or it is a mess....
$\Downarrow$
no 5^{th} force for baryons: they follow geodesics

(or need to introduce screening to save local gravity measures and keep $m_e = \text{const.}$)

$Q_b \neq 0$ but chameleon mechanism: screens interaction near massive objects
↳ 5^{th} force not in the solar system

Chameleon

scalar field which effective mass depends on environment
if ρ_m is high $\Rightarrow$ $m_\phi \uparrow$ $\Rightarrow$ field can not propagate freely (screening)

Carefull: if you decouple baryons $Q_{DM} \neq 0$ but $Q_b = 0$
you have another degree of freedom because of ρ_m

$$\dot\rho_{DM} + 3H\rho_{DM} = Q\rho_{DM}\dot\phi$$
$$\dot\rho_b + 3H\rho_b = 0 \longrightarrow x_4 = \frac{\sqrt{\rho_b}}{\sqrt{3}H}$$

Part IV

Modified Gravity

Modified gravity and dark energy: conformal transformations

Goal

- Explain accelerated expansion and possible matter-DE coupling in terms of a modification of gravity instead of an additional fluid
- Theories must respect data: local gravity constraints, e.g. perihelion shift, Lens-Tirring effect, orbit of Pioneer, Moon, ...
  cosmology: similar to ΛCDM
- Note: DE, 5^{th} force not distinguishable from modification of gravity (transform one into the other)

Theories

- GR is the only theory of gravity in 4D, 2° order eq. of motion based on one tensor field $g_{\mu\nu}$ (Lovelock theorem)
- Non-GR, modified gravity and DE theories: – $f(R)$ gravity – Gauss-Bonnet gravity – scalar-tensor theories – Braneworld models of DE
- Simplest way to introduce a new degree of freedom: conformal transformations

A powerful tool: conformal transformations

- $\boxed{\tilde{g}_{\mu\nu} = f(\phi) g_{\mu\nu}}$ frame transformation (of the metric), not a coordinate transformation
  conformal transf.: angles between 4-vectors are not affected (just rescaling of scalar product)
  same signature
  f invertible (positive definite) such to have inverse transf. $g_{\mu\nu} = f^{-1} \tilde{g}_{\mu\nu}$
  new deg. of freedom: $\phi(x^\mu)$ scalar field
  2 main applications, choose f to: (1) simplify equations or (2) modify gravity

- Key relationships: (see for e.g. Carroll)
  - $g^{\mu\nu} = f^{-1} \tilde{g}^{\mu\nu}$
  - $\sqrt{-\tilde{g}} = f^2 \sqrt{-g}$ because $\tilde{g} = f(\vdots\vdots\vdots) \Rightarrow f^4$
  - $\tilde{\nabla}_\mu \phi = \partial_\mu \phi = \nabla_\mu \phi$
  - $\tilde{\Box} = f^{-1}(2 w_{,\mu} g^{\mu\nu} \partial_\nu + \Box)$ with $w_{,\mu} \equiv f_{,\mu}/2f \overset{*}{=} \dfrac{f_{,\phi}\phi_{,\mu}}{2f}$ * recall: $f(\phi(x^\nu))$
  - $\tilde{R} = R(\tilde{g}_{\mu\nu}) = f^{-1}(R - 6 w_{,\mu} g^{\mu\nu} w_{,\nu} - 6 \Box w)$
  - $\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + C^{\rho}_{\mu\nu}$ because Γ coeff are linear in derivatives of g
    C is a tensor because difference of two connection

- Electromagnetism and conformal transformations

$$\mathcal{L}_{EM} = \sqrt{-g}\,F_{\mu\nu}F^{\mu\nu} = \underline{\sqrt{-g}}\,\underline{F_{\mu\nu}F_{\alpha\beta}\,g^{\alpha\mu}g^{\beta\nu}} = f^2\sqrt{-\tilde g}\,F_{\mu\nu}F_{\alpha\beta}\,f\tilde g^{\alpha\mu}\,f\tilde g^{\beta\nu} = \underline{\sqrt{-\tilde g}\,F_{\mu\nu}F_{\alpha\beta}\,\tilde g^{\alpha\mu}\tilde g^{\beta\nu}}$$

$\Rightarrow$ Electromagnetism is conformal invariant $\Rightarrow$ Maxwell's eq.s are not affected!
no coupling to electromagnetism

- Model with matter + scalar field (set $\frac{1}{16\pi G} \equiv 1$)

$$S = \int \sqrt{-g}\,[R + \mathcal{L}_\phi(g_{\mu\nu},\phi) + \mathcal{L}_m(g_{\mu\nu},\psi)]\,d\Omega \quad \longleftarrow \text{Einstein frame} \qquad \mathcal{L}_\phi = -\tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - V \qquad \psi = \text{matter field (baryons, DM)}$$

yes, minus

$$= \int f^{-2}\sqrt{-\tilde g}\,[f^{-1}(\tilde R - \underbrace{6\,\tilde g^{\mu\nu}w_{,\mu}w_{,\nu} + 6\,\tilde\Box w}_{\circledast}) + \mathcal{L}_\phi(f^{-1}\tilde g_{\mu\nu},\phi) + \mathcal{L}_m(f^{-1}\tilde g_{\mu\nu},\psi)]\,d\Omega$$

$$= \int \sqrt{-\tilde g}\,\{f^{-1}\tilde R + \underbrace{[f^{-2}\mathcal{L}_\phi(f^{-1}\tilde g_{\mu\nu},\phi) + f^{-1}(6\,\tilde\Box w - 6\,\tilde g^{\mu\nu}w_{,\mu}w_{,\nu})]}_{(1)} + \underbrace{f^{-2}\mathcal{L}_m(f^{-1}\tilde g_{\mu\nu},\psi)}_{(2)}\}\,d\Omega$$

$$S' = \int\sqrt{-\tilde g}\,[\underbrace{f^{-1}\tilde R}_{(3)} + \underbrace{\tilde{\mathcal{L}}_\phi(\tilde g_{\mu\nu},\phi)}_{(1)} + \underbrace{\tilde{\mathcal{L}}_m(\tilde g_{\mu\nu},\psi)}_{(2)}]\,d\Omega \quad \longleftarrow \text{Jordan frame}$$

$\circledast$ has structure of a kinetic term which can be merged with the one of $\mathcal{L}_\phi$

(1) scalar field with a different representation
(2) coupling scalar field ϕ and matter : modification of eq. of motion : 5^{th} force
(3) " " " " " R : modification of gravity (it is not the Hilbert action) (!)
scalar-tensor theory, generalized Brans-Dicke theory

Note: • S and S' describe the same physics! Just two different frames: Einstein fr. $\leftrightarrow$ Jordan fr.
no new real degree of freedom here, didn't change physics, same structure with one scalar field, same theory, just a different representation of the fields
• The modifications in $f^{-1}R$, $\tilde{\mathcal{L}}_\phi$, $\tilde{\mathcal{L}}_M$ compensate each other by construction

- $\tilde{\mathcal{L}}_\phi$ is really a canonical scalar field, it can be brought to $-\frac{1}{2}\psi_{,\mu}\psi^{,\mu} + U(\psi)$

• $\mathcal{L}_\phi = -\frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)$ canonical lagrangian

equivalent lagrangian, same eq. of motion

• $\tilde{\mathcal{L}}_\phi = -\frac{1}{2}\underline{f^{-1}}\,\tilde g^{\mu\nu}\,\underline{\left[1 - 3\left(\frac{f'}{f}\right)^2\right]}\phi_{,\mu}\phi_{,\nu} - f^{-2}V(\phi)$ $\xleftarrow{\text{equivalent to}}$ $S_\phi = \int\sqrt{-\tilde g}\,[f^{-2}\mathcal{L}_\phi(f^{-1}\tilde g_{\mu\nu},\phi) + f^{-1}(\underbrace{6\,\tilde\Box w}_{(1)} - \underbrace{6\,\tilde g^{\mu\nu}w_{,\mu}w_{,\nu}}_{(2)})]\,d\Omega$

$= -\frac{1}{2}\tilde g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - U(\varphi)$ make it canonical

$\varphi = f^{-1/2}\left[1 - 3\left(\frac{f'}{f}\right)^2\right]^{1/2}\phi_{,\mu}$ "new field"

$U(\varphi) \equiv f^{-2}(\phi)\,V(\phi)\big|_{\phi(w)}$ "new potential"

(1) integrate by part (f' pops up)
(2) obvious kinetic term $w_{,\mu}w^{,\mu}$

• $\mathcal{L}_m \rightarrow \tilde{\mathcal{L}}_m = f^{-2}\mathcal{L}_m(f^{-1}\tilde g, \psi)$ is changed $\Rightarrow$ modified eq. of motion ... 5^{th} force (coupling $M-\phi$)

Practical applications

(1) Conformal transformations to simplify math

e.g. theory with an explicit coupling

(scalar-vector theories)

$$\boxed{S = \int \sqrt{-g}\,[F(\phi)R + \mathcal{L}_\phi(g,\phi) + \mathcal{L}_M(g,\psi)]\,d\Omega}$$

(Jordan frame)

$\mathcal{L}_g = \sqrt{-g}\,F(\phi)R(g)$ Modified gravity $\mathcal{L}_g \neq \mathcal{L}_H$
$\mathcal{L}_\phi$ Canonical lagrangian $\nabla_\mu T^{\mu\nu}_{(\phi)} = 0$
$\mathcal{L}_M$ uncoupled matter field: $\nabla_\nu T^{\mu\nu}_{(M)} = 0$

conformal $\downarrow$ transformation

(coupled dark energy)

$$\boxed{\tilde{S} = \int \sqrt{-\tilde{g}}\,[\tilde{R} + \tilde{\mathcal{L}}_\phi(\tilde{g},\phi) + \tilde{\mathcal{L}}_M(\tilde{g},\psi)]\,d\Omega}$$

(Einstein frame)

$\mathcal{L}_H = \sqrt{-\tilde{g}}\,R(\tilde{g})$ Hilbert action (GR)
$\tilde{\mathcal{L}}_\phi$ with coupling $\nabla_\mu T^{\mu\nu}_{(\phi)} = -J^\nu$
$\tilde{\mathcal{L}}_M$ with coupling $\nabla_\mu T^{\mu\nu}_{(M)} = J^\nu$

S: no coupling $\Rightarrow$ e.g. standard interpretation of redshift :) non-GR $\Rightarrow F(G_{\mu\nu}) = T_{\mu\nu}$ rederive cosmology :(
S': coupling $\Rightarrow$ need to reinterpret data, e.g. redshift :(GR $\Rightarrow G_{\mu\nu} = T_{\mu\nu}$ "standard" cosmology :)
(if no screening mechanisms)

Carefull

$$\boxed{S = \int \sqrt{-g}\,[F(\phi)R + \mathcal{L}_M(g,\psi)]\,d\Omega}$$

Modified gravity + matter field : NO scalar field $\mathcal{L}_\phi$!

conformal $\downarrow$ transformation

$$\boxed{\tilde{S} = \int \sqrt{-\tilde{g}}\,[\tilde{R} + \tilde{\mathcal{L}}_\phi(\tilde{g},\phi) + \tilde{\mathcal{L}}_m(\tilde{g},\psi)]\,d\Omega}$$

Einstein-Hilbert + fluid + Coupled dark matter-fluid

↑ (Einstein-Hilbert lagrangian)

↳ recall: the conformal transformation introduces a kinetic term $6\tilde{g}^{\mu\nu}w_{,\mu}w_{,\nu} + 6\tilde{\Box}w$
this contribution does not get "swallowed" by an already present scalar field $\mathcal{L}_\phi$ (in S)
$F(\phi)$ introduces a real new deg. of freedom $\rightarrow$ Scalar field with $V(\phi)=0 \Rightarrow w_\phi = 1$

- Mixed frames: if DM and Baryons are both coupled but in a different way : $Q_b \neq Q_{DM}$
  $\Rightarrow$ you can achieve $\nabla_\mu T^{\mu\nu}_{(b)} = 0$ but $\nabla_\mu T^{\mu\nu}_{(DM)} = \delta \cdot \rho_{DM}$ with a specific $F(\phi)$
  ↳ called "physical frame"

- What to do?
  - equivalent physical representations
  - choose one frame or the other according to what you want to address
  - usually you do the math in Einstein frame, move to Jordan frame to compare with observations

(2) Generation of new theories

these two actions are two different theories, not the same physics! note, booth in Einstein frame

- $S = \int \sqrt{-g}\,[R + \mathcal{L}_M(g, \Psi)]\,d\Omega$
- $\tilde{S} = \int \sqrt{-\tilde{g}}\,[\tilde{R} + \tilde{\mathcal{L}}_m(\tilde{g}, \Psi)]\,d\Omega$ $\quad \tilde{g} = f(\phi)\,g = g(g, \phi)$ new deg. of freedom ϕ

$\delta\tilde{S} = 0$, variation of ϕ => new terms, eg. $f_{,\phi}\,\phi_{,\mu}\,dx^\mu$

different eq. of motion $\longleftarrow$ "conformal coupling"

More general transformations

- This can be generalized with non-conformal transf. e.g. $\tilde{g}_{\mu\nu} = f(\phi)\,g_{\mu\nu} + \phi_{,\mu}\phi_{,\nu}$
- There are other transformations, $\tilde{g}_{\mu\nu} = [\text{anything}]\,g_{\mu\nu}$, examples:

1) $\tilde{g}_{\mu\nu} = (f + \underbrace{\phi_{,\mu}\phi^{,\mu}}_{g_{\mu\nu}\phi^{\nu}\phi^{\mu}})\,g_{\mu\nu}$ disformal transformation

not just a factor, it is a non linear relation: quadratic relation to $g_{\mu\nu}$

All theories (1) satisfying this condition have been classified

2) $\tilde{g}_{\mu\nu} = [f + R(g_{\mu\nu})]\,g_{\mu\nu}$ quite complicated: $R(g_{\mu\nu})$ contains derivatives of $g_{\mu\nu}$

.....

∞) .....

- All freedom you want but constrain: $\tilde{g}_{\mu\nu}$ invertible, invertibility! you want inverse transf $g_{\mu\nu} = (\ldots)\,\tilde{g}_{\mu\nu}$ or you get singularities

F(R) theories

- We have seen

$R + \mathcal{L}_\phi(g,\phi) + \mathcal{L}_M(g,\psi)$ dark energy

$R + \mathcal{L}_\phi(gf,\psi) + \mathcal{L}_M(gf,\psi)$ introduce f coupling

$f^{-1}R + \mathcal{L}_\phi(g,\psi) + \mathcal{L}_M(g,\psi)$ modified gravity

Scalar-Tensor theory

we will see that

- f(R) theories

$S = \int d\Omega \sqrt{-g}\, f(R)$ the simplest, safest, most investigated one

- Motivation

· Think of the most general theory of gravity: you can put in lagrangian all possible scalars based on curvature

· Full gravity theory: $\mathcal{L} = R + f(R)$ } the simplest and safest $f(R) \simeq R + \alpha R^2 + \beta R^3 + \dots$

$+ \gamma R_{\mu\nu}R^{\mu\nu}$
$+ \sigma R R_{\mu\nu}R^{\mu\nu} + \dots$ } likely (... certainly) give rise to instabilities

· first one: Starobinski (1980, before Guth inflation) $R + \alpha R^2$ naturally provides inflation
early universe curvature was larger: $R \sim H^2 \Rightarrow R^2$ was more important at early times

- Eq. of motion in f(R), key difference with GR

$S = \int d\Omega \sqrt{-g}\, f(R) \qquad \delta S = 0 \qquad \delta\left[\sqrt{-g}\, f(R)\right] = \underbrace{f(R)\,\delta\sqrt{-g}}_{(1)} + \underbrace{\sqrt{-g}\,\delta f(R)}_{(2)}$

(1) "standard GR" but with $f(R)$ instead of $R \Rightarrow$ give rise to $f(R)$ factor

(2) newish: $(2) = \sqrt{-g}\,\frac{\delta f}{\delta R}\,\delta R = \sqrt{-g}\, f'(\underbrace{\delta g_{\mu\nu} R^{\mu\nu}}_{(2a)} + \underbrace{g_{\mu\nu}\,\delta R^{\mu\nu}}_{(2b)}) \qquad f' = f_{,R} \qquad R = g_{\mu\nu}R^{\mu\nu}$

2a: "standard GR" but with f' factor

2b: new: $g_{\mu\nu}\,\delta R^{\mu\nu} \equiv V^\mu{}_{;\mu}$ is a total divergence of a vector

in GR $\int d\Omega \sqrt{-g}\, V^\mu{}_{;\mu}$ irrelevant (integrate by part + Gauss theorem)

here $\int d\Omega \sqrt{-g}\, f'\, V^\mu{}_{;\mu}$ relevant (integrate by part) $\Rightarrow$ new term, function of f, f', f''

$$\Rightarrow \boxed{\overset{(2a)}{f'(R)}\underbrace{R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu}}_{\text{standard}}\overset{(1)}{f(R)} \underbrace{- \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R)}_{(2b)} = \underbrace{\frac{8\pi G T_{\mu\nu}}{c^4}}_{\text{standard}}}$$

eq. of motion of $f(R)$ in metric formalism

Trace: $3\Box f_{,R} + f_{,R}R - 2f = \kappa^2 T$

$(GR:\ f(R) = R \quad f'(R) = 1 \quad \nabla_\nu f' = \nabla_\nu 1 = 0)$

- A side remark: Metric / Palatini formalisms

1) Metric formalism: $S(g_{\mu\nu})$ only $g_{\mu\nu}$ is free field
recall: $\Gamma(g_{\mu\nu})$ only if you assume metric compatibility ($\nabla_\mu g^{\mu\nu}=0$) and Torsion free ($T^\gamma_{\mu\nu}=0$)

2) Palatini formalism: $S(\Gamma^\alpha_{\mu\nu}, g_{\mu\nu})$ also Γ is a free field but with $\sqrt{-g}R$ you still get GR

With modified gravity lagrangians (eg. $f(R)$ gravity), the two formalisms give $\neq$ eq. of motions

- How many $f(R)$ theories?

Thousands! $\rightarrow$ common points: respect conservation laws
we want $\ddot{a}>0$ at late times (not like Starobinski)
not too far from ΛCDM behavejour

Possible $f(R)$

- $f(R)=R+\alpha R^2$ $\rightarrow$ inflation, OK but not good for DE (Starobinski)
- $f(R)=R+1/R$ $\rightarrow$ surely an effect at late times
you get DE late time $\ddot{a}>0$ but not a good past
during matter era it goes through a singularity becaus of $1/R$
- $f(R)=R-\mu R_c \dfrac{(R/R_c)^{2n}}{(R/R_c)^{2n}+1}$ n, R_c = free constant parameters of the theory μ = dimensionless
this one works

$\rightarrow$ Without even deriving the eq. of motion, you see immediately:
1) no singularity
2) if R is very small (past) $\Rightarrow$ $f(R)\rightarrow R$ (GR) :)
3) " " " " large (present) $\Rightarrow$ $f(R)\rightarrow R-\mu R_c$ $\mu R_c \equiv \Lambda \sim H_0^2$ (data) :)
GR – cosmological const.

$\Rightarrow$ connects current acceleration with an healthy past

$$f(R)=\frac{R+R\left(\frac{R}{R_c}\right)^{2n}-\mu R_c\left(\frac{R}{R_c}\right)^{2n}}{(R/R_c)^{2n}+1}\simeq R$$

- More on good/bad models later...

- **Classification of $f(R)$ theories**: Phase-space analysis

0) Consider: FLRW (flat) $\Rightarrow R = 6(2H^2+\dot H)$; $f(R)$, ρ_m, ρ_r rename $F \equiv f' \equiv f_{,R}$

1) Rederive Friedmann eq., more complicated now with H^3, H^4 terms

2) define dimensionless parameters: $x_1 \equiv -\frac{\dot F}{HF}$ $\quad x_2 \equiv -\frac{f}{6FH^2}$ $\quad x_3 \equiv \frac{R}{6H^2}$ $\quad x_4 = \frac{k^2\rho_r}{3FH^2} = \Omega_r$ radiation

more than just 2 because of other derivatives

such that $\Omega_m = 1 - x_1 - x_2 - x_3 - x_4$ $\quad \Omega_r \equiv x_4$ $\quad \Omega_{DE} \equiv x_1 + x_2 + x_3$

energy density: in $\dot F$, in f, in R

3) From K-G. eq. to set of 1° order differential eq.s

$$\begin{aligned}
\frac{dx_1}{dN} &= -1 - x_3 - 3x_2 + x_1^2 - x_1x_3 + x_4, \\
\frac{dx_2}{dN} &= \frac{x_1x_3}{m} - x_2(2x_3 - 4 - x_1), \\
\frac{dx_3}{dN} &= -\frac{x_1x_3}{m} - 2x_3(x_3 - 2), \\
\frac{dx_4}{dN} &= -2x_3x_4 + x_1x_4,
\end{aligned}$$

with $m(R) \equiv \frac{d\ln F}{d\ln R} = \frac{R f_{,RR}}{f_{,R}} = \frac{x_3}{x_2}\frac{f f_{,RR}}{f_{,R}^2} = m(x_2, x_3)$!*

to get a self contained autonomous set of eq.s i.e. all depending on x_1, x_2, x_3, x_4 $\Rightarrow$ introduce

$$r \equiv -\frac{d\ln f}{d\ln R} = -\frac{R f_{,R}}{f} = \frac{x_3}{x_2} \quad \longrightarrow \quad R = \frac{x_3}{x_2}\frac{f}{f_{,R}} \;*$$

$$\left(x_2 = -\frac{f}{6FH^2} = -\frac{f}{6F}\frac{6x_3}{R} \qquad \frac{x_3}{x_2} = -\frac{f'R}{f} \right)$$

4) Usual analysis of critical points, stability

- A long list of crit. points...

not interesting : unstable (kinetic dominated)

matter points : saddle $\Omega_M \sim 1$

accelerated points : stable $\nearrow$ $\Omega_\phi \to 1$ $\quad \Omega_M \to 0$ (attractor)

$\searrow$ $\Omega_\phi \to$ const $\quad \Omega_M \to 1-$const. (scaling)

- The useful ones are:

P_1: $\Omega_m = 0$, $w_{eff} = -1$ $\qquad \Lambda$ like

P_5: $\Omega_m = 1 - \frac{m(7+10m)}{2(1+m)^2}$, $\quad w_{eff} = -\frac{m}{1+m}$ $\qquad$ Matter: $m \simeq 0$: $\Omega_m \simeq 1$ $\quad w_{eff} \simeq 0$

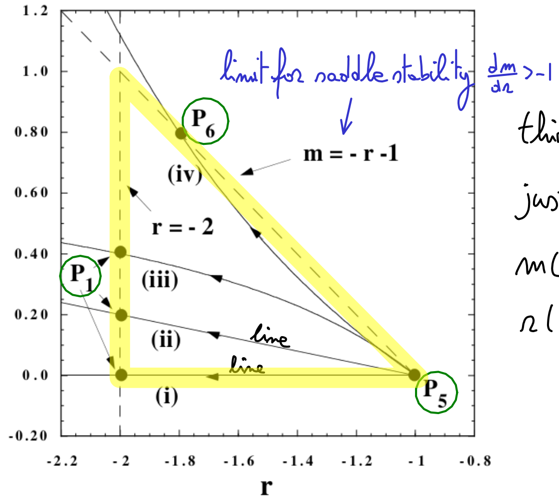
saddle: for $\left.\frac{dm}{dr}\right|_{r=-1} > -1$

P_6: $\Omega_m = 0$, $\quad w_{eff} = \frac{2-5m-6m^2}{3m(1+2m)}$; can provide for acceleration $w_{eff} < -\frac{1}{3}$

P_5 an P_6 along line $m(r) = -r-1$

5) We look for cosmology with the right sequence : (radiation) → (matter) → (acceleration)

all good $f(R)$ theories lay in this region →



this is not a phase-space
just values of m, r given the evolution of R

$m(R)$ ↘
$r(R)$ ↗ $m(r)$ leads all solutions

• Conditions for valid $f(R)$ theories in metric formalism

1) $f_{,R} > 0$ for $R > R_0 > 0$ R_0 = Ricci at $z=0$ to have a de Sitter final attractor + it avoids anti-gravity

2) $f_{,RR} > 0$ for $R \geq R_0$ consistency with local gravity measures + stability of cosmological perturbations

3) $f(R) \to R - 2\Lambda$ for $R \gg R_0$ consistency with local gravity tests + presence of matter domination ΛCDM like

4) $0 < m(r) < 1$ at $r = -2$ $m(r) \equiv \frac{R f_{,RR}}{f_{,R}}$, $r \equiv -\frac{R f_{,R}}{f}$ stability of late time de Sitter

eg. $f(R) = R - \frac{\alpha}{R^n}$ $\alpha > 0$, $n > 0$ breaks (2)

• Example of $f(R)$ theories that goes from P_5 to P_1

(i) • $m(r) = \frac{R f_{,RR}}{f_{,R}} \overset{!}{=} 0$ $\forall$ time $\Rightarrow$ $f_{,RR} = 0$ $f_{,R} = \alpha$ $f = \alpha R + \beta$: $\boxed{f(R) = R - 2\Lambda}$

$\Rightarrow$ m quantifies deviations from ΛCDM

• $m(r) = \text{const}$ $\Rightarrow$ $\frac{\cancel{R} f_{,RR}}{f_{,R}} = \frac{m}{R}$ $\ln f_{,R} = m \ln R + \beta$ $f_{,R} = \alpha R^m$ $f = \alpha R^{m+1} + \beta$ $\boxed{f(R) = \alpha R^{1+m} - 2\Lambda}$ $\sim \Lambda$CDM

(ii) $m(r) = \frac{1-c}{c} r + b - 1$ straight line $\Rightarrow$ $\boxed{f(R) = (R^b - \Lambda)^c}$ $c \geq 1$ and $bc \approx 1$ to allow a saddle matter point

(iii) $m(r) = p \frac{(1+r)}{r}$ $\Rightarrow$ $\boxed{f(R) = R - \mu R_c (R/R_c)^p}$ $0 < p < 1$ $\mu > 0$ $R_c > 0$

• Example of $f(R)$ theories going from P_5 to P_6

constrains :

(iv) $m(r) = -C(r+1)(r^2 + ar + b)$ $C = 2n(2n+1)/\mu^{2n}$

$m'(r) = -C(3r^2 + 2ar + b + 2r + a)$ $m'(-1) = -C(\cancel{3}^{1} - \cancel{2}a + b - \cancel{2} + \cancel{a}) = -C(1 - a + b) \overset{!}{>} -1$, $m'(-2) = C(3a - b - 8) < -1$

- $f(R)$ in Einstein frame

- conformal transformation: $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ $\Omega^2 = F$ positive function
- $S_E = \int d\Omega \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + S_M(\Omega^{-2} \tilde{g}_{\mu\nu}, \Psi_M)$

  where we define a new field and potential $\kappa\phi \equiv \sqrt{\frac{3}{2}} \ln F$ $V(\phi) \equiv \frac{RF - f}{2\kappa^2 F^2}$
- conformal coupling $\Omega^2 = F = e^{-2Q\phi}$: $\kappa \cancel{\phi} \sqrt{\frac{2}{3}} = -2Q\cancel{\phi}$ $Q = -\kappa/\sqrt{6}$ const.

  Q is const. and relatively large (order 1) => without $V(\phi)$, field propagates freely and 5^{th} force propagate for breach local gravity constraints
- But... can choose f such that V (originating from gravity) provides screening mechanism.

  get Q_{eff}, V_{eff} more later....

Scalar-Tensor theories

Summary

$$S_{ST} = \int d\Omega \sqrt{-g}\left[\frac{1}{2} f(\varphi, R) - \frac{1}{2}\xi(\varphi)(\nabla\varphi)^2\right] + S_M(g_{\mu\nu}, \Psi_M)$$

e.g.:

$f(\varphi, R) = f(R)$	$\xi(\varphi) = 0$	$f(R)$ gravity
* $f(\varphi) = \varphi R$	$\xi(\varphi) = \frac{\omega_{BD}}{\varphi}$	Brans-Dicke
$f(\varphi) = \varphi R - 2U(\varphi)$	$\xi(\varphi) = \frac{\omega_{BD}}{\varphi}$	Brans-Dicke with potential
* $f(\varphi, R) = F(\varphi)R - 2U(\varphi)$	$\xi(\varphi) = \dots$	simplest generalization of Brans-Dicke
$f(\varphi, R) = 2e^{-\varphi}R - 2U(\varphi)$	$\xi(\varphi) = -2e^{-\varphi}$	dilaton potential (from effective string theory)

Brans-Dicke (1961) *

$$S_{BD} = \int d\Omega \sqrt{-g}\left\{\frac{1}{2}\Psi R - \frac{\omega_{BD}}{2\Psi}(\nabla\Psi)^2\right\} + S_M(g_{\mu\nu}, \Psi_M)$$

Simplest generalization of BD *

$$S_{ST} = \int d\Omega \sqrt{-g}\left[\frac{1}{2}F(\varphi)R - \frac{1}{2}\xi(\varphi)(\nabla\varphi)^2 - U(\varphi)\right] + S_M(g_{\mu\nu}, \Psi_M)$$

$$S_{(E)} = \int d\Omega \sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R} - \frac{1}{2}(\tilde{\nabla}\phi)^2 - V(\phi)\right] + S_M(\tilde{g}_{\mu\nu}F^{-1}, \Psi_M) \quad \text{with} \quad d\phi = d\varphi\sqrt{\frac{3}{2}\left(\frac{F_{,\varphi}}{F}\right)^2 + \frac{\xi}{F}} \; *$$

$$V = U/F^2$$

Jordan frame $\tilde{g}_{\mu\nu} = f\, g_{\mu\nu}$ Einstein frame

Define coupling strength

$$Q \equiv -\frac{F_{,\phi}}{2F} \qquad Q \stackrel{!}{=} \text{const.} \quad \text{special case (universal coupling)} \quad \Rightarrow \quad F(\varphi) = e^{-2Q\phi}$$

$$* \Rightarrow \quad \xi = F\left(\frac{d\phi}{d\varphi}\right)^2\left(1 - \frac{3}{\cancel{2}}\cancel{4}^{\,6}Q^2\right)$$

Back to Jordan with explicit coupling

$$S_{ST} = \int d\Omega \sqrt{-g}\left[\frac{1}{2}F(\varphi)R - \frac{1}{2}\xi(\varphi)(\nabla\varphi)^2 - U(\varphi)\right] + S_M(g_{\mu\nu}, \Psi_M)$$

$$= \int d\Omega \sqrt{-g}\left[\underline{\frac{1}{2}e^{-2Q\varphi}R - U(\varphi)} - \frac{1}{2}e^{-2Q\varphi}(1 - 6Q^2)(\nabla\varphi)^2\right] + S_M(g_{\mu\nu}, \Psi_M)$$

$$\nabla\phi(\varphi(x^\nu)) = \frac{d\phi}{d\varphi}\nabla\varphi$$

this is BD with potential and $3 + 2\omega_{BD} = \frac{1}{2Q^2}$ GR for: $Q \to 0 \sim \omega_B \to \infty$

f(R) theories

$$S_{f(R)} = \int d\Omega \sqrt{-g}\,\frac{1}{2}f(R) = \int d\Omega\sqrt{-g}\left(\frac{1}{2}F(R)R - U\right) \qquad U \equiv \frac{FR - f(R)}{2} \qquad Q \stackrel{!}{=} 1/\sqrt{6} \qquad e^{-2Q\phi} \equiv F \qquad \phi \equiv \sqrt{\frac{3}{2}}\ln F$$

One theory, special cases:

$$\underset{\text{general}}{\text{S-T}} \longrightarrow \underset{Q=\text{const}}{\text{CDE}} \longrightarrow \underset{Q=\frac{1}{\sqrt{6}}}{f(R)}$$

5th force, potential:

$$\tilde{g}_{\mu\nu} = e^{-2Q\varphi}g_{\mu\nu} \simeq -(1 - 2\Phi \underline{- 2Q\varphi})g_{\mu\nu} \qquad \boxed{\Phi_{ST} \equiv \Phi + Q} \qquad F_{\text{extra}} = -Q\nabla\varphi$$

$Q \ll 1$, 1° order

Scalar-tensor theories

- We build an even larger class of theories, which includes the previous ones
- Possibilities are:
  - **scalar-Tensor theory** we start simple, then → Horndeski: the most general scalar tensor theory
  - vector " "
  - tensor " " } instability → ghosts production, i.e. no bounds on energy levels

- Example of scalar-tensor theory: $f(R)$; we will see that this is the case

- First one ever proposed: Brans-Dicke (1961) $S_{BD} = \int d\Omega \sqrt{g} \left\{ \frac{1}{2} \psi R - \frac{\omega_{BD}}{2\psi} \overset{\psi_{,\nu}\psi^{,\nu}}{(\nabla\psi)^2} \right\} + S_M(g_{\mu\nu}, \Psi_M)$

  ω_{BD} 1 additional dimensionless free parameter

  ψ plays the role of a dynamical "gravitational constant" G

  maybe G is related to the density of the universe

  (Dirac saw a numerical coincidence of $G, \hbar, c$ maybe there is a link)

- if ω_{DB} very large ⇒ kinetic term would dominate ⇒ $\nabla\psi$ very small

  ⇒ $\psi \sim$ const. ⇒ $\sqrt{-g}\, R \sim GR$

  ⇒ lower bound on $\omega_{BD} \geq 10^{5\div4}$ or unacceptable because of local gravity constraints

- Now we will generalize that

- $$\boxed{S_{ST} = \int d\Omega \sqrt{-g} \left[\frac{1}{2} f(\varphi, R) - \frac{1}{2} \xi(\varphi) (\nabla\varphi)^2 \right] + S_M(g_{\mu\nu}, \Psi_M)}$$

  f, ξ arbitrary functions

  Jordan frame: modified gravity + conserved matter

- Examples:

$f(\varphi, R) = f(R)$	$\xi(\varphi) = 0$	$f(R)$ gravity
$f(\varphi) = \varphi R$	$\xi(\varphi) = \frac{\omega_{BD}}{\varphi}$	Brans-Dicke
$f(\varphi) = \varphi R - 2U(\varphi)$	$\xi(\varphi) = \frac{\omega_{BD}}{\varphi}$	Brans-Dicke with potential
$f(\varphi, R) = F(\varphi) R - 2U(\varphi)$	$\xi(\varphi) = \dots$	simplest generalization of Brans-Dicke
$f(\varphi, R) = 2e^{-\varphi} R - 2U(\varphi)$	$\xi(\varphi) = -2e^{-\varphi}$	dilaton potential (from effective string theory)

- We now consider $f(\varphi,R)=F(\varphi)R-2U(\varphi)$

$$S_{ST}=\int d\Omega\sqrt{-g}\left[\frac{1}{2}F(\varphi)R-\frac{1}{2}\xi(\varphi)(\nabla\varphi)^2-U(\varphi)\right]+S_M(g_{\mu\nu},\Psi_M)$$

- We also identify the F and ξ associated to a constant matter-field coupling Q

1) Go to Einstein frame:

conformal transformation $\tilde{g}_{\mu\nu}=F g_{\mu\nu}$ (i.e. positive to keep gravity attractive)

in an even more general case with $f(\varphi,R)$ $\tilde{g}_{\mu\nu}=\Omega^2(\varphi)g_{\mu\nu}$ with $\Omega^2(\varphi)\equiv F(\varphi)=\frac{\delta f}{\delta R}$

$$\Rightarrow S_{(E)}=\int d\Omega\sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R}-\frac{1}{2}(\tilde{\nabla}\phi)^2-V(\phi)\right]+S_M(\tilde{g}_{\mu\nu}F^{-1},\Psi_M)$$

"standard" — contains additional kinetic terms — interaction coupling Q

where we redefined the $\varphi\to\phi$ and $V\to U$

$d\phi=d\varphi\sqrt{\frac{3}{2}\left(\frac{F_{,\varphi}}{F}\right)^2+\frac{\xi}{F}}$ integrate $\Rightarrow\phi$

$V=U/F^2$

2) Define coupling strength $Q\equiv-\frac{F_{,\phi}}{2F}=-\frac{F_{,\varphi}}{2F}\frac{d\varphi}{d\phi}=-\frac{F_{,\varphi}}{2F}\left[\frac{3}{2}\left(\frac{F_{,\varphi}}{F}\right)^2+\frac{\xi}{F}\right]^{-1/2}$

alternative definition: $Q\equiv-\frac{F_{,\varphi}}{2F}$ with $F_{,\phi}=F_{,\varphi}\frac{d\varphi}{d\phi}$

3) Case with $Q\overset{!}{=}\text{const.}$ (universal)

this assumption is not too restrictive: Q should anyway evolve slowly
use a fixed Q for a given interval of time
i.e. find the $F(\varphi)$ for which $Q=\text{const.}$

$$\begin{cases}\underbrace{Q}_{\text{const.}}=\frac{-F_{,\phi}}{2F}\quad\Longrightarrow\quad F(\varphi)=e^{-2Q\phi}\\[2mm]\frac{d\phi}{d\varphi}=\sqrt{\frac{3}{2}\left(\frac{F_{,\varphi}}{F}\right)^2+\frac{\xi}{F}}\rightarrow\frac{\xi}{F}=\left(\frac{d\phi}{d\varphi}\right)^2-\frac{3}{2}\overbrace{\left(\frac{F_{,\phi}}{F}\right)^2}^{=4Q^2}\left(\frac{d\phi}{d\varphi}\right)^2\quad\Longrightarrow\quad\xi=F\left(\frac{d\phi}{d\varphi}\right)^2\left(1-\cancel{\tfrac{3}{2}}\cancel{4}^{6}Q^2\right)\end{cases}$$

4) Back to Jordan frame:

use the general lagrangian of scalar-tensor theories
just to have it in a form with explicit coupling Q

$$S_{ST}=\int d\Omega\sqrt{-g}\left[\frac{1}{2}F(\varphi)R-\frac{1}{2}\xi(\varphi)(\nabla\varphi)^2-U(\varphi)\right]+S_M(g_{\mu\nu},\Psi_M)$$

$$=\int d\Omega\sqrt{-g}\left[\frac{1}{2}e^{-2Q\varphi}R-U(\varphi)-\frac{1}{2}e^{-2Q\varphi}(1-6Q^2)(\nabla\varphi)^2\right]+S_M(g_{\mu\nu},\Psi_M)$$

$F(\phi)=e^{-2Q\phi}$
$\nabla\phi(\varphi(x^\nu))=\frac{d\phi}{d\varphi}\nabla\varphi$

- if $Q\to0$ minimally coupled scalar field (chameleon)
- $Q=0\Rightarrow GR$ i.e. Q characterizes the deviation from GR

- Link between Q and w_{BD} in Brans-Dicke with potential

$$S_{BD} = \int d\Omega \sqrt{-g} \left[\frac{1}{2}\psi R - \frac{w_{BD}}{2\psi}(\nabla\psi)^2 - U(\psi) \right] + S_M(g_{\mu\nu}, \Psi_M)$$

equivalent for:

$$S_{(J)} = \int d\Omega \sqrt{-g} \left[\frac{1}{2}F(\phi)R - U(\phi) - \frac{1}{2}F(\phi)(1-6Q^2)(\nabla\phi)^2 \right] + S_M(g_{\mu\nu}, \Psi_M)$$

$$\Rightarrow \boxed{\psi = F = e^{-2Q\phi}}$$

$$\nabla\psi = -2Q\nabla\phi\, e^{-2Q\phi}$$

$$\frac{w_{BD}}{\psi}(\nabla\psi)^2 = w_{BD}\cancel{e^{2Q\phi}}\, 4Q^2 \cancel{(\nabla\phi)^2} \cancel{e^{-4Q\phi}} \stackrel{!}{=} \cancel{e^{-2Q\phi}}(1-6Q^2)\cancel{(\nabla\phi)^2} \Rightarrow \boxed{3 + 2w_{BD} = \frac{1}{2Q^2}}$$

GR for: $Q \to 0 \sim w_{BD} \to \infty$

Resulting field eq.s

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = \frac{1}{\psi}T_{\mu\nu} - \frac{1}{\psi}g_{\mu\nu}U(\psi) + \frac{1}{\psi}(\nabla_\mu\nabla_\nu\psi - g_{\mu\nu}\Box\psi) + \frac{\omega_{\rm BD}}{\psi^2}\left[\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}(\nabla\psi)^2\right]$$

- $f(R)$ theories are a sub-set of scalar-tensor teories (Brans-Dicke)

- Scalar-Tensor:

$$S_{ST} = \int d\Omega \sqrt{-g} \Big(\underbrace{\frac{1}{2}e^{-2Q\phi}}_{(1)} R - \frac{1}{2}\underbrace{(1-6Q^2)}_{(2)} e^{-2Q\phi}(\nabla\phi)^2 - \underbrace{U}_{(3)} \Big) + S_M(g_{\mu\nu}, \Psi_M) \quad \text{(Jordan)}$$

- Write $S_{f(R)}$ with the same shape

$$S_{f(R)} = \int d\Omega \sqrt{-g}\, \frac{1}{2}f(R) = \int d\Omega \sqrt{-g}\left[\underline{\frac{1}{2}f(R)} + \frac{1}{2}F(R)R - \underline{\frac{1}{2}F(R)R}\right] = \int d\Omega\sqrt{-g}\left(\frac{1}{2}F(R)R - U\right)$$

(3) $U \equiv \frac{FR - f(R)}{2}$ define a potential

(2) $Q \stackrel{!}{=} 1/\sqrt{6}$ to get rid of term (2)

(1) $e^{-2Q\phi} \equiv F$ define coupling in this way

$\Leftrightarrow \phi = \sqrt{\frac{3}{2}} \ln F$ define scalar field (yes, with (-), by definition of ϕ)

$\Rightarrow$ $f(R)$ is the simplest scalar-tensor theory where there is no additional kinetic term $[(2) \stackrel{!}{=} 0]$

a kinetic term is naturally embedded as it naturally emerges out of the conformal transf.

$\downarrow$

from coupling between R and field ϕ (in term 1)

- Is there a difference between theories?

  - Same theory, just a "sub-sets": $f(R) \supset CDE \supset ST$
    $Q = \frac{1}{\sqrt{6}}$ $Q = \text{const}$ general

  - The only kind of difference

    matematically equivalent

    In S-T and $f(R)$ we have in mind a universal coupling (same Q for DM and baryons) → Jordan frame

    in CDE we have in mind a not universal coupling where baryons can also NOT be coupled
    → not to screw up observations (Phisical frame, $\nabla_\mu T^{\mu\nu}_{\text{baryons}} = 0$)

- Interpreting the coupling strength:

  identify the gravitational potential => force (Einstein frame)

  Weak field limit, perturbation theory:

$$\tilde{g}_{\mu\nu} = e^{-2Q\varphi} g_{\mu\nu} = e^{-2Q\varphi}[-(1-2\Phi)g_{\mu\nu} + \dots] \qquad \Phi \text{ Newtonian potential}$$

$$Q \ll 1 \to \quad \simeq (1-2Q\varphi)[-(1-2\Phi)g_{\mu\nu} + \dots]$$

$$1^\circ \text{ order} \to \quad \simeq -(1-\underline{2\Phi - 2Q\varphi})g_{\mu\nu} \qquad \text{matter particles in this metric}$$

=> $\boxed{\Phi_{ST} \equiv \Phi + Q\varphi}$ potential in scalar-tensor theory Newtonian + correction

=> $\boxed{\vec{a} = -\vec{\nabla}\Phi_{ST}}$ Force has extra term $\vec{F}_{\text{extra}} = -Q\vec{\nabla}\varphi$ strength is also given by $\vec{\nabla}\varphi$
=> spatial dependency of φ

- Getting your cosmology: usual phase-space analysis

  usually: Q as a fixed parameter
  $U(\phi)$ (Jordan frame) potential is a given function

  case: $\lambda \equiv -\frac{U_{,\phi}}{U} = \text{const.}$ => $U(\phi) = e^{-\lambda\phi}$, i.e. parameters: $\{Q, \lambda\}$ like for coupled DE

  => same classes of solutions with critical points
  - matter era
  - kinetic unstable points
  - modified matter era $\Omega_m \to 1-\varepsilon$
    $\Omega_\phi \to \varepsilon$
  - Accelerated solutions → scaling Ω_ϕ, Ω_m const
    ↘ field dominated $\Omega_\phi \to 1$ minor modification of ΛCDM like

Horndeski theory

- We could generalize more and more....

  e.g.: $f(\phi)R$, $f(\phi, \Box\phi)R$, $R + \Box R + \phi_{,\mu}\phi^{,\mu}R + R_{,\mu}$ in general, we just need $\mathcal{L} \in \mathbb{R}$ invariant

- But... we discard all $\mathcal{L}$ giving instabilities:

  ↳ classical inst.: eg. infinitely fast cosmic expansion, get singularity very soon

  ↳ quantum inst.: e.g. "normally" e^- in atom have a ground state (energy bound) much deeper

  but, if in a theory, energy states are unbound from below

  $\Rightarrow$ e^- could go to lower energy level $\rightsquigarrow \gamma$

  " " " " $\rightsquigarrow \gamma$

  " " " " $\rightsquigarrow \gamma$ boundless

  .....

  $\Rightarrow$ infinite source of photons

  $\Rightarrow$ particle production out of nothing (ghosts)

- A way out: Ostrogosky theorem

  - Every $\mathcal{L}$ that leads to eq. of motion higher than 2^{nd} order do not have a ground state

    i.e. no energy states bound from below

    i.e. Unstable

  - 3^{rd} order ... no! keep even orders or you have problems with time reversal symmetry

  - 4^{th} order ... 4 deg. of freedom $\phi, \dot{\phi}, \ddot{\phi}, \dddot{\phi}$ (init. conditions)

    $\Rightarrow$ large phase-space: impossible to avoid trajectories escaping to ∞ along one of these directions

- Careful: degenerate cases

  - eg. system with 2 fields ϕ, ψ $\Rightarrow$ 2 KG eq.s $\begin{cases} \Box\phi + V_{,\phi} = 0 \\ \Box\psi + V_{,\psi} = 0 \end{cases}$ perhaps even coupling between ϕ, ψ: $V(\phi, \psi)$

    $\Rightarrow$ Two 2^{nd} order diff. eq.s can be combined together in an higher order equation

    ↳ do you see? This "higher order" problem is just an "illusion", in reality the problem is 2° order

  - if you go to higher orders you need to demonstrate that you have a degenerate case, i.e. that you can "break" the problem in more fields as above

- <u>Horndeski action</u>

The most general scalar-tensor $(\phi, g_{\mu\nu})$ theory leading to eq. of motion up to 2nd order

$$\boxed{S = \int d\Omega \sqrt{-g} \sum_{i=2}^{5} \mathcal{L}_i(\varphi, R) + S_M}$$

sum of 4 Lagrangians based on arbitrary functions $K(\phi,X)$, $G_3(\phi,X)$, $G_4(\phi,X)$, $G_5(\phi,X)$ $\quad X \equiv -\frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$

$$\left.\begin{aligned} &\mathcal{L}_2 = K(\phi, X) \qquad \text{eg. } \underbrace{\mathcal{L}_2 = X - V}_{\text{canonical}},\ \underbrace{\mathcal{L}_2 = X^3\phi + V}_{\text{non-canonical.....}}, \ \ldots \\ &\mathcal{L}_3 = -G_3(\phi, X)\Box\phi \end{aligned}\right\} \text{kinetic terms}$$

$$\left.\begin{aligned} &\mathcal{L}_4 = G_4(\phi,X)\overset{*}{R} + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \quad (\nabla_\mu\nabla_\nu\phi)^2 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi \quad G_{4,X} = \partial_X G_4 \\ &\mathcal{L}_5 = G_5(\phi,X)\overset{*}{G}_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5,X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right] \end{aligned}\right\} \text{coupling terms}$$

* G_4, G_5 coupling of ϕ with R and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ => gravity modifications "gravitational"

very large class but unique

- <u>various parts can produce terms with derivatives higher than 2°</u>

eg. $(\Box\phi)^2$ contains a $(\ddot\phi)^2 \curvearrowright$ Euler-Lagrange eq. you get $\frac{d^2}{dt^2}\left[\frac{\delta\ddot\phi^2}{\delta\ddot\phi}\right] = \frac{d^2}{dt^2}(2\ddot\phi) = 2\ddddot\phi$!

<u>but....</u> the equations are constructed such that all these terms cancel each other in the final eq. of motion

so you can not change $\mathcal{L}$ at all! eg. even by changing just $\frac{1}{6} \to \frac{1}{7}$ instabilities in $\mathcal{L}$ do not cancel out

- <u>Examples:</u>

1) standard gravity with non-canonical kinetic term: $G_4 = \frac{1}{2}$, $G_5 = \text{const}$
2) canonical form: $K = X - V(\phi)$, $G_3 = \text{const.}$ $\Big\}$ ϕCDM
3) ΛCDM: (1) + $K = -2\Lambda$
4) Brans-Dicke with non-canonical kinetic term: $G_4 = G_4(\phi)$, $G_5 = \text{const.}$
5) Original Brans-Dicke: $K = \frac{\omega_{BD}}{\phi}X$, $G_3 = 0$, $G_4(\phi) = \frac{\phi}{2}$
6) $f(R)$: $G_4(\phi) = \frac{1}{2}e^{2\phi/\sqrt{6}}$ $\quad K(\phi) = -\frac{1}{2}(R f_{,R} - f)$ $\quad \phi = \frac{\sqrt{6}}{2}\log(1 + f_{,R})$ $\Rightarrow$ $\mathcal{L} = \frac{1}{2}(R - f(R))$

- <u>Working with Horndesky</u>

- Remaining generic, with arbitrary functions, G_i, you can work out solutions for the background

for perturbations, (structure formation) you need explicit functions to work out solutions

- Theoretically: look at what the various terms do $\to$ predict observable features
  Observationally: try to constrain terms, eg. do we need all of them?

↳ Example, gravitational waves (GW): see next section

▫ Does it produce new effects?

- Mostly, solution similar to scalar-tensor theories but....
  .... possible modification of the speed of gravitational waves!

- In GR: linearize eq.s:

  scalar perturbation $g_{\mu\nu} = (1-2\phi) g_{00} + (1+2\phi) g_{ii}$

  tensor perturbation $h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{21} & h_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ GR: 2 polarization modes

  TT gauge

  spin 2 (quadrupole)

  $\Box h_{\mu\nu} = K T_{\mu\nu}$ wave solution, propagate with speed $c=1$

- In Hordesky: $c_T = 1 + \underbrace{\frac{2X(2G_{4,X} - 2G_{5,\phi} - (\ddot{\phi} - \dot{\phi}H)G_{5,X}}{2(G_4 - 2XG_{4X} + XG_{5\phi} - \dot{\phi}HXG_{5,X})}}$ (!)

- Strong observational constrain

  NS-BH merger: GW + photons constrains $c_T = 1 \pm 10^{-15}$ (!)

  $c_T = 1$ for $\left.\begin{array}{l} G_{4,X} = 0 \quad \text{i.e. } G_4 = \text{const} \\ G_{5,X} = 0 \quad \text{i.e. } G_5 = \text{const} \end{array}\right\}$ i.e. standard scalar-tensor theory

  or for $X = 0$ i.e. static field

▫ Can we generalize even more?

  yes... bejond Hordesky, modifies how matter interacts with the field

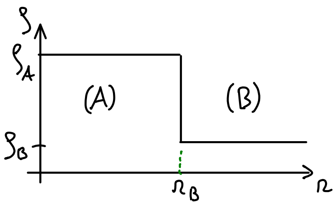
Screaning mechanism

Recap and motivation

- DE coupled with matter / modified gravity: matter eq. of motion is affected
  5^{th} force, effective grav. potential $\Phi(r) = \Phi_N(r) - \underbrace{Q\,\phi(r)}_{=\Phi_Q(r)}$
- Φ_Q can not contraddict local gravity measures $\Rightarrow$ 2 possibilities:
  1) $|Q| \ll 1$ (5^{th} force always small)
  2) $\Phi_Q(\bar{x})$ contains a screaning term depending on the environment
     bare coupling Q can be large but in eq. of motion of bodies appears an effective coupling $Q_{eff}(\rho(\bar{x}))$
     $Q_{eff} \ll Q$ where $\rho(\bar{x})$ is "large" $\Rightarrow$ field acquires an heavy mass and it can not propagate freely

For simplicity, we focus on screaning only, leave cosmology aside

assume static metric and spherical symmetry (e.g. Earth) $\Rightarrow$ only r dependency $\phi(r)$
we should use Schwarzschild but we even assume Minkowski metric (neglect backreaction)
self gravity would give minor contribution, we focus on screaning only



ρ_B homogeneous background
r_B = radious of the body $\quad \rho = \rho_A = \text{const} \quad M = \frac{4\pi}{3}\rho_A r_B^3$

Einstein Frame, Coupled Dark-Energy ($Q \stackrel{!}{=} \text{const}$) $\quad$ "$'$" $\equiv d/dN$

$$\nabla_\mu T^{\mu\nu}_{(m)} = Q T_m \nabla^\nu \phi \qquad \rho_m' + 3\rho_m = Q\rho_m\phi' \qquad (1)$$
$$(\nu = 0)$$
$$\nabla_\mu T^{\mu\nu}_{(\phi)} = -Q T_m \nabla^\nu \phi \qquad \rho_\phi' + 3(\rho_\phi + P_\phi) = -Q\rho_m\phi' \qquad (2)$$

(1) solution: $\rho_m = \underline{\rho_{m0}}\, a^{-3} e^{Q(\phi-\phi_0)} \stackrel{*}{=} \hat{\rho}_m e^{Q\phi}$ $\quad$ * $\phi - \phi_0 \to \phi$ rescaling the field (redefinition of ϕ)

(2) plug ρ_ϕ, P_ϕ: $\Box\phi - V_{,\phi} = Q\rho_m$ $\quad$ ϕ' cancels $\quad$ general, it applies to any metric
+ because of the $(-+++)$ signature in $\Box$ which contains a $-\frac{d}{d\tau}$

$$\Box\phi = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \qquad \text{spherical symmetry + weak gravity background (Minkowski)}$$

$$\Box\phi - (V_{,\phi} + Q\rho_m) \equiv \Box\phi - V_{eff,\phi} = 0 : \qquad V_{eff,\phi} \equiv V_{,\phi} + Q\hat{\rho}_m e^{Q\phi} \qquad \boxed{V_{eff}(\phi) = V(\phi) + \hat{\rho}\, e^{Q\phi}} \quad (!)$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = V_{eff,\phi}}$$

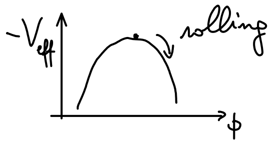
↳ determines the "dynamics" of ϕ: "r evolution"
↳ dependency on $\hat{\rho}(\bar{x})$!

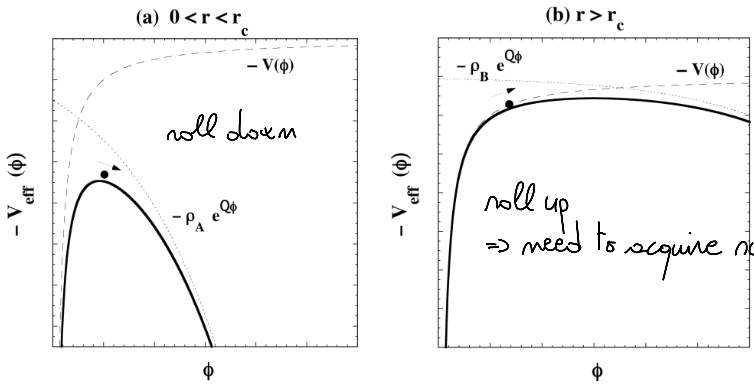
note: in a cosmosmological contex $\hat{\rho} \propto a^{-3}$ is not static (!) but small time evolution, we neglect

- Dynamic of ϕ

$\Box\phi - V'_{eff} = 0$ minus! $\Rightarrow$ the system behaves as

(!!)





roll up

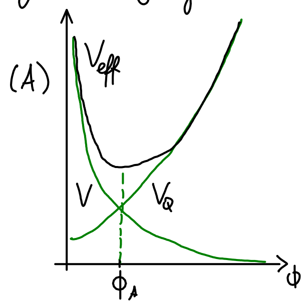
$\Rightarrow$ need to acquire sufficient kinetical energy in region $\Omega_A < \Omega < \Omega_B$

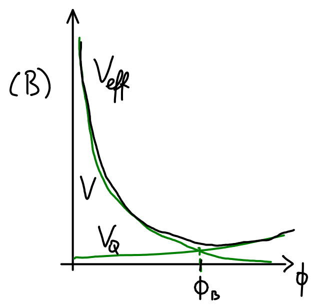
First qualitative assessment of $\phi(r)$

- Study the 3 regions (A, C, B) "separately"

• for an explicit example Assume $V(\phi) = e^{-\lambda\phi}$: $V_{eff}(\phi) = V(\phi) + \hat{\rho}\, e^{Q\phi} = e^{-\lambda\phi} + \hat{\rho}\, e^{Q\phi}$

A: high density region (interior) $r < r_A$ B: low density (exterior) $r > r_B$





if $\phi \sim \phi_A \Rightarrow \phi$ nearly frozen around minima

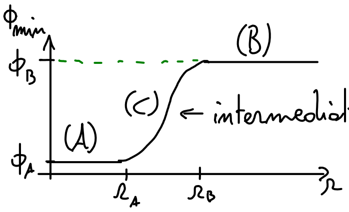
ϕ rolls down $r = r_A$ where $Q\hat{\rho}_A e^{Q\phi}$ becomes relevant

• V_{eff} has a minima where the field will sit

Position of $V(\phi)$ minima depends on $\hat{\rho}(r)$, i.e. depends on radius

$V'_{eff} = V_{,\phi}(\phi) + Q\,\hat{\rho}\, e^{Q\phi_A} = -\lambda\, e^{-\lambda\phi} + \hat{\rho}\, Q\, e^{Q\phi} \overset{!}{=} 0 \qquad \phi = (Q+\lambda)^{-1} \ln\left(\frac{\lambda}{\hat{\rho}\, Q}\right) \longrightarrow \boxed{\phi_A < \phi_B}$

$\hat{\rho}_A > \hat{\rho}_B$



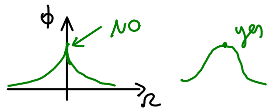
$\leftarrow$ intermediate regime : numerical solution or further approximations (later...)

• The mass of the field depends on r as well $m^2 \equiv \frac{d^2 V_{eff}}{d\phi^2}$ (curvature) $\longrightarrow \boxed{m_A^2 > m_B^2}$

under certain conditions m_A is large enough to provide screaning, massive field in dense region

Solving for $\phi(r)$ where V is minimum

1) impose conditions: $\left.\frac{d\phi}{dr}\right|_{r=0} \overset{!}{=} 0$ to avoid a cusp:



$\lim_{r\to\infty}\phi(r) = \phi_B$ to set the background far from the overdensity

2) treat separately regions A, B, C and then join smoothly where they meet

- Region (A): $0<r<r_A$

Approximate potential around minimum of ϕ_A with quadratic term (1° non-zero term of Taylor expansion)

$V_{eff} \simeq \frac{1}{2} m_A^2 (\phi-\phi_A)^2 + \text{const.}$ $\quad m_A^2 = \left.\frac{d^2 V_{eff}}{d\phi^2}\right|_A$

$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = m_A^2(\phi-\phi_A)$ i.e. linearized KG eq. 3 solutions: $\phi = \phi_A = \text{const.}$, $\phi = A_\pm \frac{e^{\pm m_A r}}{r}$

$\Rightarrow \boxed{\phi} = \phi_A + A_{(-)}\frac{e^{-m_A r}}{r} + A_{(+)}\frac{e^{-m_A r}}{r} = \boxed{\phi_A + A\left(\frac{e^{-m_A r}}{r} - \frac{e^{m_A r}}{r}\right)}$ general solution

singularity at $r=0$ $\Rightarrow$ $A_{(-)} = -A_{(+)}$ to avoid singularity boundary; condition $\left.\frac{d\phi}{dr}\right|_{r=0} = 0$ is satisfied

- Region (C): $r_A<r<r_B$

$V_{eff,\phi} \equiv V_{,\phi} + Q\hat{\rho}_m e^{Q\phi}$ $\quad |V_{,\phi}| \ll |Q\hat{\rho}_A e^{Q\phi}|$ is satisfied; $Q\phi \ll 1$ for most dark energy potentials

$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = V_{eff,\phi} \simeq \hat{\rho}_A Q e^{Q\phi} \simeq Q\hat{\rho}_A \quad \Rightarrow \quad \cancel{r^2}\frac{d\phi}{dr} = Q\hat{\rho}_A r^{\cancel{3}}\frac{1}{3} + \frac{C}{r^2}$ $\quad \boxed{\phi(r) = \frac{1}{6}Q\hat{\rho}_A r^2 - \frac{C}{r} + D}$

- Region (B): $r>r_B$ $\quad \phi$ rolls up to maximum value if it has enough kinetic energy $\frac{d\phi}{dr}$ $\Rightarrow$ V' negligible

$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = V_{eff,\phi}$ $\quad \cancel{\frac{1}{r^2}}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \simeq 0$ $\quad r^2\frac{d\phi}{dr} = -E$ $\quad d\phi = \frac{-E}{r^2}dr$ $\quad \boxed{\phi = \frac{E}{r} + C}$ $\quad \phi(r\to\infty) = \phi_B$ $\quad \phi_B = C$

A, C, D, E found by imposing continuity of $\phi(r)$, $\phi(r')$ at $r=r_A$, $r=r_B$

$$\phi(r) = \phi_A - \underbrace{\frac{1}{m_A(e^{-m_A r_A} + e^{m_A r_A})}\left[\phi_B - \phi_A + \frac{1}{2}Q\rho_A(r_A^2 - r_c^2)\right]}_{A} \frac{e^{-m_A r} - e^{m_A r}}{r} \qquad (0<r<r_A) \quad \text{eq. A}$$

$$\begin{aligned}\phi(r) = \; & \phi_B + \frac{1}{6}Q\rho_A(r^2 - 3r_B^2) + \frac{Q\rho_A r_A^3}{3r} \\ & - \left[1 + \frac{e^{-m_A r_A} - e^{m_A r_A}}{m_A r_A(e^{-m_A r_A} + e^{m_A r_A})}\right]\left[\phi_B - \phi_A + \frac{1}{2}Q\rho_A(r_A^2 - r_B^2)\right]\frac{r_A}{r}\end{aligned} \qquad (r_A<r<r_B) \quad \text{eq. C}$$

$$\begin{aligned}\phi(r) = \; & \underbrace{\phi_B}_{C} - \underbrace{\Bigg[r_A(\phi_B-\phi_A) + \frac{1}{6}Q\rho_A r_B^3\left(2+\frac{r_A}{r_B}\right)\left(1-\frac{r_A}{r_B}\right)^2 + \frac{e^{-m_A r_A} - e^{m_A r_A}}{m_A(e^{-m_A r_A} + e^{m_A r_A})}\left\{\phi_B - \phi_A + \frac{1}{2}Q\rho_A(r_A^2 - r_B^2)\right\}\Bigg]}_{E}\frac{1}{r}\end{aligned} \qquad (r>r_B) \quad \text{eq. B}$$

The effective coupling Q_{eff}

- We care of what happens outside the body $\Rightarrow$ region (B), look at eq. B
- we need $(\phi_B - \phi_A)$: from boundary condition between region (A) and (C)

region (A): $V_{eff,\phi} \simeq m_A^2(\phi(r) - \phi_A)$
region (C): $V_{eff,\phi} \simeq Q\hat{\rho}_A$

$\Bigg\} \quad m_A^2(\phi(r_A) - \phi_A) = Q\hat{\rho}_A$ ← eq. A — this also sets the value of r_A

$$\cancel{\phi_A} - \frac{1}{m_A(e^{-m_A r_A} + e^{m_A r_A})}\left[\phi_B - \phi_A + \frac{1}{2}Q\hat{\rho}_A(r_A^2 - r_B^2)\right]\frac{e^{-m_A r_A} - e^{m_A r_A}}{r_A} - \cancel{\phi_A} = \frac{Q\hat{\rho}_A}{m_A^2}$$

$$\phi_B - \phi_A + \frac{1}{2}Q\hat{\rho}_A(r_A^2 - r_B^2) = \frac{Q\hat{\rho}_A}{m_A} r_A \frac{(e^{-m_A r_A} + e^{m_A r_A})}{e^{m_A r_A} - e^{-m_A r_A}} \quad (1)$$

$\Phi_B \equiv \frac{M}{8\pi r_B} = \frac{1}{6}\hat{\rho}_A r_B^2 \qquad M = \frac{4}{3}\pi r_B^3 \rho_A$

- Plug in eq. B

$$\boxed{\phi(r) = \phi_B - 2Q_{eff}\frac{GM}{r}} \qquad \boxed{Q_{eff} = Q\left\{1 - \frac{r_A^3}{r_B^3} + 3\frac{r_A}{r_B}\frac{1}{(m_A r_B)^2}\left[\frac{m_A r_A(e^{m_A r_A} + e^{-m_A r_A})}{e^{m_A r_A} - e^{-m_A r_A}} - 1\right]\right\}}$$

we identified / defined an effective coupling Q_{eff} acting outside the body!

Resulting 5th force

$\vec{F}_\phi = -Q\vec{\nabla}\phi \qquad r > r_B \qquad$ interested outside the body $\qquad \phi(r) = \phi_B - 2Q_{eff}\frac{GM}{r}$

$$F_\phi(r) = -Q\frac{d\phi}{dr} = -Q\underbrace{\cancel{\frac{d\phi_B}{dr}}}_{=0} + \underbrace{2QQ_{eff}\frac{GM}{r^2}}_{(*)}$$

Condition to have screaning

Not all theories provide screaning

- screaning means: 5th force is negligible even if Q is large, i.e. $\boxed{|Q_{eff}| \ll 1}$
- to have screaning the theory must satisfy: $\boxed{r_B - r_A \ll r_B}$ (thin shell condition) and
  $\boxed{m_A r_B \gg 1}$ (field is heavy inside r_B)

Chameleon mechanism

Thin-shell parameter

Expand (eq. 1) for $\Delta r \equiv r_B - r_A \ll r_B$, $m_A r_B \gg 1$: $\frac{\Delta r}{r_B} \ll 1$ and $\frac{1}{m_A r_B} \ll 1$

$$\phi_B - \phi_A + \frac{1}{2}Q\hat{\rho}_A(r_A^2 - r_B^2) = \frac{Q\hat{\rho}_A}{m_A} r_A \frac{(\cancel{e^{-m_A r_A}} + e^{m_A r_A})}{e^{m_A r_A} - \cancel{e^{-m_A r_A}}} \qquad \hat{\rho}_A r_B^2 = 6\Phi_B$$

$$\phi_B - \phi_A + \frac{1}{\cancel{2}}Q\frac{6\Phi_B}{r_B^2}(r_A - r_B)\underbrace{(\cancel{r_A} + r_B)}_{\sim 2r_B} = \frac{Q6\Phi_B}{m_A r_B^2} r_A \qquad \div 6Q\Phi_B$$

thin-shell parameter $\quad \varepsilon_{th} \equiv \frac{\phi_B - \phi_A}{6Q\Phi_B} \simeq \underbrace{\frac{(r_B - r_A)}{r_B}}_{(1)} + \underbrace{\frac{r_A}{m_A r_B^2}}_{(2)} \simeq \frac{\Delta r}{r_B}$ if (2) ≪ (1) perché dovrebbe essere così?

$\Rightarrow \boxed{Q_{eff} \simeq 3Q\cdot\varepsilon_{th}} \Rightarrow$ if $\varepsilon_{th} \ll 1 \to Q_{eff} \ll Q \Rightarrow$ screaning!

Part V

Relativistic linear cosmic Structure Formation

Scalat-Vector-Tensor (SVT) decomposition

(cosmic structure formation)

- Flat FLRW metric ($K=0 \Rightarrow \chi = f_K = w$)

$$ds^2 = -c^2 dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j = a^2(\tau)\left(-d\tau^2 + \delta_{ij}\,dx^i dx^j\right) \qquad \tau \equiv \text{conformal time} \qquad i,j = 1,2,3$$

- Add perturbation to the metric (Newton gauge)

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + a^2 g^{(1)}_{\mu\nu} \qquad |g^{(1)}_{\mu\nu}| \ll |g^{(0)}_{\mu\nu}| \qquad g^{(1)} = \begin{pmatrix} -2\psi & w_i \\ w_i & 2\phi\delta_{ij} + h_{ij} \end{pmatrix}$$ most generic perturbed metric

$$\left.\begin{cases} \psi(x^\nu),\ \phi(x^\nu) : \text{two scalar fields} & 2 \\ w_i(x^\nu) : \text{3-vector field} & 3 \\ h_{ij}(x^\nu) : \text{3-tensor, traceless, symmetric } -1+6 = 5 \end{cases}\right\}$$ 10 degrees of freedom ✓

sign, matter of definition

one constrain — Because the metric is σ

- 4-interval

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)c^2 d\tau^2 + \underbrace{2w_i\,c\,d\tau\,dx^i}_{w_i(d\tau dx^i + dx^i d\tau)} + \left[(1+2\phi)\delta_{ij} + h_{ij}\right] dx^i dx^j\right\}$$

- Decompose w_i in longitudinal and transverse components (Helmotz's theorem)

$$\bar{w} \equiv \bar{w}^S + \bar{w}^V = \bar\nabla E + \bar\nabla \times \bar{A}$$

E = scalar field — 1

$\bar{A}$ = divergenless vector $\bar\nabla\bar{A} = 0$ — 2

} deg. of free.

$\bar{w}^S$ = scalar mode component $\quad \bar\nabla\times\bar{w}^S = \varepsilon_{ijk}\,\delta^j w^k_{(s)} = 0 \quad$ curl free component $\Rightarrow$ longitudinal $\quad \varepsilon$ = Levi-Civita

$\bar{w}^V$ = vector mode component $\quad \bar\nabla\bar{w}^V = \delta^i w^V_i = 0 \quad$ divergence-free " $\Rightarrow$ transverse

- In Fourier space: $\hat{\bar{w}} = \underset{(1)}{-i\bar{k}\hat{E}} \underset{(2)}{- i\bar{k}\times\hat{\bar{A}}}$

1) $/\!/\ \bar{k}$ mode — scalar = longitudinal

2) $\perp \bar{k}$ mode — vector = transverse

$\bar{k}$, with $|\bar{k}|=1$, gives the direction of the contributions

- Decompose h_{ij} in analogy to $\bar{w}$

$$h_{ij} = h^S_{ij} + h^V_{ij} + h^T_{ij}$$

$/\!/$) $h^{(S)}_{ij}$ scalar component $\quad \varepsilon_{ijk}\,\delta^j \underline{\delta^l h^k_l} = \varepsilon_{ijk}\,\delta^j S^k = 0$ * $\quad$ curl free divergence (longitudinal) : 1 d.o.f.

S^k: longitudinal vector

$\perp$) $h^{(V)}_{ij}$ vector " $\quad \delta^j \underline{\delta^i h^{(V)}_{ij}} = \delta^j r_j = 0$ ** $\quad$ divergence free divergence (transverse) : 2 d.o.f.

r^j: transverse vector

T) $h^{(T)}_{ij}$ tensorial " $\quad h^T_{ij} = h^T_{ji},\ \delta^i h^T_{ij} = 0,\ h^{T\,i}{}_i = 0 \quad$ symmetric, transverse, traceless : 2 d.o.f.

(s) and (v) not sufficient, h^T_{ij} all that is left

- In Fourier space:

  - $h_{ij}^{(S)} = \underbrace{\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)}_{\equiv D_{ij}} B \xrightarrow{\text{Fourier}} \hat{h}_{ij}^{s} = \left[(ik_i)(ik_j) - \frac{1}{3}\delta_{ij}(ik_\ell)(ik^\ell)\right]B(\vec{k})$

    $= \left(-k_ik_j + \frac{1}{3}\delta_{ij}|\vec{k}|^2\right)B = -|\vec{k}|^2\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)B(\vec{k}) = \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)B'(\vec{k}) \quad B' \equiv -|k|^2 B$

    scalar mode component, $B \in \mathbb{R}$

    generates irrotational velocities perturbations

  - $h_{ij}^{(V)} = \frac{1}{2}\left(\partial_i h_j + \partial_j h_i\right) \xrightarrow{\text{Fourier}} \hat{h}_{ij}^{v} = \frac{i}{2}\left(k_i h_j + k_j h_i\right)$

    one scalar is not enough: needs vector, h_i

    generates rotational velocities perturbations

  - $h_{ij}^{(T)}$ the only mode present in vacuum — traceless-transverse (TT) gauge

    generates tensorial perturbation: gravitational waves

- Excitation of these modes
  - Each perturbation mode needs specific sources to be generated, e.g. GW time varying quadrupole moment
  - These tree modes are decoupled, i.e. they evolve independently => can be treated separately

    $G_{0i} = \kappa T_{0i}$ equations have transverse and longitudinal modes

    taking the curl $\vec{\nabla}\times\sim$ => only transverse component remains

    " " divergence $\vec{\nabla}\sim$ " longitudinal " " } decoupled ✓
  - Density perturbation $\delta(\vec{x},t)$ is a scalar => excites only the scalar (longitudinal) part

    rotational modes => " " intrinsically vectorial modes

    anisotropic matter perturbations => " " tensorial mode (GW)
  - If at $t=0$ rotational and vorticity modes $=0$ they remain zero (unless $\vec{\nabla}s \neq 0$ entropy)
  - If present => decreases as a^{-1} i.e. die out...

$\mapsto$ Consider scalar modes only!

- Now consider the "scalar" components only: $\Psi,\ \phi,\ \varpi^s \equiv \nabla E,\ S^{(s)}_{ij} = D_{ij}B$

$$g^{(1)s} = \begin{pmatrix} 2\Psi & w_i \\ w_i & S_{ij} \end{pmatrix} = \begin{pmatrix} -2\Psi & w_i^{(s)} \\ w_i^{(s)} & 2\phi\delta_{ij} + h_{ij}^{(s)} \end{pmatrix} = \begin{pmatrix} 2\Psi & E_{,i} \\ E_{,i} & 2\phi\delta_{ij} + D_{ij}B \end{pmatrix}$$

- Apply gauge choice $\Rightarrow$ $10-4=6$ deg. of freedom $\xi^\nu(x^\mu) = 4$ functions of coordinates

$$x'^\mu = x^\mu + \xi^\mu \qquad \xi^\mu = \left\{ \begin{array}{l} \xi^0 \\ \xi^i = \xi^i_s + \xi^i_v \qquad \xi^i_s = \alpha \bar{k} \quad \hat{k}_i \xi^i_v = 0 \end{array} \right\} \begin{array}{l} 2 \text{ scalar},\ 2 \text{ vector modes} \\ \xi^0,\ \xi^i_s \qquad \xi^i_v \end{array}$$

$\alpha \in \mathbb{R}$

cos'era quello?

- Degrees of freedom:

			(a) scalar modes	(b) vector modes	(c) tensor modes
$g_{\mu\nu}$	Ψ, ϕ	[2] $\to$	2		
	w_i	[3] $\to$	1	2	
	S_{ij}	[5] $\to$	1	2	2
			4	4	2
in specific gauge:			$\downarrow -2$ 2	$\downarrow -2$ 2	2

$\Rightarrow$ [10] deg. of freed.

$\Rightarrow$ [6] physical deg. of freedom

(a) "Newtonian" ; gravity as gradient of a potential ; but 2 scalars not 1!

(b) Gravitomagnetism ; vector type potential ; in GR the momentum sources gravity

(c) Gravitational waves ; the only mode that exists in vacuum

In conformal Newton gauge (Poisson gauge, longitudinal gauge, shear free gauge)

$$\left\{ \begin{array}{l} w_i^s \overset{!}{=} 0 \quad (E=0) \\ B \overset{!}{=} 0 \end{array} \right. \Rightarrow \boxed{ds^2\big|_{\text{scalar}} = a^2(t)\left[-(1+2\Psi)c^2 d\tau^2 + (1+2\phi)\delta_{ij}dx^i dx^j\right]}$$

Einstein field eq.s $\Rightarrow \Psi = -\phi$!

- In spatially flat gauge

$$\left\{ \begin{array}{l} \phi = 0 \\ B = 0 \ \text{ll ?} \end{array} \right. \Rightarrow \boxed{ds^2\big|_{\text{scalar}} = a^2(t)\left[-(1+2\Psi)c^2 d\tau^2 - 2\underbrace{w_i^s}_{\partial_i E} c\, d\tau\, dx^i + \delta_{ij}dx^i dx^j\right]}$$

$$S^s_{ij} \overset{!}{=} 0 \overset{!}{=} S^v_{ij} \qquad (1s + 2v)$$

3 constrains The tensorial is left.

Note: some people use $1-2\phi$ (i.e. $\phi \to -\phi$ as convention)

Appendix

- Inverse of perturbation metric

for notation convenience: $g = g^{(0)} + g^{(1)} \equiv g^{(0)} + h$

1° order

$$g^{\mu\gamma} g_{\gamma\nu} \overset{!}{=} \delta^\mu_\nu : \quad (g^{(0)\mu\gamma} + h^{\mu\nu})(g^{(0)}_{\gamma\nu} + h_{\gamma\nu}) \simeq g^{(0)\mu\gamma}(g^{(0)}_{\gamma\nu} + h_{\gamma\nu}) + h^{\mu\nu} g^{(0)}_{\gamma\nu} \overset{!}{=} \delta^\mu_\nu \quad \Big\rbrace \cdot g^{(0)\gamma\nu}$$

$$g^{(0)\mu\nu}(1 + g^{(0)\partial\nu} h_{\gamma\nu}) + h^{\mu\nu} = g^{(0)\mu\nu} \qquad g^{(0)\mu\nu} g^{(0)\gamma\nu} h_{\gamma\nu} + h^{\mu\nu} = 0 \qquad h \to g^{(1)}$$

$$\Rightarrow g^{(1)\mu\nu} = - g^{(0)\mu\nu} g^{(0)\gamma\nu} g^{(1)}_{\partial\nu} \quad \text{at 1° order}$$

in perturbation theory, indices are rised / lowered with the background metric

- meaning: $\bar{w} = \bar{w}^{(s)} + \bar{w}^{(v)}$:

$\bar{w}^{(s)} \equiv \bar{\nabla} E \qquad \bar{w}^{(v)} \equiv \bar{\nabla} \times \bar{A} \qquad \bar{A}$ = divergenless vector $\bar{\nabla}\bar{A} = 0$

$$\bar{\nabla}\bar{w} = \bar{\nabla}^2 E + \bar{\nabla}\underbrace{\cancel{(\bar{\nabla} \times \bar{A})}}_{=0} \qquad \Rightarrow \quad \bar{w}^{(s)} \text{ quantifies the divergence of } \bar{w}$$

$$\bar{\nabla} \times \bar{w} = \underbrace{\cancel{\bar{\nabla} \times (\bar{\nabla} E)}}_{=0} + \bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}\underbrace{\cancel{(\bar{\nabla}\bar{A})}}_{=0} - \bar{\nabla}^2 \bar{A} \quad \Rightarrow \quad \bar{w}^{(v)} \text{ quantifies the curl of } \bar{w}$$

Linear structure formation

Dynamic of the components filling up the universe:

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} & \text{eq. of motion of tensor field } g_{\mu\nu} \\ D_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\nu\gamma} T^{\gamma\nu} + \Gamma^\nu_{\nu\gamma} T^{\mu\gamma} = 0 & \text{" " " " source } T_{\mu\nu} \end{cases}$$

solve simultaneously

Specify Energy-momentum tensor

on large scale the universe is neutral: no Electro-Magnetic $T^{EM}_{\mu\nu} = 0$

$T_{\mu\nu} = T^{DM}_{\mu\nu} + T^{B}_{\mu\nu} + T^{\gamma}_{\mu\nu}$

$T_{\mu\nu} = (\rho + \frac{P}{c^2}) u_\mu u_\nu + P g_{\mu\nu} + \Sigma_{\mu\nu}$ Ideal fluid, for now consider one component only

$\Sigma_{\mu\nu}$ anisotropic stress

traceless and flow orthogonal: $\Sigma^\mu{}_\mu = 0$ $\Sigma^\mu{}_\nu u^\nu = 0$

- Perturbative approach (1° order) (linear regime)
  - We consider small perturbations around the background

    $\rho = \hat{\rho} + \delta\rho \quad P = \hat{P} + \delta P \quad u^\mu = \bar{u}^\mu + \delta u^\mu$
  - Perturbed metric: consider scalar perturbations only

    $g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} \qquad ds^2 = a^2(\tau)\left[-(1+2\Psi)c^2 d\tau^2 + (1+2\Phi)\delta_{ij}\, dx^i dx^j\right]$ Newton Gauge
  - Perturbed E.-M. tensor:

    $T_{\mu\nu} = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} \qquad T^{(1)}_{\mu\nu} = \rho\left[\delta(1+c_s^2) u_\mu u_\nu + (1+w)(\delta u_\nu u_\mu + u_\mu \delta u_\nu) + c_s^2 \delta\, g_{\mu\nu}\right]$

    $(u^\mu) = \frac{1}{a}\left(c(1-\Psi), \bar{v}\right) \quad (u_\mu) = a\left(-c(1+\Psi), -c\bar{w} + \bar{v}\right)$

- Relevant perturbation quantities are

$\delta(x^\mu) \equiv \frac{\delta\rho(x^\mu)}{\hat{\rho}} \equiv \frac{\rho(x^\mu) - \hat{\rho}}{\hat{\rho}} \equiv D(t)\,\delta(x^i, 0)$ density contrast, $\delta(x^i, t)$, and growth/decay function, D

$\theta(x^\mu) \equiv D_i v^i(x^\mu)$ velocity divergence field, $\theta(x^i, t)$ $\quad v^i \equiv \frac{dx^i}{d\tau}$

in linear regim! δ is a random (Gaussian) field with $\langle\delta\rangle = 0$ and modes are decoupled

- Linearize and solve eq. of motion of $g_{\mu\nu}$ and Matter (r and m)

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad D_\mu T^{\mu\nu} = 0$$

• Linearize Einstein eq.s $\quad G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ $\quad$ Keep 1° order terms only, $c=1$, $"{}'" = \frac{d}{d\tau}$, $\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\tau}$

conformal H

$G^{(1)}_{\mu\nu} = \frac{8\pi G}{c^4} T^{(1)}_{\mu\nu} \rightleftarrows$ eq. of motion of $g_{\mu\nu}$

$$\begin{cases} (1)\quad \bar\nabla^2\phi + 3\mathcal{H}(\mathcal{H}\Psi - \phi') = -4\pi G a^2 \delta\rho & \mu\nu = 0,0 \\ (2)\quad \bar\nabla^2(\phi' - \mathcal{H}\Psi) = 4\pi G a^2 (1+w)\rho\theta & 0,i \\ (3)\quad \Psi = -\phi \quad \text{(not the case in modified gravity)} & i,j \quad (\text{because } \delta T^i{}_j = 0) \\ (4)\quad \phi'' + 2\mathcal{H}\phi' - \mathcal{H}\Psi' - (\mathcal{H}^2 + 2\mathcal{H}')\Psi = -4\pi G a^2 c_s^2 \delta\rho & ii \end{cases}$$

• Matter eq. of motion $\quad \nabla_\mu T^{\mu\nu} = 0$

$(\nu=0)\quad \nabla_\mu T^{\mu 0} = 0 \rightarrow (\delta\rho)' + 3\mathcal{H}(\delta\rho + \delta P) = -(\rho + P)(\theta + 3\phi')$ $\quad \rho' + 3\mathcal{H}(\rho + P) = 0 \quad w \equiv \frac{P}{\rho}$

$$(5)\quad \boxed{\delta' + 3\mathcal{H}(c_s^2 - w)\delta = -(1+w)(\theta + 3\phi')}$$

Perturbed continuity eq.

$\delta' = -\theta - 3\phi'$ non relativistic with $w=0$ $c_s=0$

velocity potential

$(\nu=i)\quad \nabla_\mu T^{\mu i} = 0 \quad \rightarrow \quad \delta q' + 3\mathcal{H}\delta q = -a\,\delta P - a(\rho+P)\Psi \qquad \delta q \equiv a(\rho+P)V \qquad v^i \equiv \nabla^i V \qquad w \equiv \frac{P}{\rho}$

plug δq, divergence ∇_i

$$(6)\quad \boxed{\theta' + \left[\mathcal{H}(1-3w) + \frac{w'}{1+w}\right]\theta = -\bar\nabla^2\left(\frac{c_s^2}{1+w}\delta + \Psi\right)}$$

relativistic Euler eq

$\theta' + \mathcal{H}\theta = -\bar\nabla^2(c_s^2\delta + \Psi)$ non relativistic matter

• Go to Fourier space $\qquad a'(\rho+P)V + a(\rho+P)'V + a(\rho+P)V$

$f(\bar x) = \int d^3k\, f(\bar k)\, e^{i\bar k \bar x} \qquad f = \phi, \Psi, \delta, \theta \qquad$ functions are sum of plane waves

$\bar k$ = Fourier mode (comoving) $\quad \lambda = \frac{2\pi}{k}\cdot a$ physical scale

$f(\bar x, t) \to f_k(t)$

$\bar\nabla \to i\bar k \qquad \nabla_i f(\bar x) = \int d^3k\, i\bar k\, \nabla_i \bar x\, f(\bar k) = \int d^3k\, i k\, f(\bar k)$

$\bar\nabla^2 \to -k^2$

we can "drop" the exponentials: Continuity and Euler eq.s are linear ⇒ no mode coupling

i.e. k-modes (eg. δ_k) evolve independently

i.e. there is no power transfer from one mode to the other

⇒ Continuity and Euler eq.s are valid for each mode

$$\begin{cases} (1)\quad k^2\phi_k + 3\mathcal{H}(\phi_k' - \mathcal{H}\Psi_k) = 4\pi G a^2 \rho\delta_k \\ (2)\quad k^2(\phi_k' - \mathcal{H}\Psi_k) = -4\pi G a^2(1+w)\rho\theta_k \\ (3)\quad \Psi_k = -\phi_k \\ (4)\quad \phi_k'' + 2\mathcal{H}\phi_k' - \mathcal{H}\Psi_k' - (\mathcal{H}^2 + 2\mathcal{H}')\Psi_k = -4\pi G a^2 c_s^2 \rho\delta_k \end{cases}$$

$$\boxed{k^2\phi_k = 4\pi G a^2\rho\left[\delta_k + 3\mathcal{H}(w+1)\theta_k/k^2\right]} \quad (A)$$

Relativistic Poisson eq.

(1),(3) ⇝ (4)

$$\boxed{\phi_k'' + 3\mathcal{H}(1+c_s^2)\phi_k' + (c_s^2k^2 + 2\mathcal{H}' + 3\mathcal{H}^2c_s^2 + \mathcal{H}^2)\phi_k = 0}$$

(B) Potential eq.

$$\begin{cases} (5)\quad \delta_k' + 3\mathcal{H}(c_s^2 - w)\delta_k = -(1+w)(\theta_k + 3\phi_k') \\ (6)\quad \theta_k' + \left[\mathcal{H}(1+3w) + \frac{w'}{1+w}\right]\theta_k = k^2\left(\frac{c_s^2}{1+w}\delta_k + \Psi_k\right) \end{cases}$$

Relativistic hydrodynamic in dynamic universe

Super-horizon scales $k \ll \mathcal{H}$ $\lambda \gg R_H$ To combine the rest, easier with approximations ...

$w = \text{const.} \Rightarrow c_s^2 = w$ (matter, radiation domination) ; use: $\mathcal{H}' = -\frac{1}{2}(1+3w)\mathcal{H}^2$, $\Psi = -\phi$

(eq. B): $\phi'' + 3\mathcal{H}(1+c_s^2)\phi' = 0$

2 solutions: $\phi' = 0 \Rightarrow \phi = \text{const.}$; the other one is decaying with time (irrelevant)

(eq. 1): $3\mathcal{H}^2\phi = 4\pi G a^2 \rho \delta_k$ $\quad \cancel{8\pi G \rho a^2}\,\phi = \cancel{4\pi G a^2}\rho\delta_k \quad 2\phi = \delta_k \Rightarrow \delta_k = \text{const.}$

$\uparrow \mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2$ Friedmann

Important: during transition $w_r = \frac{1}{3} \to 0 = w_m$ this is not the case, ϕ evolves! $\phi' \neq 0$

$\Rightarrow$ gravitational redshift!

Sub-horizon scales $k \gg \mathcal{H}$

- $w = 0$ in absence of perturbations but $c_s^2 = \frac{\delta P}{\delta \rho} \ll 1$
- Combine hydro equations (5-6) with Poisson eq. (A)

(6) $\boxed{\theta_k' + \mathcal{H}\theta_k = k^2(c_s^2\delta_k - \phi_k)}$

(A) $\boxed{k^2\phi_k \simeq 4\pi G a^2 \rho \delta_k \overset{*}{=} \frac{3}{2}\mathcal{H}^2\delta_k}$ $\quad \xrightarrow{\frac{d}{d\tau}} \quad \phi' = \frac{3}{2k^2}(2\mathcal{H}\mathcal{H}'\delta_k + \mathcal{H}^2\delta_k') = \frac{3\mathcal{H}^2}{2k^2}\delta_k\left(2\frac{\mathcal{H}'}{\mathcal{H}} + \frac{\delta_k'}{\delta_k}\right)$ $\quad$ * Friedmann

(5) $\delta_k' + 3\cancel{\mathcal{H}c_s^2}\delta_k = -(\theta_k + 3\phi_k')$ $\quad \boxed{\delta_k' = \underbrace{-\theta_k}_{(a)} \underbrace{- \frac{9\mathcal{H}^2}{2k^2}\delta_k\left(2\frac{\mathcal{H}'}{\mathcal{H}} + \frac{\delta_k'}{\delta_k}\right)}_{(b)} \overset{*}{\simeq} -\theta_k}$ $\quad$ * Newtonian limit

$\frac{d(5)}{d\tau}$ $\quad$ energy conservation eq. $\quad$ (a) continuity in Newtonian limit
(b) relativistic contributions

$$\delta_k'' = -\theta_k' \overset{(6)}{=} -\mathcal{H}\theta_k + k^2(c_s^2\delta_k - \phi_k) \overset{(5),(A)}{=} -\mathcal{H}\delta' + k^2c_s^2\delta_k - \frac{3}{2}\mathcal{H}^2\delta_k \qquad \boxed{\delta_k'' + \mathcal{H}\delta' + \left(k^2c_s^2 - \frac{3}{2}\mathcal{H}^2\right)\delta_k = 0} \quad (!)$$

density contrast perturbation eq.

Damped harmonic oscillator

- Meaning of c_s: $\mathcal{H} \to 0$ (Minkowski limit) $\delta_k'' + k^2c_s^2\delta_k = 0 \Rightarrow c_s = $ sound speed
- Growing mode for $k^2c_s^2 - \frac{3}{2}\mathcal{H}^2 < 0$, i.e. $\lambda > \lambda_J \equiv a\frac{2\pi}{k_J} = ac_s\sqrt{\frac{8\pi^2}{3\mathcal{H}^2}} = c_s\sqrt{\frac{\pi}{G\rho}}$ $\quad \omega_k = $ Real
  - $\uparrow$ physical, not comoving $\quad \nwarrow$ Friedmann
- Damped oscillations for $\lambda < \lambda_J$ $\quad \omega_k = $ complex
- Very small scales $c_s k \ll \mathcal{H}$: $\delta_k'' + \mathcal{H}\delta' - \frac{3}{2}\mathcal{H}^2\delta_k = 0 \Rightarrow$ solutions: $\delta_- = B a^{-3/2}$ $\quad \delta_+ = A \cdot a$
- Photons: $c_s = c/\sqrt{3} \Rightarrow \lambda_J \approx H^{-1} - R_H \Rightarrow$ no growth of structures during rad. era within the horizon

Evolution of ϕ: $\quad$ Poisson eq: $k^2\phi_k = 4\pi G a^2 \rho\left[\delta_k + 3\cancel{\mathcal{H}\theta_k/k^2}\right]$ $\quad w = 0$ $\rho \propto a^{-3}$ $\Rightarrow \phi_k = \text{const.}\ \cancel{a^2 a^{-3} a}$

2 species: matter and Λ

$$\delta_k'' + \mathcal{H}\delta' + \left(\cancel{k^2 c_s^2} - \frac{3}{2}\mathcal{H}^2\right)\delta_k = 0$$

generalize for matter + Λ for which $c_s = 0$ (DM) $(c_s=0)$

$$\delta_m'' + \mathcal{H}\delta_m' - \frac{3}{2}\mathcal{H}^2\left(\Omega_m\delta_m + \underbrace{\cancel{\Omega_\Lambda\delta_\Lambda}}_{=0}\right) = 0$$

$\delta_\Lambda = 0$ because Λ does not cluster

- Effect of Λ on perturbations growth

solution with Ω_m = const rough (!) approximation $\Rightarrow$ $\delta_m \propto a^m$ $\quad m_\pm = \frac{1}{4}\left(-1 \pm \sqrt{1+24\Omega_m}\right)$

for $\Omega_m \to 0$ (Λ dominates) : $m_\pm = \frac{1}{4}(-1 \pm 1) = \begin{cases} -1/2 & (-)\ \delta_m \text{ decaying solution} \\ 0 & (+)\ \delta_m = \text{const} \Rightarrow \text{it means that } \Lambda \text{ slows down growth} \end{cases}$

- Better approximation of numerical solution of eq.

growth rate : $f \equiv \dfrac{d\log\delta_m}{d\log a} \approx \Omega_m^\gamma(a)$ $\quad \gamma \approx 0{,}55$ growth index

$$\Omega_m(a) \equiv \frac{\rho_m}{\rho_{cr}} = \frac{\Omega_{m,0}\, a^{-3}}{\Omega_m a^{-3} + 1 - \Omega_m} \quad \text{for } \Lambda\text{CDM}$$

growth function : $G(a) \equiv \dfrac{\delta_m(a)}{\delta_{m,0}} = \exp\displaystyle\int_1^a f(a)\, d\log a \approx \exp\int_1^a \Omega_m^\gamma(a)\, d\log a$

$\Rightarrow \mathcal{H}^2\delta$ not const $\Rightarrow \phi$ not const.

Appendix

To compute $T_{\mu\nu}^{(1)}$ in perturbed $T_{\mu\nu} = T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)}$ you need $u_\nu = u_\nu^{(0)} + u_\nu^{(1)}$

- $(u^\mu) \equiv \underbrace{\frac{dx^\mu}{ds}}_{\text{proper time}} = \frac{dx^\mu}{d\tau}\frac{d\tau}{ds}$

  $\left\{ \frac{dx^\mu}{d\tau} = \left(1, \frac{dx^i}{d\tau}\right)^T \right.$ $v^i \equiv \frac{dx^i}{d\tau} = a\frac{dx^i}{dt}$ velocity in conformal time $\tau = x^0$ ds = proper time

  $\frac{ds}{d\tau} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = a\sqrt{+(1+2\psi)c\cancel{\frac{d\tau^2}{d\tau^2}} + 2w_i \cancel{\frac{d\tau}{d\tau}} v^i - (1-2\phi)\bar{v}^2 + 2S_{ij}v^iv^j}$

  $$= \frac{(1, v^i)}{a(\tau)\sqrt{(1+2\psi) + 2w_i v^i - (1-2\phi)\bar{v}^2 - 2S_{ij}v^iv^j}}$$ $\left(= \frac{\gamma}{a}(c, v^i)\right.$ like sp. relat. but γ factor is perturbed$\left.\right)$

  $* \simeq \frac{1}{a}(1, v^i)\underbrace{(1+2\psi)^{-1/2}}_{\text{taylor}} \simeq \frac{1}{a}(1, v^i)(1-\psi) \underbrace{\simeq}_{1^\circ\text{order (p.}\psi\ 2^\circ\text{order)}} \boxed{\frac{1}{a}(1-\psi, v^i)}$

  $*$ at 1° order ($v \ll c$, $\psi \ll 1$, $\phi \ll 1$) ψ^2, $\bar{v}\cdot\bar{w}$ are 2° order

- (u_μ): $u_0 = g_{00}u^0 + g_{0i}u^i \simeq -a^{\cancel{2}}(1+2\psi)c\frac{(1-\psi)}{\cancel{a}} - a^{\cancel{2}}w_i\frac{v^i}{\cancel{a}} \simeq -ac(1+\cancel{2}\psi - \cancel{\psi}) = -ac(1+\psi)$

  $u_i = g_{i0}u^0 + g_{ij}u^j \simeq -a^{\cancel{2}}w_i\frac{c}{\cancel{a}}(1-\psi) + a^2(1+2\phi)\delta_{ij}\frac{v^j}{\cancel{a}} + a^2 2S_{ij}\frac{v^j}{a} \underbrace{\simeq}_{1^\circ\text{order}} -acw_i + av_i$

- $u_\mu u^\mu = -c^2(1-\psi^2) + \bar{v}(\bar{v} - c\bar{w}) \simeq -c^2$

$\models\!\!>$ $\boxed{(u^\mu) = \frac{1}{a}(c(1-\psi), \bar{v}) \quad (u_\mu) = a(-c(1+\psi), -c\bar{w} + \bar{v}) \quad u_\mu u^\mu = -c^2}$ $\psi, \bar{w}, \bar{v}$ carries the perturbations at 1°order

- Interpreting v^i, physical separation r^i

  $\frac{dr^i}{dt} = \frac{d(ax^i)}{dt} = \dot{a}x^i + a\frac{dx^i}{dt} = aHx^i + a\frac{dx^i}{dt} = \overset{(1)}{\mathcal{H}x^i} + \overset{(2)}{v^i}$ (!)

  (2) $\hookrightarrow$ peculiar velocity (perturbation) $v^i \equiv \frac{dx^i}{d\tau}$

  (1) $\longrightarrow$ Hubble flow $\mathcal{H} \equiv aH$ conformal Hubble

Perturbation theory with a coulped scalar field

Multicomponent universe

Consider two components, matter + DE generic DE $\Rightarrow w_{DE}(a),\ c^2_{s,DE}(a)$

- $\delta_t = \sum_i \Omega_i \delta_i \qquad \theta_t = \sum_i \frac{1+w_i}{1+w_{eff}} \Omega_i \theta_i$ total perturbations
- $w_{eff} \equiv \frac{P_t}{\rho_t} = \sum_i \Omega_i w_i \qquad c^2_{s,t} = \frac{\sum_i c^2_{s_i} \Omega_i \delta_i}{\delta_t} = \frac{\sum_i c^2_{s_i} \Omega_i \delta_i}{\sum_i \Omega_i \delta_i}$ total eq. of state and sound speed

$\frac{\mathcal{H}'}{\mathcal{H}} = 1 + \frac{H'}{H} = -\frac{1}{2}(1+3w_{eff})$ w_{eff} obeys this expression (from Freedman + $P=\rho w$)

Gravity equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$

- Gravity is sourced by contribution of all components: ρ_{tot}, w_{eff}, $\mathcal{H}^2 = \frac{8\pi G}{3}\overset{(!)}{\rho_t} a^2$ (Friedmann eq.)

$\Rightarrow$ 1 set of gravity equations!

- Einstein eq. $(\mu,\nu = 0,0)$ recal: here "$'$" $= \frac{d}{d\tau}$

$$k^2\phi_k + 3\mathcal{H}(\phi_k' - \mathcal{H}\psi_k) = 4\pi G a^2 \rho_t \delta_k = \frac{3}{2}\mathcal{H}^2 \delta_k$$

$$\phi_k = \frac{3}{2}\frac{\mathcal{H}^2}{k^2}\delta_k + 3\frac{\mathcal{H}}{k^2}(\mathcal{H}\psi_k - \mathcal{H}\phi')$$

$$\boxed{\phi_k = 3\hat\lambda^2\left(\frac{1}{2}\delta_k + \psi_k - \phi'\right)} \quad (1)$$

$\frac{d}{dN} = \frac{1}{\mathcal{H}}\frac{d}{d\tau}$ $\quad \tau \to N$ $\quad$ "$'$" $\to \mathcal{H}$ "$'$" $\quad$ now "$'$" $= d/dN$

$\hat\lambda \equiv \mathcal{H}/k$

recal $\delta = \Omega\delta$

- Einstein eq. $(\mu,\nu = 0,i)$

$$k^2(\phi_k' - \mathcal{H}\psi_k) = -4\pi G a^2 (1+w_{eff})\rho\,\theta_{k,t} \quad \text{total}$$

$$\cancel{\mathcal{H}}\phi_k' = \cancel{\mathcal{H}}\psi_k - \frac{3}{2}\frac{\mathcal{H}^2}{k^2}(1+w_{eff})\cancel{\mathcal{H}}v_{k,t}$$

$$\boxed{\phi_k' = \psi_k - \frac{3}{2}\hat\lambda^2(1+w_{eff})v_{k,t}} \quad (2)$$

$v \equiv \theta_k/\mathcal{H}$ New velocity divergence definition

- Link $\phi \leftrightarrow \psi$ $(\mu,\nu = i,j)$ (3) $\boxed{\psi_k = -\phi_k}$ in GR (absence of anisotropic stress in $T_{\mu\nu}$)

$\psi_k = -\phi_k + \sigma_k$ $\quad$ in general $\psi_k = \phi_k + \sigma_k$

- Poisson eq.

$$k^2\phi_k = 4\pi G a^2 \rho\left[\delta_k + 3\mathcal{H}(w_{eff}+1)\theta_k/k^2\right] \qquad k^2 = \mathcal{H}^2/\hat\lambda^2 \qquad \mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2 \text{ Friedmann}$$

$$\frac{\cancel{\mathcal{H}^2}}{\hat\lambda^2}\phi_k = \frac{3}{2}\cancel{\mathcal{H}^2}\left[\delta_k + 3\cancel{\mathcal{H}}(w_{eff}+1)\cancel{\mathcal{H}}v_k\frac{\hat\lambda^2}{\cancel{\mathcal{H}^2}}\right]$$

$$\boxed{\phi_k = \frac{3}{2}\hat\lambda^2\left[\underbrace{\delta_k + 3(w_{eff}+1)v_k\hat\lambda^2}_{\equiv \Delta_k}\right]} \quad (A)$$

$\equiv \Delta_k$ gauge invariant combination, for comparison to observations

$\Delta_k \to \delta_k$ on small scales

Hydro sector $\nabla_\nu T^{\mu\nu} = 0$ (not interaction with DE!)

- DE is largely unknown ⇒ generic expressions for $w(\tau)$ and $c_s^2(\tau)$
- Dark energy affects structure formation trough $\mathcal{H}$ (dynamic of the background)
- Not only that... DE is a fluid and therefore it could cluster!

⇒ 2 sets of hydro equations! Because δ_M, δ_{DE} are independent

• Continuity (one fore M and one for DE)

$$\delta_k' + 3\mathcal{H}(c_s^2 - w)\delta_k = -(1+w)(\theta_k + 3\phi_k')$$

$$\cancel{\mathcal{H}}\delta_k' + 3\cancel{\mathcal{H}}(c_s^2 - w)\delta_k = -(1+w)\left(\frac{\theta_k}{\mathcal{H}} + 3\cancel{\mathcal{H}}\phi_k'\right)$$

$$\boxed{\delta_k' + 3(c_s^2 - w)\delta_k = -(1+w)(\vartheta_k + 3\phi_k')}$$

$\frac{d}{dN} = \frac{1}{\mathcal{H}}\frac{d}{d\tau}$ "$'$" $\to \mathcal{H}$ "$'$" now "$'$" $= d/dN$

$\vartheta \equiv \theta_k/\mathcal{H}$

• Euler eq. (one fore M and one for DE)

$$\theta_k' + \left[\mathcal{H}(1-3w) + \frac{w'}{1+w}\right]\theta_k = k^2\left(\frac{c_s^2}{1+w}\delta_k + \psi_k\right)$$

$$\cancel{\mathcal{H}}\left(\vartheta_k'\cancel{\mathcal{H}} + \vartheta_k\frac{\mathcal{H}'}{\mathcal{H}}\right) + \left[\cancel{\mathcal{H}}(1-3w) + \frac{\cancel{\mathcal{H}}w'}{1+w}\right]\cancel{\mathcal{H}}\vartheta_k = \frac{k^2}{\mathcal{H}^2}\left(\frac{c_s^2}{1+w}\delta_k + \psi_k\right)$$

$$\left(\vartheta_k' + \vartheta_k\frac{\mathcal{H}'}{\mathcal{H}}\right) + \left[1 - 3w + \frac{w'}{1+w}\right]\vartheta_k = \frac{1}{\hat\lambda^2}\left(\frac{c_s^2}{1+w}\delta_k + \psi_k\right)$$

$$\boxed{\vartheta_k' = \left[3w - 1 - \frac{w'}{1+w} - \frac{\mathcal{H}'}{\mathcal{H}}\right]\vartheta_k + \frac{1}{\hat\lambda^2}\left(\frac{c_s^2}{1+w}\delta_k + \psi_k\right)}$$

acceleration friction pressure gravity

$\theta' \to \mathcal{H}\theta' = \mathcal{H}(\vartheta'\mathcal{H} + \vartheta\mathcal{H}')$

$\hat\lambda \equiv \frac{\mathcal{H}}{k}$

$\frac{\mathcal{H}'}{\mathcal{H}} = -\frac{1}{2}(1+3w_{eff})$ from Freedman

• Combine them (generic DE!)

∘ from (1), (2), (5), (6)

$$\boxed{\phi'' + \left[3c_{s,e}^2 + \frac{1}{2}(5 - 3w_{eff})\right]\phi' + \left[(3+\hat\lambda^{-2})c_{s,e}^2 - 3w_{eff}\right]\phi = 3(c_{s,e}^2 - w_{eff})\sigma + \sigma'}$$

∘ from (5)−(6)

• $$\boxed{\delta_r'' - \frac{1}{2}(1+3w_{eff})\delta_r' + \frac{1}{3}\hat\lambda^{-2}\delta_r = \frac{4}{3}\hat\lambda^{-2}\phi + 2(1+w_{eff})\phi' - 4\phi''}$$

for radiation ($w = c_s^2 = 1/3$) $\hat\lambda \ll 1$ $\sigma = 0$

$$\boxed{\frac{d^2\Theta_k}{d\tau^2} + \frac{1}{3}k^2\Theta_k = \frac{k^2}{3}\phi_k - \frac{d^2\phi_k}{d\tau^2}}$$

for temperature anysotropies Θ $\delta_r = 4\Theta$ (Boltzmann law) (B_r)

for $w_{eff} \simeq 1/3$ radiation domination → CMB!!

• $$\boxed{\delta_m'' + \frac{1}{2}(1-3w_{eff})\delta_m' = -\hat\lambda^{-2}\psi - \frac{3}{2}(1-3w_{eff})\phi' - 3\phi''}$$

for matter = dust ($w_M = 0$) (B_m)

note $c_{s,e}^2$, w_{eff} containing DE: $c_{s,DE}^2(\tau)$, $w_{DE}(\tau)$!

here I stress δ_m only because it is observable: eg. galaxy clustering, cluster counts, ...

• $\lambda \ll 1,\ \Psi = -\Phi\ (\sigma = 0)$

$$\delta_m'' + \frac{1}{2}(1-3w_{eff})\delta_m' = \hat{\lambda}^{-2}\Phi \cancel{- \frac{3}{2}(1-3w_{eff})\Phi' - 3\Phi''} \qquad (B_m)$$

$$\Phi_k = \frac{3}{2}\hat{\lambda}^2\left[\delta_k \cancel{+ 3(w_{eff}+1)v_k\hat{\lambda}^2}\right] = \frac{3}{2}\hat{\lambda}^2(\Omega_m\delta_m + \Omega_{DE}\delta_{DE}) \qquad (A)$$

$$c^2_{s,t} = c^2_{s,DE}\left[1 - \frac{3\hat{\lambda}^2\Omega_m\delta_m}{2\Phi}\right] \qquad \text{eq. 11.30}$$

$$\delta_m'' + \frac{1}{2}(1-3\overset{(!)}{w_{eff}})\delta_m' = \frac{3}{2}(\Omega_m\delta_m + \overset{(!)}{\overline{\Omega_{DE}\delta_{DE}}})$$

$$c^2_{s,t} = c^2_{s,DE}\left[1 - \frac{\Omega_m\delta_m}{(\Omega_m\delta_m + \Omega_{DE}\delta_{DE})}\right]$$

not clustering DE : $\delta_{DE} = 0 \quad \Rightarrow \quad w_{eff} = \Omega_{DE}w_{DE}$

$$c^2_{s,t} = 0$$

DE as a scalar field

- Need to specify $w_{(\varphi)}(a)$, $c^2_{s(\varphi)}(a)$

- $w_\varphi(a) = \frac{P_\varphi(a)}{\rho_\varphi(a)}$ $c_s^2 = \frac{\delta P_\varphi}{\delta\rho_\varphi}$: need perturbative approach

- Scalar field: $\rho_\varphi = -\frac{1}{2} g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi)$ $\delta\rho_\varphi = H^2(\varphi'\delta\varphi' - \varphi'^2\phi) + V_{,\varphi}\delta\varphi$ (a)

  $P_\varphi = -\frac{1}{2} g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi)$ $\delta P_\varphi = H^2(\varphi'\delta\varphi' - \varphi'^2\phi) - V_{,\varphi}\delta\varphi$ (b)

  1° order terms using perturbed metric

  $\theta_\varphi = -\frac{ik^i\delta T^{(\varphi)}_{0i}}{(1+w_\varphi)\rho_\varphi} = \hat{\lambda}^{-2}\frac{\delta\varphi}{\varphi'}$ (d)

  clearly $\delta\phi = 0$ (no clustering) $c_s^2 = 1$

  yes H, not $\mathcal{H}$

  $m_\varphi^2 \equiv \frac{d^2V}{d\varphi^2}$ dark energy effective mass $\hat{m}^2 \equiv m_\varphi^2/H^2$ $\hat{V} \equiv V/H^2$ dimensionless versions

- Perturbation eq. for φ :

  take (5): $\delta_k' + 3(c_s^2 - w)\delta_k = -(1+w)(\vartheta_k + 3\phi_k')$, plug a,b,c, $\frac{\mathcal{H}'}{\mathcal{H}} = -\frac{1}{2}(1+3w_{\text{eff}})$, $\delta_\varphi = \frac{\delta\varphi}{\varphi}$

$$\boxed{\delta\varphi'' + \left(2 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\delta\varphi' + (\hat{\lambda}^{-2} + \hat{m}_\varphi^2)\delta\varphi - \varphi'(3\phi' - \Psi') + 2\hat{V}_{,\varphi}\Psi = 0}$$

↳ damped oscillations for scales $\hat{\lambda} < 1/\hat{m}_\varphi$ ⇒ no contribution to total gravitational potential
⇒ approximate it as homogeneous

↳ larger scales, behaviour depends on $\hat{m}_\phi$

$\hat{m}_\varphi < 1$ ($m_\varphi < H$) ϕ and $\delta\phi$ evolves slowly ⇒ $\phi' \simeq [3(1+w_\phi)\Omega_\phi]^{1/2}$ for $w \simeq -1$

$\hat{m}_\varphi > 1$ ($m_\varphi > H$) oscillations even on large scales
background will oscillate as well, w_{eff} will depart from the one of DE
⇒ field can act as dark matter

Including coupling

- Modifications of hydro eqs easy, just GR + an extra force

$$\nabla_\mu T^{\mu\nu}_{(DM)} = Q_{DM}(\varphi)\, T_{(DM)} \nabla^\nu\varphi$$
$$\nabla_\mu T^{\mu\nu}_{(b)} = Q_b(\varphi)\, T_{(b)} \nabla^\nu\varphi$$
$$\nabla_\mu T^{\mu\nu}_{(\varphi)} = -\left(Q_{DM}(\varphi)\, T_{(DM)} + Q_b(\varphi)\, T_{(b)}\right)\nabla^\nu\varphi$$

⇒ you get additional terms in continuity and Euler eq.

- Perturbation eq. for several coupled fluids "i" $i = DM, b$ $m(\varphi)$ $\hat{V}(\varphi)$ unperturbed

$$\boxed{\begin{aligned}\delta\varphi'' + \left(2 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\delta\varphi' + (\hat{\lambda}^{-2} + \hat{m}_\phi^2)\delta\varphi - \varphi'(3\phi' - \Psi') + 2\hat{V}_{,\varphi}\Psi = -3\textstyle\sum_i Q_i(1-3c_{s,i}^2)\Omega_i\delta_i - 6\sum_i Q_i(1-3w_i)\Omega_i\Psi \\ -3\textstyle\sum_i(1-3w_i)Q_{i,\varphi}\Omega_i\delta\varphi\end{aligned}}$$

Dust + Q universal, same for DM and b : $= -3Q\Omega_m\delta_m - 6Q\Omega_m\Psi$

$\boxed{\text{Getting a physical understanding}}$

we are not considering the propagation of this field (the waves), it is there static ("frozen")

Consider : quasi-static limit (all time derivatives negligible)

small scales $(\hat{\lambda}^2 \ll 1)$: $\Phi_k = \frac{3}{2}\hat{\lambda}^2 \left[\delta_k^N + \cancel{3(w_\phi+1)\vartheta_k\hat{\lambda}^2}\right] \Rightarrow (\Phi_k \ll \delta_k^N)$

$\Psi = -\varphi$

only DE and $m = (DM+b)$

$$\cancel{\delta\varphi''} + \cancel{\left(2+\frac{\mathcal{H}'}{\mathcal{H}}\right)\delta\varphi'} + (\underbrace{\hat{\lambda}^{-2}}_{\text{dominant}} + \underbrace{\cancel{\hat{m}_\phi^2}}_{(!)})\delta\varphi - \cancel{\varphi'(3\Phi'-\Psi')} + \cancel{2\hat{V}_{,\varphi}\overset{-\Phi}{\Psi}} = -3Q\Omega_m\delta_m^N + \cancel{6Q\Omega_m\overset{\Phi}{\Psi}}$$

$\boxed{\hat{\lambda}^{-2}\delta\varphi = -3Q\Omega_m\delta_m^N}$ $\Rightarrow$ $\delta\varphi$ obeys a Poisson eq.

$\Rightarrow$ field $\delta\varphi$ contributes as a gravitational potential when $Q \neq 0$ (modified gravity!)

$\boxed{\Psi \to \hat{\Psi} \equiv \Psi + Q\delta\varphi}$ To account for coupling in structure formation eq.s, just replace (!)

Astonishing! we got a Poisson eq. out of a Klein-Gordon eq., why?!

- In ordinary gravity:

  2 d.o.f. = 2 polarization modes, propagates only gravitons,

  no scalar particle out of gravity, in fact there is no KG eq. which arises from scalar fields

- Here:

  - now we have a scalar field $\delta\varphi$ $\Rightarrow$ KG eq. called a propagating deg. of freedom because $\delta\varphi$ has its own dynamics

  - but in the quasi-static approx we neglect it, and take only the static part of the potential

  $\Rightarrow$ a Poisson eq. can be seen as a static limit of a K.G. eq. (just the "Coulomb part")

  - this scalar field is a generalized gravity, i.e. gravity is not just carried by a spin-2 particle it is also carried by a scalar particle obeying a K.G. eq.

Other modification : recall... the varying particle mass (from continuity eq.)

Mass of the particle can be interpreted as changing with time (because of coupling with φ)

$\rho_m = n_{m0} \cdot m\, a^{-3} e^{Q\varphi} \simeq n_{m0} \cdot m\, a^{-3} \cdot (1+Q\varphi)$

$\Rightarrow$ Additional term in Euler (which is a force eq.: $\bar{F} = \dot{\bar{p}} = m\dot{\bar{v}} + \underline{\dot{m}\,\bar{v}}$)

$\boxed{Q\varphi'}$ (!) it will go into the friction term "change in inertia"

Final perturbation eq. with coupling

General:
$$\begin{cases} \delta_k' + 3(c_s^2 - w)\delta_k = -(1+w)(\vartheta_k + 3\phi_k') \\ \vartheta_k' = \left[3w - 1 - \frac{w'}{1+w} - \frac{\mathcal{H}'}{\mathcal{H}}\right]\vartheta_k + \frac{1}{\hat\lambda^2}\left(\frac{c_s^2}{1+w}\delta_k + \Psi_k\right) \\ \phi_k = \frac{3}{2}\hat\lambda^2\left[\delta_{k,t} + 3(w+1)\vartheta_k\hat\lambda^2\right] \qquad \delta_i = \Omega_i\delta_i \\ \delta\varphi'' + \left(2 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\delta\varphi' + (\hat\lambda^{-2} + \hat m_\phi^2)\delta\varphi - \varphi'(3\phi' - \Psi') + 2\hat V_{,\varphi}\Psi = -3Q\Omega_m\delta_m - 6Q\Omega_m\Psi_m \end{cases}$$

only DE and DM

Consider as before: quasi static limit, small scales, $\Psi = -\phi$ (GR), use $\frac{\mathcal{H}'}{\mathcal{H}} = -\frac{3}{2}(1 - w_{eff})$

Furthermore: $w = 0$, $c_s = 0$ (DM), $\delta_{k\varphi} \ll \delta_{k,DM}$ as you screw up observation of galaxies

$$\begin{cases} \delta_k' = -\vartheta_k \\ \vartheta_k' = -\frac{1}{2}(1 - 3w_{eff}\overset{(c)}{)}\vartheta_k + \frac{1}{\hat\lambda^2}\overset{(2)}{\Psi_k} \\ \overset{(2)}{\phi_k} = \frac{3}{2}\hat\lambda^2(\delta_{k\,DM} + \cancel{\delta_{k\varphi}}) \\ \hat\lambda^{-2}\delta\varphi = -3Q\Omega_m\delta_m \end{cases} \xrightarrow{\text{coupling}} \begin{cases} \delta_k' = -\vartheta_k \\ \vartheta_k' = -\frac{1}{2}(1 - 3w_{eff} + \overset{*}{\underline{2Q\varphi_k'}})\vartheta_k + \frac{1}{\hat\lambda^2}(\Psi_k + Q\delta\varphi_k) \\ \underbrace{-(\Psi_k}_{-\phi} + Q\delta\varphi_k) = \frac{3}{2}\hat\lambda^2\delta_k \end{cases}$$

$\rightarrow \phi = \frac{3}{2}\hat\lambda^2\Omega_m\delta_m(1 + 2Q^2)$

i.e. $\phi = \phi_N(1+2Q^2)$

$G \rightarrow G(1+2Q^2)$

$$\boxed{\begin{array}{l} (a)\ \Psi \rightarrow \hat\Psi \equiv \Psi + Q\delta\varphi = \Psi - 3Q^2\Omega_m\delta_m\hat\lambda^2 \\ (b)\ Q\varphi' \text{ in friction term of Euler} \end{array}}$$ (Q^2 dependency!)

matter perturbation eq.: $\delta_m'' + \frac{1}{2}\delta_m'(1 - 3w_{eff} + 2Q\varphi') - \frac{3}{2}(1 + 2Q^2)\Omega_m\delta_m = 0$

good approx.: growth rate $f \equiv \frac{d\log\delta_m}{d\log a} \approx \Omega_m^\gamma(a) \quad \gamma = 0{,}54(1 + \alpha Q^2)$

Further generalizations

e.g. decouple baryons $\Rightarrow$ additional eq. $\nearrow$ DM, $\searrow$ b

e.g. Q not constant

you get different eq.s but the modifications will appear in the same points!

Screening

Scale at which $\hat\lambda^2 \sim \hat m^2$, $\hat\lambda^2 \ll 1$ i.e. now we do NOT neglect $\hat m$

Poisson for $\delta\varphi$ / Poisson for ϕ

$$\begin{cases} (\hat\lambda^{-2} + \hat m^2)\,\delta\varphi = -3Q\Omega_m \delta_{km} \\ \hat\lambda^{-2}\phi_k = \frac{3}{2}\Omega_m \delta_{km} \end{cases} \rightarrow \hat\Psi = \Psi_N + Q\,\delta\varphi = -\frac{3}{2}\Omega_m \delta_m \hat\lambda^2 \Big(1 + \underbrace{\frac{3Q^2}{\hat\lambda^2(\hat\lambda^{-2} + \hat m^2)}}_{(!)}\Big)$$

Coupling, Q, gives you an effective gravitational constant $G_{eff} = G_N\left(1 + \frac{3Q^2\hat\lambda^2}{(\hat\lambda^{-2} + \hat m^2)}\right)$ $F = G_{eff}\frac{M m}{r^2}$

To understand what $\hat\Psi$ does, consider a point massive particle

We work out $\Omega_m \delta_m$ to make it work for a point particle

Point particle: $\rho(\bar x) = m\,\delta_D(\bar x - \bar x_0)$ m = mass, $\bar x_0$ = position of particle

Real space: $\int \rho(\bar x)\,d^3x = \int m\,\delta_D(\bar x - \bar x_0)\,d^3x = m$

Fourier space: $\rho_k \propto \int d^3x\,\rho(\bar x)\,e^{-i\bar k\bar x} \propto \int d^3x\, m\,\delta_D(\bar x - \bar x_0)\,e^{-i\bar k\bar x} \propto m$ (I dropped normalization of Fourier transf.) the point is $\propto m$

Particle = Earth $\Rightarrow \rho \gg \hat\rho$ $\quad \Omega_m \delta_{km} = \frac{\cancel{\rho_m}}{\rho_{cr}}\frac{\delta_k \cancel{\hat\rho}}{\cancel{\hat\rho}} = \frac{\delta_k}{\rho_{cr}} \propto m\frac{8\pi G}{3H^2}$

$$\hat\Psi_k = -\frac{\cancel{3}}{\cancel{2}}\, m\,\frac{\cancel{8}^{\,4}\pi G}{\cancel{3}H^2}\,\frac{1}{\hat\lambda^2}\left(1 + \frac{3Q^2\hat\lambda^2}{(\hat\lambda^{-2} + \hat m^2)}\right)$$ to Real space

$$\hat\Psi(\bar r) = \int d^3k\,\hat\Psi_k\,e^{i\bar k\bar r} = \int d^3k\, A\cdot\left(1 + \frac{3Q^2\hat\lambda^2}{(\hat\lambda^{-2} + \hat m^2)}\right) = -\frac{GM}{r}\left(1 + 2Q^2 e^{-\hat m r}\right) \simeq \begin{cases} 1 + 2Q^2 & r \gg \hat m \\ 1 & r \ll \hat m \end{cases}$$

Yukawa potential carried by a scalar particle

- $\hat m^{-1}$ = range of the interaction
- Q = strength of interaction
- in general $Q(\varphi)$, $m(\varphi)$ $\Rightarrow$ possible dependency on position and space

Note: there is an additional k dependency!
now growth with different scales will evolve differently!

Summary

Scalar-Vector-Tensor decomposition

$$ds^2 = a^2(t)\{-(1+2\Psi)c^2 d\tau^2 + 2 w_i \, c \, d\tau \, dx^i + [(1+2\Phi)\delta_{ij} + h_{ij}]dx^i dx^j\}$$

$$\bar{w} \equiv \bar{w}^s + \bar{w}^v \qquad \hat{\bar{w}}^s = -i\bar{k}\hat{E} \quad , \quad \hat{\bar{w}}^v = -i\bar{k}\times\hat{\bar{A}} \quad \bar{\nabla}\bar{A} = 0$$

$$h_{ij} = h^s_{ij} + h^v_{ij} + h^T_{ij} \qquad \hat{h}^s_{ij} = \left(k_i k_j - \tfrac{1}{3}\delta_{ij}\right)B(\bar{k}) \quad , \quad \hat{h}^v_{ij} = \tfrac{i}{2}(k_i h_j + k_j h_i) \quad , \quad h^T_{ij} = h^T_{ji}, \quad \partial^i h^T_{ij} = 0, \quad h^{Ti}{}_i = 0$$

Perturbative approach

- $\rho = \hat{\rho} + \delta\rho \quad P = \hat{P} + \delta P \quad u^\mu = \bar{u}^\mu + \delta u^\mu \quad g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} \quad T_{\mu\nu} = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}$

  $ds^2\big|_s = a^2(t)[-(1+2\Psi)c^2 d\tau^2 + (1+2\Phi)\delta_{ij}dx^i dx^j]$ Newton's gauge: $w_i^s \overset{!}{=} 0 \quad B \overset{!}{=} 0$

  $T^{(1)}_{\mu\nu} = \rho[\delta(1+c_s^2)u_\mu u_\nu + (1+w)(\delta u_\nu u_\mu + u_\mu \delta u_\nu) + c_s^2 \delta \, \delta_{\mu\nu}]$

  $(u^\mu) = \frac{1}{a}(c(1-\Psi), \bar{v}) \quad (u_\mu) = a(-c(1+\Psi), -c\bar{w} + \bar{v}) \quad u_\mu u^\mu = -c^2$

- $\delta(x^\mu) \equiv \dfrac{\delta\rho(x^\mu)}{\hat{\rho}} \equiv \dfrac{\rho(x^\mu) - \hat{\rho}}{\hat{\rho}}$ density contrast field , velocity divergence field $\theta(x^\mu) \equiv \nabla_i v^i(x^\mu)$

Solve eq.s of motion

"$'$" $= d/d\tau$

$G^{(1)}_{\mu\nu} = \dfrac{8\pi G}{c^4} T^{(1)}_{\mu\nu} \qquad \nabla_\mu T^{\mu\nu}_{(1)} = 0$ move to Fourier space $\partial_j \to i k_j \quad j = 1,2,3 \quad f \to f_k$

$$\begin{cases} (1) & k^2\phi_k + 3\mathcal{H}(\phi_k' - \mathcal{H}\Psi_k) = 4\pi G a^2 \rho \delta_k \\ (2) & k^2(\phi_k' - \mathcal{H}\Psi_k) = -4\pi G a^2 (1+w)\rho\theta_k \\ (3) & \Psi_k = -\phi_k \\ (4) & \phi_k'' + 2\mathcal{H}\phi_k' - \mathcal{H}\Psi_k' - (\mathcal{H}^2 + 2\mathcal{H}')\Psi_k = -4\pi G a^2 c_s^2 \rho \delta_k \end{cases}$$

$\Psi_k = -\phi_k + \sigma_k$

$$\boxed{k^2\phi_k = 4\pi G a^2 \rho[\delta_k + 3\mathcal{H}(w+1)\theta_k/k^2]} \quad (A)$$ Relativistic Poisson eq.

$(1),(3) \rightsquigarrow (4)$

$$\boxed{\phi_k'' + 3\mathcal{H}(1+c_s^2)\phi_k' + (c_s^2 k^2 + 2\mathcal{H}' + 3\mathcal{H}^2 c_s^2 + \mathcal{H}^2)\phi_k = 0}$$

(B) Potential eq.

$$\begin{cases} (5) & \delta_k' + 3\mathcal{H}(c_s^2 - w)\delta_k = -(1+w)(\theta_k + 3\phi_k') \\ (6) & \theta_k' + \left[\mathcal{H}(1+3w) + \dfrac{w'}{1+w}\right]\theta_k = k^2\left(\dfrac{c_s^2}{1+w}\delta_k + \Psi_k\right) \end{cases}$$

Relativistic hydrodynamic in dynamic universe

Define: $\hat{\lambda} \equiv \mathcal{H}/k$, $v \equiv \theta_k/\mathcal{H}$, Change "time" variable : $\tau \to N \Rightarrow$ "$'$" $\to \mathcal{H}\cdot$"$'$"

$$\begin{cases} (1) & \phi_k = 3\hat{\lambda}^2\left(\tfrac{1}{2}\delta_k + \Psi_k - \phi'\right) \\ (2) & \phi_k' = \Psi_k - \tfrac{3}{2}\hat{\lambda}^2(1+w_{eff})v_{k,t} \end{cases}$$

(A) $\phi_k = \frac{3}{2}\hat{\lambda}^2[\delta_k + 3(w_{eff}+1)v_k\hat{\lambda}^2]$ now "$'$" $= d/dN$

(5) $\delta_k' + 3(c_s^2 - w)\delta_k = -(1+w)(v_k + 3\phi_k')$

(6) $v_k' = \left[3w - 1 - \dfrac{w'}{1+w} - \dfrac{\mathcal{H}'}{\mathcal{H}}\right]v_k + \dfrac{1}{\hat{\lambda}^2}\left(\dfrac{c_s^2}{1+w}\delta_k + \Psi_k\right)$

Combine

$\phi'' + [3c_{s,t}^2 + \frac{1}{2}(5 - 3w_{eff})]\phi' + [(3+\hat{\lambda}^{-2})c_{s,t}^2 - 3w_{eff}]\phi = 3(c_{s,t}^2 - w_{eff})\sigma + \sigma'$ from (1), (2), (5), (6)

$\delta_r'' - \frac{1}{2}(1+3w_{eff})\delta_r' + \frac{1}{3}\hat{\lambda}^{-2}\delta_r = \frac{4}{3}\hat{\lambda}^{-2}\phi + 2(1+w_{eff})\phi' - 4\phi''$ for radiation $(w = c_s^2 = 1/3)$ $\hat{\lambda} \ll 1$ $\sigma = 0$

$\delta_m'' + \frac{1}{2}(1-3w_{eff})\delta_m' = -\hat{\lambda}^{-2}\Psi - \frac{3}{2}(1-3w_{eff})\phi' - 3\phi''$ for matter = dust $(w_m = 0)$

DE as a scalar field

$$w_\varphi(a) = \frac{P_\varphi(a)}{\rho_\varphi(a)} \qquad c_s^2 = \frac{\delta P_\varphi}{\delta \rho_\varphi}$$

$$\rho_\varphi = -\frac{1}{2} g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi)$$
$$P_\varphi = -\frac{1}{2} g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi)$$

$$\delta\rho_\varphi = H^2(\varphi'\delta\varphi' - \varphi'^2\Phi) + V_{,\varphi}\delta\varphi \quad (a)$$
$$\delta P_\varphi = H^2(\varphi'\delta\varphi' - \varphi'^2\Phi) - V_{,\varphi}\delta\varphi \quad (b)$$
$$\theta_\varphi = -\frac{ik^i\delta T^{(\varphi)}_{0i}}{(1+w_\varphi)\rho_\varphi} = \hat\lambda^{-2}\frac{\delta\varphi}{\varphi'} \quad (d)$$

Additional eq. for dynamical field

$$(7)\quad \delta\varphi'' + \left(2+\frac{\mathcal{H}'}{\mathcal{H}}\right)\delta\varphi' + (\hat\lambda^{-2} + \hat m_\phi^2)\delta\varphi - \varphi'(3\Phi' - \Psi') + 2\hat V_{,\varphi}\Psi = 0 \qquad (5), a, b, c, \quad \frac{\mathcal{H}'}{\mathcal{H}} = -\frac{1}{2}(1+3w_{eff}), \quad \delta_\varphi \equiv \frac{\delta\varphi}{\varphi}$$

Coupled dark energy / Modified gravity

$$\nabla_\mu T^{\mu\nu}_{(DM)} = Q_{DM}(\phi)T_{(DM)}\nabla^\nu\phi$$
$$\nabla_\mu T^{\mu\nu}_{(b)} = Q_b(\phi)T_{(b)}\nabla^\nu\phi$$
$$\nabla_\mu T^{\mu\nu}_{(\phi)} = -\left(Q_{DM}(\phi)T_{(DM)} + Q_b(\phi)T_{(b)}\right)\nabla^\nu\phi$$

$$(7)\quad \delta\varphi'' + \left(2+\frac{\mathcal{H}'}{\mathcal{H}}\right)\delta\varphi' + (\hat\lambda^{-2} + \hat m_\phi^2)\delta\varphi - \varphi'(3\Phi' - \Psi') + 2\hat V_{,\varphi}\Psi = -3\sum_i Q_i(1-3c_{s,i}^2)\Omega_i\delta_i - 6\sum_i Q_i(1-3w_i)\Omega_i\Psi$$
$$-3\sum_i(1-3w_i)Q_{i,\varphi}\Omega_i\delta\varphi$$

Sub-horizon - Quasi static limit

- $\hat\lambda^{-2}\delta\varphi = -3Q_{DM}\Omega_{DM}\delta_{DM}$ (7) : $\delta\varphi$ obeys a Poisson eq. $\Rightarrow$ $\delta\varphi$ is a gravitational potential because of Q
  $\Psi \to \hat\Psi \equiv \Psi + Q\delta\varphi$
- $\rho_m \simeq m_{m0}\cdot m a^{-3}\cdot(1+Q\varphi)$: mass evolution $\Rightarrow$ add $Q\varphi'$ in friction term of Euler eq.

$$\begin{cases} \delta_k' = -v_k \\ v_k' = -\frac{1}{2}(1-3w_{eff} + \underline{2Q\varphi_k'})v_k + \frac{1}{\hat\lambda^2}(\Psi_k + Q\delta\varphi_k) \\ -(\Psi_k + Q\delta\varphi_k) = \frac{3}{2}\hat\lambda^2\delta_k \;\to\; \phi = \frac{3}{2}\hat\lambda^2\Omega_{DM}\delta_{DM}(1+2Q^2) \end{cases}$$

$$\delta_m'' + \frac{1}{2}\delta_m'(1-3w_{eff} + 2Q\varphi') - \frac{3}{2}(1+2Q^2)\Omega_m\delta_m = 0$$

$$f \equiv \frac{d\log\delta_m}{d\log a} \approx \Omega_m^\gamma(a) \qquad \gamma = 0{,}54(1+\alpha Q^2)$$

Part VI

Non-linear cosmic structure formation

Non-linear perturbation theory

Full non-linear structure formation studied through numerical simulations
Now we go to 2^{nd} order
Standard Eulerian perturbation theory
linear scales: $\lambda = 10-50\ Mpc$ $\quad k < 0{,}1\ h/Mpc$
mildly non-linear perturbations $\quad 0{,}1 < k < 0{,}3\ h/Mpc$

Assumptions

- keep scalar perturbations only, i.e. irrotational fluid (this breaks at some small scale)
- higher order perturbations occur on scales much smaller than the horizon $\lambda \ll R_H$
- Non relativistic particles
- Dust: $w = 0 = c_s^2$

Newtonian mechanics and gravity (full, not 1° order)

$$\begin{cases} (1)\quad \partial_t \rho + \bar\nabla_r(\rho\bar u) = -\underbrace{\rho\bar\nabla\bar v}_{\simeq 0} & \text{continuity} \\ (2)\quad \partial_t u + (\bar u\bar\nabla_r)\bar u = -\cancel{\frac{\bar\nabla P}{\rho}} - \nabla_r\phi_N & \text{Euler} \\ (3)\quad \bar\nabla^2\phi_N = 4\pi G\rho & \text{Poisson} \end{cases}$$

where: r = proper coordinates (physical)
$\bar u \equiv \dot a x + v(\bar x, t)$
$\bar v \equiv a\dot{\bar x}$ peculiar velocity

- Neglect shear term in Euler eq. (spherical perturbations remain spherical)
- derivatives with respect to $N \equiv \ln a$ $\quad \frac{\delta}{\delta N} = \text{“}'\text{”}$
- change to co-moving coordinates, $\bar\nabla_x = a\bar\nabla_r \Rightarrow \left.\frac{\delta f(\bar x, t)}{\delta t}\right|_r = \left.\frac{\delta f}{\delta t}\right|_x - \frac{\dot a}{a}(\bar x\bar\nabla_x)f$, $\quad \phi \equiv -\left(\phi_N + \frac{1}{2}a\ddot a\bar x^2\right)$

$$\begin{cases} (1)\quad \mathcal{H}\delta' = -\nabla^i[(1+\delta)v_i] \\ (2)\quad \mathcal{H}v_i' = -\mathcal{H}v_i - v_j\nabla^j v_i + \nabla_i\phi \\ (3)\quad \bar\nabla^2\phi = -\frac{3}{2}\Omega_m\mathcal{H}^2\delta \end{cases}$$

- decompose $\bar v \equiv \bar v^{\parallel} + \bar v^{\perp}$ $\quad$ neglect $\bar v^{\perp}$ associated to vorticity
(if $\bar v^{\perp} = 0$ initially, it remains zero for a vanishing shear stress)
- renormalized velocity divergence field $\theta \equiv \nabla^i v^i/\mathcal{H}$
- total derivative $\frac{d\delta}{dN} \equiv \delta' + \frac{v^i}{\mathcal{H}}\nabla_i\delta \Rightarrow$ continuity $\frac{d\delta}{dN} = -\theta(1+\delta)$

Combine eq.s (1)-(2)-(3): $\boxed{\underbrace{\delta'' + \left(1 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\delta' - \frac{3}{2}\Omega_m\delta}_{\text{as in linearized eq.s}} = \underbrace{\frac{4}{3}\frac{1}{1+\delta}(\delta')^2 + \frac{3}{2}\Omega_m\delta^2}_{\text{non-linear terms (!)}}}$ full (all orders) (A)

$\delta(t)$

- Einstein-deSitter universe

to have an easy analytical solution

Einstein-deSitter: $\Omega_m = 1$, $\Omega_k = 0$, $\frac{\mathcal{H}'}{\mathcal{H}} = -\frac{1}{2}$, $f \equiv \frac{\delta'}{\delta} \simeq \Omega_m^\gamma(\eta) = 1$ at all times

Non linear terms:

$$\frac{4}{3}\frac{1}{1+\delta}(\delta')^2 + \frac{3}{2}\Omega_m \delta^2 = \Big(\frac{4}{3}\underbrace{\frac{1}{1+\delta}}_{3^{rd}\text{ order}}\underbrace{\Big(\frac{\delta'}{\delta}\Big)^2}_{f} + \frac{3}{2}\Omega_m\Big)\delta^2 \simeq \Big(\frac{4}{3}\overset{=1}{f^2} + \frac{3}{2}\overset{=1}{\Omega_m}\Big)\delta^2 = \frac{17}{6}\delta^2 \quad \text{up to } 2^\circ \text{ order}$$

Expand $\delta(\vec{x}, \eta)$ in a perturbative series:

$\delta = G_{(1)}\delta_{(1)} + G_{(2)}\delta^2_{(1)} + \dots$ first and second order growth rates: $G_{(1)}$, $G_{(2)}$

Solve for $G_{(2)}$ up to second order

plug in eq.(A): $G_{(1)}\delta_{(1)}$ will give you a zero (recall... it is not contributing to higher orders

$$\cancel{G''_{(1)}\delta_{(1)}} + G''_{(2)}\delta^2_{(1)} + \Big(1 + \frac{\mathcal{H}'}{\mathcal{H}}\Big)\Big(\cancel{G'_{(1)}\delta_{(1)}} + G'_{(2)}\delta^2_{(1)}\Big) - \frac{3}{2}\Omega_m\Big(\cancel{G_{(1)}\delta_{(1)}} + G_{(2)}\delta^2_{(1)}\Big) \simeq \frac{17}{6}\Big(G_{(1)}\delta_{(1)} + G_{(2)}\delta^2_{(1)}\Big)^2$$

$$G''_{(2)}\cancel{\delta^2_{(1)}} + \frac{1}{2}G'_{(2)}\cancel{\delta^2_{(1)}} - \frac{3}{2}G_{(2)}\cancel{\delta^2_{(1)}} \simeq \frac{17}{6}\Big(G_{(1)}\cancel{\delta_{(1)}} + \underbrace{\cancel{G_{(2)}\delta^2_{(1)}}}_{4^{th}\text{ order}}\Big)^2$$

ansatz $G_{(2)} = \alpha G^2_{(1)}$ $\Rightarrow$ $\boxed{G_{(2)} = \frac{17}{21}G^2_{(1)}}$ where $G_{(1)} = \eta^1$ in Einstein-deSitter

Other cosmologies?

Solve numerically but... in any case $G_{(2)} \propto G^2_{(1)}$ is a very good approx!

Go to Fourier space

Need to evaluate δ_k modes to evaluate power spectrum $\langle \delta(\bar{k})\delta^*(\bar{k}')\rangle \equiv (2\pi)^3 P(k)\,\delta(\bar{k}-\bar{k}')$

Now... things get messy...

- Full conservation law: $\partial_t \rho + \bar{\nabla}_r(\rho \bar{v}) = 0$ $\quad \boxed{\dot{\delta} = -\bar{\nabla}[(1+\delta)\bar{v}]}$ $\quad$ "$\dot{}$" = conformal

  ("any" cosmology)

- F. transform: $f(\bar{x}) = \int d^3k\, f_k\, e^{i\bar{k}\bar{x}}$

$$\int d^3k_1\, \dot{\delta}_{k_1} e^{i\bar{k}_1\bar{x}} = \bar{\nabla}\left[\left(1+\int d^3k\, \delta_k e^{i\bar{k}\bar{x}}\right)\int d^3k_2\, \bar{v}_{k_2} e^{i\bar{k}_2\bar{x}}\right] = -i\int \bar{k}_2 \bar{v}_{k_2} e^{i\bar{k}_2\bar{x}} d^3k_2 - i\int (\bar{k}_1+\bar{k}_2)\,\delta_{k_1}\bar{v}_{k_2} e^{i(\bar{k}_1+\bar{k}_2)\bar{x}} d^3k_1 d^3k_2$$

- Back Fourier transform to extract the δ_k and $\bar{u}_k$ modes

left: $\displaystyle \int d^3k_1 \frac{d^3x}{(2\pi)^3}\,\dot{\delta}_{k_1} e^{i(\bar{k}_1-\bar{k}_3)\bar{x}} = \int d^3k_1\, \dot{\delta}_{k_1}\delta_D(\bar{k}_1-\bar{k}_3) = \dot{\delta}_{k_3}$ $\qquad \displaystyle \delta_D(\bar{k}-\bar{k}') = \int \frac{d^3x}{(2\pi)^3} e^{i(\bar{k}-\bar{k}')}$ Delta Dirac

right: $\displaystyle -i\int \bar{k}_2\bar{v}_{k_2} e^{i(\bar{k}_2-\bar{k}_3)\bar{x}} d^3k_2 \frac{d^3x}{(2\pi)^3} - i\int (\bar{k}_1+\bar{k}_2)\,\delta_{k_1}\bar{v}_{k_2} e^{i(\bar{k}_1+\bar{k}_2-\bar{k}_3)\bar{x}} d^3k_1 d^3k_2 \frac{d^3x}{(2\pi)^3}$

$$\Rightarrow \boxed{\dot{\delta}_k = -i\bar{k}\underset{\substack{2^\circ\text{order}\\(*)}}{\bar{v}_k} - i\int d^3k_1\, d^3k_2\, (\bar{k}_1+\bar{k}_2)\overbrace{\underset{\substack{1^\circ\text{order!}\\(***)}}{\delta_{k_1}}\underset{\substack{1^\circ\text{order!}\\(**)}}{\bar{v}_{k_2}}}^{2^\circ\text{order}}\,\delta_D(\bar{k}_1+\bar{k}_2-\bar{k})}$$

$k_3 \to k$

Keep only 2^{nd} order terms $\quad \delta = \varepsilon\delta^{(1)} + \varepsilon^2\delta^{(2)} + \dots$, $\quad \bar{v} = \varepsilon \bar{v}^{(1)} + \varepsilon \bar{v}^{(2)}$

- Using (*) $\theta \equiv i\bar{k}\bar{v}$, (**) $-i\dfrac{\bar{k}}{k^2}\theta = \bar{v}$, (***) $\delta_u = -\underbrace{\dfrac{\theta_k}{aHf}}_{\equiv\vartheta} = -\dfrac{\vartheta_k}{f} = \dfrac{\vartheta^\circ_k \overbrace{G_{(1)}(t)}^{\text{linear growth}}}{f}$ from linear theory

$$\dot{\delta}_k = -\theta + i\int d^3k_1\, d^3k_2\, (\bar{k}_1+\bar{k}_2)\frac{-\vartheta_{k_1}}{f_{k_1}}\left(-i\frac{\bar{k}_2\theta_{k_2}}{k_2^2}\right)\delta_D(\bar{k}_1+\bar{k}_2-\bar{k})$$

$\Big\}\ \div\, aH \equiv \mathcal{H}$, $\quad \dfrac{G_1G_2}{f_1} = \dfrac{G_1G_2G_1'}{G_1} = G_1'G_2$, $\quad f = \dfrac{G'}{G}$

$$\vartheta + \delta_k' = \int d^3k_1\, d^3k_2\, (\bar{k}_1+\bar{k}_2)\frac{\bar{k}_2}{k_2^2}\vartheta^\circ_{k_1}\vartheta^\circ_{k_2} G'_{(1)1} G_{(1)2}\,\delta_D(\bar{k}_1+\bar{k}_2-\bar{k})$$

Symmetrize kernel

$$(\bar{k}_1+\bar{k}_2)\frac{\bar{k}_2}{k_2^2} \longrightarrow K_c = \frac{1}{2}\left[(\bar{k}_2+\bar{k}_1)\frac{\bar{k}_1}{k_1^2} + (\bar{k}_1+\bar{k}_2)\frac{\bar{k}_2}{k_2^2}\right] = \frac{1}{2}(\bar{k}_1+\bar{k}_2)\left(\frac{\bar{k}_1}{k_1^2}+\frac{\bar{k}_2}{k_2^2}\right)$$

k_1 and k_2 have same meaning

$$\Rightarrow \boxed{\overbrace{\vartheta + \delta_k' = \int d^3k_1\, d^3k_2\, K_c(\bar{k}_1,\bar{k}_2)\,\vartheta^\circ_{k_1}\vartheta^\circ_{k_2}\, G'_{(1)1} G_{(1)2}\,\delta_D(\bar{k}_1+\bar{k}_2-\bar{k})}^{\text{correction at } 2^{nd}\text{ order} \equiv c}} \quad (A)$$

Delta function (enforces energy conservation)

Time evolution

Quantity that we are tracking

Modes are coupled!!! Kernell specific for the starting eq.

The 2^{nd} order correction is based on 1^{st} order quantities!

$\Rightarrow$ use linear theory to get $G_{(1)i}$, evaluate integral to get 2^{nd} order correction

Poisson eq. in Fourier space

Take divergence of Euler eq., plug in Poisson eq.
Follow the same procedure
You get the same structure also for Euler eq.

$$\boxed{\vartheta' + F\vartheta + S\delta = -\underbrace{\int d^3k_1\, d^3k_2\, K_E(\bar{k}_1, \bar{k}_2)\, \vartheta^\circ_{k_1} \vartheta^\circ_{k_2}\, G'_{(1)1} G'_{(1)2}\, \delta_D(\bar{k}_1 + \bar{k}_2 - \bar{k})}_{\equiv E}} \quad (B)$$

For E-deS

$$F(z) = 1 + \frac{H'(z)}{H(z)}, \quad S(z) = \frac{3}{2}$$

$$K_E = \frac{\bar{k}_1 \bar{k}_2 (\bar{k}_1 + \bar{k}_2)}{2k_1^2 k_2^2}$$

Solve the system of equations for $G_{(2)}$

- $(A)' \curvearrowright (B) \Rightarrow \delta'' + F(z)\delta' - S(z)\delta = C' - E + F(z)C$
- plug $\delta \simeq \delta_{(1)} + \delta_{(2)} = G_{(1)}\delta_0 + G_{(2)}\delta_0 + \dots$ as before $G_{(1)}$ vanish becaus they satisfy the linear eq.

$$\delta''_{(2)} + F\delta'_{(2)} - S\delta_{(2)} = \underbrace{(G''_{(2)} + 2FG_{(2)} - SG_{(2)})}_{\equiv R^2(z)}\delta$$

- Solution : $\delta^{(2)} = G^{(2)}(z)\mathsf{R}(z)^{-2}[C' - E + F(z)C]$

$$= \frac{G^{(2)}(z)}{\mathsf{R}(z)^2} \int \delta_1\delta_2[(G_1G_2'' + G_1'G_2')K_C + G_1'G_2'K_E + F(z)G_1G_2'K_C]\delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d^3k_1 d^3k_2$$

In E-deS universe

$G_{(1)} = G'_{(1)} = G''_{(1)} = a$ and $G_{(2)} = a^2$, plug in $\uparrow$ $\quad \delta = G\tilde{\delta} \quad \tilde{\delta}$: init condition

$$\delta_{(2)} = G_{(1)}^2 \int d^3k_1\, d^3k_2\, F_{(2)}^{EdeS}(\bar{k}_1, \bar{k}_2)\, \tilde{\delta}_1^{(1)} \tilde{\delta}_2^{(1)}\, \delta_D(\bar{k}_1 + \bar{k}_2 - \bar{k})$$

$$\vartheta_{(2)} = G_{(1)}^2 \int d^3k_1\, d^3k_2\, G_{(2)}^{EdeS}(\bar{k}_1, \bar{k}_2)\, \tilde{\vartheta}_1^{(1)} \tilde{\vartheta}_2^{(1)}\, \delta_D(\bar{k}_1 + \bar{k}_2 - \bar{k})$$

$\uparrow$ assuming G independent on the mode and using $GG' = GfG = G^2$ ($f=1$ EdS)

$$F_{(2)} = -\left[\frac{5}{7} + \frac{1}{2}\bar{k}_1\bar{k}_2\left(\frac{1}{k_1^2} + \frac{1}{k_2^2}\right) + \frac{2}{7}\frac{(\bar{k}_1\bar{k}_2)^2}{k_1^2 k_2^2}\right]$$

$$G_{(2)} = \frac{3}{2} + \frac{\bar{k}_1\bar{k}_2}{2k_1k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{4}{7}\left(\frac{\bar{k}_1\bar{k}_2}{k_1k_2}\right)^2$$

note: $F_2(\bar{k}, -\bar{k}) = G_2(\bar{k}, -\bar{k}) = 0$

Going at 3rd order

we need 3rd order to evaluate the power spectrum, more later....

$$\delta = \varepsilon\delta_{(1)} + \varepsilon^2\delta_{(2)} + \varepsilon^3\delta_{(3)}$$

$$\delta^{(3)} = \int d^3k_1\, d^3k_2\, d^3k_3\, F_3(\bar{k}_1, \bar{k}_2, \bar{k}_3)\, \delta_{k_1}\delta_{k_2}\delta_{k_3}\, \delta_D\left(\sum_{i=1}^{3}\bar{k}_i - \bar{k}\right)$$

Going to higher order

total mess... go for numerical simulations

Power spectrums

Power spectrum: captures statistical properties of the fluctuations of a field for each k mode
it is the Fourier transform of the correlation function
it can be measured by studying distribution of galaxies, clusters

$\boxed{\langle f(k) f^*(k')\rangle = (2\pi)^3 P(k)\,\delta(\bar{k}-\bar{k}')}$ modes are not correlated at 1° order

3D Power spectrum in general

$\boxed{\delta = \varepsilon\delta_{(1)} + \varepsilon^2\delta_{(2)} + \varepsilon^3\delta_{(3)}}$

3rd order expansion

density power spectrum (one-loop spectrum)

$$\boxed{\langle \delta_k \delta_{k'}^* \rangle} = \langle(\varepsilon\delta_{(1)} + \varepsilon^2\delta_{(2)} + \varepsilon^3\delta_{(3)})(\varepsilon\delta^*_{(1)} + \varepsilon^2\delta^*_{(2)} + \varepsilon^3\delta^*_{(3)})\rangle$$
$$= \underbrace{\varepsilon^2\langle\delta_{(1)}\delta^*_{(1)}\rangle}_{P_{Linear}} + \underbrace{\cancel{2\varepsilon^3\langle\delta_{(1)}\delta^*_{(2)}\rangle}}_{P_{12}} + \underbrace{\varepsilon^4\langle\delta_{(2)}\delta^*_{(2)}\rangle}_{P_{22}} + \underbrace{2\varepsilon^4\langle\delta_{(1)}\delta^*_{(3)}\rangle}_{2P_{13}} + O(\varepsilon^5)$$

4th order contributions ⇒ need also $\delta_{(3)}$!

$P_L(k) = \langle\delta_{(1)}\delta^*_{(1)}\rangle$ linear contribution

$$P_{12}(k) = \langle\delta_k^{(1)}\delta_k^{(2)*}\rangle = \langle\delta_k^{(1)}\cdot G^2_{(1)}\int d^3k_1\, d^3k_2\, F^{EdeS}_{(2)}(\bar{k}_1,\bar{k}_2)\,\delta^{(1)*}_{k_1}\delta^{(1)*}_{k_2}\,\delta_D(\bar{k}_1+\bar{k}_2-\bar{k})\rangle$$
$$= \int d^3k_1\, d^3k_2\, F_{(2)}(\bar{k}_1,\bar{k}_2)\,\langle\delta_k^{(1)}\delta^{(1)*}_{k_1}\delta^{(1)*}_{k_2}\rangle\,\delta_D(\bar{k}_1+\bar{k}_2-\bar{k})$$

$$P_{22}(k) = \langle\delta_k^{(2)}\delta_k^{(2)}\rangle = \int \underbrace{d^3k_1\, d^3k_2\, d^3k_3\, d^3k_4}_{\text{2D integral because of }\delta_D} F_{(2)}(\bar{k}_1,\bar{k}_2)\, d^3k_3\, d^3k_4\, F_{(2)}(\bar{k}_3,\bar{k}_4)\,\langle\delta^{(1)}_{k_1}\delta^{(1)}_{k_2}\delta^{(1)*}_{k_3}\delta^{(1)*}_{k_4}\rangle\,\delta_D(\bar{k}_1+\bar{k}_2-\bar{k})\,\delta_D(\bar{k}_3+\bar{k}_4-\bar{k})$$

$$P_{13}(k) = \langle\delta_k^{(1)}\delta_k^{(3)}\rangle = \int d^3k_1\, d^3k_2\, d^3k_3\, F_{(3)}(\bar{k}_1,\bar{k}_2,\bar{k}_3)\,\langle\delta_k^{(1)}\delta^{(1)*}_{k_1}\delta^{(1)*}_{k_2}\delta^{(1)*}_{k_3}\rangle\,\delta_D(\bar{k}_1+\bar{k}_2+\bar{k}_3-\bar{k})$$

Assume a probability distribution for the amplitude of the modes

here: Gaussian initial conditions (well motivated by many inflationary models and data)

recall: modes are independent in the linear regime ⇒ fluctuation will be Gaussian as well

can exploit Wick's theorem: multi-variate Gaussian with covariance matrix C

$$\langle f f^*\rangle = N\int f_a f_b \exp\left(-\tfrac{1}{2}k_i C^{-1}_{ij} k_j\right) \equiv C_{ab}$$

N = normalization

$$\langle f_a f_b f_c\rangle = N\int f_a f_b f_c \exp\left(-\tfrac{1}{2}k_i C^{-1}_{ij} k_j\right) = 0 \quad \text{as all odd orders}$$

$$\langle f_a f_b f_c f_d\rangle = \langle f_a f_b\rangle\langle f_c f_d\rangle + \langle f_a f_c\rangle\langle f_b f_d\rangle + \langle f_a f_d\rangle\langle f_b f_c\rangle = C_{ab}C_{cd} + C_{ac}C_{bd} + C_{ad}C_{bc}$$

$$P_{22} = \int\langle\delta_1^{(1)}\delta_2^{(1)}\delta_3^{*(1)}\delta_4^{*(1)}\rangle\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta_D(\mathbf{k}-\mathbf{k}_3-\mathbf{k}_4)(F_2 d^3k_1 d^3k_2)(F_2 d^3k_3 d^3k_4)$$

Wick's theorem

$$= \int(\underbrace{\cancel{\langle\delta_1^{(1)}\delta_2^{(1)}\rangle}}_{=0}\cancel{\langle\delta_3^{*(1)}}\delta_4^{*(1)}\rangle + \langle\delta_1^{(1)}\delta_3^{*(1)}\rangle\langle\delta_2^{(1)}\delta_4^{*(1)}\rangle + \langle\delta_1^{(1)}\delta_4^{*(1)}\rangle\langle\delta_2^{(1)}\delta_3^{*(1)}\rangle)\ldots$$

Power spectrum definition ⊛

$$= \int (0 + P(k_1)\delta_D(\mathbf{k}_1-\mathbf{k}_3)P(k_2)\delta_D(\mathbf{k}_2-\mathbf{k}_4) + P(k_1)\delta_D(\mathbf{k}_1-\mathbf{k}_4)P(k_2)\delta_D(\mathbf{k}_2-\mathbf{k}_3))\ldots$$

$d\bar{k}_1$, $d\bar{k}_2$

$$= \int P(k_1)\underset{(1)}{\delta_D(\mathbf{k}_1-\mathbf{k}_3)}P(k_2)\underset{(2)}{\delta_D(\mathbf{k}_2-\mathbf{k}_4)}\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta_D(\mathbf{k}-\underset{=k_1}{\mathbf{k}_3}-\underset{=k_2}{\mathbf{k}_4})(F_2 d^3k_1 d^3k_2)(F_2 d^3k_3 d^3k_4) + (3\leftrightarrow 4)$$

(1) $\bar{k}_1 = \bar{k}_3$
(2) $\bar{k}_2 = \bar{k}_4$

$$= 2\int P(k_1)P(k_2)\underset{(3)}{\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}\underset{(4)}{\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}F^2{}_2 d^3k_1 d^3k_2$$

(3) (4) $\bar{k}_2 = \bar{k}-\bar{k}_1$

$$= 2\int P(k_1)P(|\mathbf{k}-\mathbf{k}_1|)F^2{}_2 d^3k_1$$

⊛ i.e. exploit non correlation of 1° order $\delta_D(\bar{k}_1-\bar{k}_2)$

$$P_{13} = 3P(k)\int P(k_1)F_3(\mathbf{k},\mathbf{k}_1,-\mathbf{k}_1)d^3k_1$$

$\mapsto$ One-Loop power spectrum $\boxed{\langle \delta\delta^* \rangle = P_L + P_{22} + 2P_{13} + \dots}$ $\langle \delta_k \delta^*_{k'} \rangle = (2\pi)^3 P(k)\,\delta_D(k-k')$

P_L = from linear theory

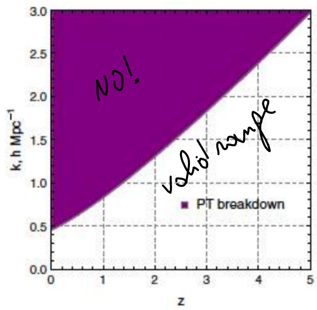
$$\left.\begin{aligned} P_{22} &= 2\int P(k_1)\,P(|\vec{k}-\vec{k}_1|)\,F^2_{(2)}(\vec{k},\vec{k}-\vec{k}_1)\,d^3k_1 \\ P_{13} &= 3P(k)\int P(k_1)\,F_3(\vec{k},\vec{k}_1,-\vec{k}_1)\,d^3k_1 \end{aligned}\right\}$$

2^{nd} order corrections $=$ 4^{th} order in power spectrum

Velocity power spectrum

Same procedure : $\langle v_k v^*_{k'} \rangle = \dots$

linear but with no BAOs



Chudaykin+ (2020)

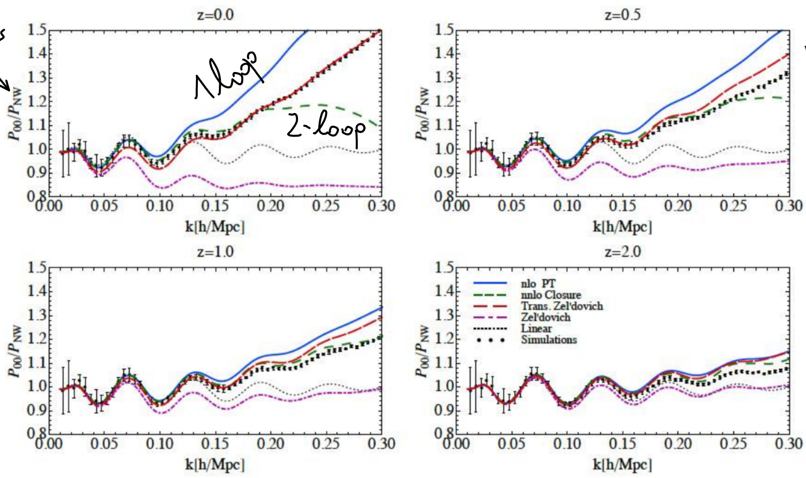


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Figure 1. $P_{00}(k)$ power spectrum term is plotted at four redshifts $z = 0.0,\ 0.5,\ 1.0$ and 2.0. We show linear result (black, dotted), one loop PT (blue, solid), two loop closure (green, dashed), corrected Zel'dovich (red, long-dashed) of [33] , simple Zel'dovich (magenta, dot-dashed) and simulation measurements (black dots). The error bars show the variance among realizations in simulations. The power spectrum is divided by no-wiggle fitting formula from [34], to reduce the dynamic range.

Appendix

- In the linear regime you can easily convert δ, $\vec{v}$, and $\theta \equiv \vec{\nabla}\vec{v}$

Full continuity : $\dot{\delta} = -\vec{\nabla}(1+\delta)\vec{v} \xrightarrow{\text{Linear}} \dot{\delta} = -\vec{\nabla}\vec{v} \xrightarrow{\text{Fourier}} \dot{\delta}_k = -i\vec{k}\vec{v}$

Decompose $\vec{v}$: $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$ neglect $\vec{v}_\perp$ $\qquad \vec{v} = \frac{\vec{k}}{k} v$

$\boxed{\dot{\delta}_k = -ikv}$ (1)

Using the growth function $f \equiv \frac{\delta'}{\delta}$: $\qquad \dot{\delta} = aH\delta' = aHf\delta$ (2)

Combine (1)-(2) : $v = iaHf\delta_k \frac{1}{k} \qquad v = iaHf\delta_k\frac{1}{k} \qquad \vec{v} = \frac{\vec{k}}{k} v \quad \Rightarrow \quad \boxed{\vec{v} = iaHf\delta_k \frac{\vec{k}}{k^2}}$

With respect to θ $\quad -i\vec{k}\vec{v} = -i\theta = iaHf\delta_k \frac{\vec{k}\vec{k}}{k^2} \quad \Rightarrow \quad \boxed{\theta = -aHf\delta_k}$

3 equivalent expressions

Summary

Full eqs: continuity, Euler, Poisson (newtonian)

$$\begin{cases} \mathcal{H}\delta' = -\nabla^i[(1+\delta)v_i] \\ \mathcal{H}v_i' = -\mathcal{H}v_i - v_j\nabla^j v_i + \nabla_i\phi \\ \nabla^2\phi = -\frac{3}{2}\Omega_m\mathcal{H}^2\delta \end{cases}$$

$$\boxed{\underbrace{\delta'' + \left(1+\frac{\mathcal{H}'}{\mathcal{H}}\right)\delta' - \frac{3}{2}\Omega_m\delta}_{\text{as in linearized eq.s}} = \underbrace{\frac{4}{3}\frac{1}{1+\delta}(\delta')^2 + \frac{3}{2}\Omega_m\delta^2}_{\text{non-linear terms (!)}}}$$

$\simeq \frac{17}{6}\delta^2$ Einstein-deSitter

Expand $\delta(\vec x, a)$ in a perturbative series

$\delta = G_{(1)}\delta_{(1)} + G_{(2)}\delta_{(1)}^2 + \dots$ ansatz $G_{(2)} = \alpha G_{(1)}^2$ $\Rightarrow$ $\boxed{G_{(2)} = \frac{17}{21}G_{(1)}^2}$

Go to Fourier space

$\theta \equiv i\vec k\vec v$ (linear!) $\delta_k = \frac{v_k^\circ G_{(1)}(t)}{f}$ $-i\frac{\vec k}{k^2}\theta = \vec v$

$\dot\delta = -\nabla[(1+\delta)\vec v] \longrightarrow \dot\delta_k = -i\vec k\vec v_k - i\int d^3k_1\, d^3k_2\, (\vec k_1+\vec k_2)\,\delta_{k_1}\vec v_{k_2}\,\delta_D(\vec k_1+\vec k_2-\vec k)$ ← Keep only 2^{nd} order terms

Continuity $\div aH \equiv \mathcal{H}$ ($\boxed{\vartheta + \delta_k' = \underbrace{\int d^3k_1\, d^3k_2\, K_C(\vec k_1,\vec k_2)\, \vartheta^\circ_{k_1}\vartheta^\circ_{k_2}\, G'_{(1)1}G_{(1)2}\,\delta_D(\vec k_1+\vec k_2-\vec k)}_{\equiv C}}$

mode coupling!!

Euler $\boxed{\vartheta' + F\vartheta + S\delta = -\underbrace{\int d^3k_1\, d^3k_2\, K_E(\vec k_1,\vec k_2)\, \vartheta^\circ_{k_1}\vartheta^\circ_{k_2}\, G'_{(1)1}G'_{(1)2}\,\delta_D(\vec k_1+\vec k_2-\vec k)}_{\equiv E}}$

$K_C = \frac{1}{2}(\vec k_1+\vec k_2)\left(\frac{\vec k_1}{k_1^2}+\frac{\vec k_2}{k_2^2}\right)$ $K_E = \frac{\vec k_1\vec k_2(\vec k_1+\vec k_2)}{2k_1^2k_2^2}$ $F(z) = 1+\frac{\mathcal{H}'(z)}{\mathcal{H}(z)}$, $S(z) = \frac{3}{2}$

Solve the system of equations for $G_{(2)}$

- $(A)' \curvearrowright (B)$ $\delta \simeq \delta_{(1)} + \delta_{(2)} = G_{(1)}\delta_0 + G_{(2)}\delta_0$ $\Rightarrow$ $\delta_{(2)}'' + F\delta_{(2)}' - S\delta_{(2)} = \underbrace{(G_{(2)}'' + 2FG_{(2)} - SG_{(2)})}_{\equiv R^2(z)}\delta$
- Solution: $\delta^{(2)} = G^{(2)}(z)\mathsf{R}(z)^{-2}[C' - E + F(z)C]$

$$= \frac{G^{(2)}(z)}{\mathsf{R}(z)^2}\int \delta_1\delta_2[(G_1G_2'' + G_1'G_2')K_C + G_1'G_2'K_E + F(z)G_1G_2'K_C]\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1d^3k_2$$

In E-deS universe

$G_{(1)} = G'_{(1)} = G''_{(1)} = a$ and $G_{(2)} = a^2$, plug in ↑ $\delta = G\tilde\delta$ $\tilde\delta$: init condition

$\delta_{(2)} = G_{(1)}^2\int d^3k_1\, d^3k_2\, F_{(2)}^{EdeS}(\vec k_1,\vec k_2)\,\tilde\delta_1^{(1)}\tilde\delta_2^{(1)}\,\delta_D(\vec k_1+\vec k_2-\vec k)$

$\vartheta_{(2)} = G_{(1)}^2\int d^3k_1\, d^3k_2\, G_{(2)}^{EdeS}(\vec k_1,\vec k_2)\,\tilde\vartheta_1^{(1)}\tilde\vartheta_2^{(1)}\,\delta_D(\vec k_1+\vec k_2-\vec k)$

$F_{(2)} = -\left[\frac{5}{7} + \frac{1}{2}\vec k_1\vec k_2\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right) + \frac{2}{7}\frac{(\vec k_1\vec k_2)^2}{k_1^2k_2^2}\right]$

$G_{(2)} = \frac{3}{2} + \frac{\vec k_1\vec k_2}{2k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{4}{7}\left(\frac{\vec k_1\vec k_2}{k_1k_2}\right)^2$

note: $F_2(\vec k,-\vec k) = G_2(\vec k,-\vec k) = 0$

One-Loop power spectrum $\langle\delta\delta^*\rangle = P_L + P_{22} + 2P_{13} + \dots$ $\langle\delta_k\delta_{k'}^*\rangle = (2\pi)^3P(k)\delta_D(k-k')$

P_L = from linear theory

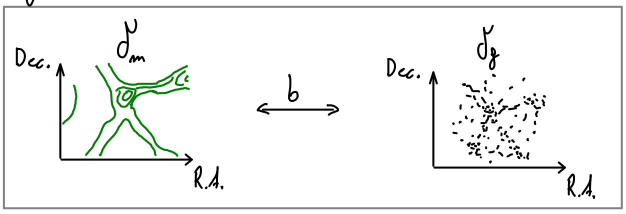
$$\left.\begin{aligned} P_{22} &= 2\int P(k_1)P(|\vec k-\vec k_1|)F_{(2)}^2(\vec k,\vec k-\vec k_1)\,d^3k_1 \\ P_{13} &= 3P(k)\int P(k_1)F_3(\vec k,\vec k_1,-\vec k_1)\,d^3k_1 \end{aligned}\right\}$$

2^{nd} order corrections = 4^{th} order in power spectrum

The bias

We have the equations for δ_m = matter (most of which is dark matter)
What we observe are galaxies, i.e. baryons
Galaxies they sample δ_m but their formation depends on a lot of baryonic physics
↳ "galaxies are a biased tracer of the underlying density matter field"

galaxy formation is a complex matter, still many unknowns
link δ_m with δ_g through the bias : $\boxed{\delta_g = b\,\delta_m}$



b should contain all astrophysical aspects linked to galaxy formation (complicated)
different types of galaxies are biased in different ways, e.g. red ones (ellipticals) are more clustered than blue ones
People started with b being a constant factor (to be fit with data), fine in the linear regime
At higher order b is a non linear function
Moreover δ_g might not only depend on δ_m, it might also depend on velocity field, gravitational potential, ...

$$\boxed{\delta_g(\delta_m, \Phi_g, \Phi_v) = \underbrace{\cancel{c} + b_1\delta_m + \frac{b_2}{2}\delta_m^2}_{\text{Taylor expansion at 2° order}} + \underbrace{b_{\mathcal{G}_2}\mathcal{G}_2 + \cancel{\frac{b_3}{6}\delta_m^3} + \cancel{b_{\delta\mathcal{G}_2}\delta\mathcal{G}_2} + \cancel{b_{\mathcal{G}_3}\mathcal{G}_3} + b_{\Gamma_3}\Gamma_3 + \cancel{R_*^2\partial^2\delta_m}}_{\text{Other physical dependencies}}}$$ Chudaykin+(2020)

$$\mathcal{G}_2(\Phi_g) \equiv (\partial_i\partial_j\Phi_g)^2 - (\partial_i^2\Phi_g)^2 \qquad \Gamma_3 \equiv \mathcal{G}_2(\Phi_g) - \mathcal{G}_2(\Phi_v)$$

The barred termes are not needed because degenera with other terms (they are redundant)
=> A 2° order you just need 4 bias terms : $\underline{b_1, b_2, b_{\mathcal{G}_2}, b_{\Gamma_3}}$

! Free parameters in the cosmological analysis !

What we measure is $\boxed{\langle \delta_g(\delta_m, \Phi_g, \Phi_v)\, \delta_g^*(\delta_m, \Phi_g, \Phi_v)\rangle}$ with the procedure we followed earlier

Redshift correction

Other complication: we do not measure physical distances, we measure redshifts z
on top of cosmological redshift (→ distance) we have proper motions, $\bar{v}$! (velocity field)

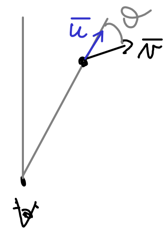
$u = \bar{v}\,\frac{\bar{r}}{r}$ peculiar velocity component along the line of sight

$v_{obs} = H_0 r + u$ Hubble-Lemaître law ($H_0 = 1 = c$)

$v_{obs} = H_0 r + u - u_0$ include your own proper motion (redshift space)

$\bar{v}_k = i H_0 f \delta_k \frac{\bar{k}}{k^2}$ linear theory: $v \propto \delta$! (from continuity eq.)

⇒ v is not another free variable



From "real" cosmological distance (cosmological redshift) to observed
this is a change in coordinates ⇒ change in observed density $\delta_g = b\,\delta_m$

$$\boxed{\delta_{obs,k} = \delta_k\,(1+\beta\cos^2(\theta))}$$

in redshift space ↗ ↖ in cosmological redshift

β redshift distortion parameter
β linked to $v \propto f\delta_m \propto f\,b\,\delta_g$ "real"

$\boxed{\beta = \frac{f}{b}}$ f = growth factor
b = bias (1° order

Final correction to the galaxy power spectrum

At linear level:

$$P_g(k,\mu,z) = b^2(1+\beta\cos^2(\theta))^2 P_m(k,z) \equiv Z_1(k,b,\mu,z)^2 P_m(k,z)$$

bias ↑ ↑ redshift distortion

At second order:

same procedure but with very complicated kernels

$$P_{\rm gg,RSD}(z,k,\mu) = Z_1^2(\mathbf{k})P_{\rm lin}(z,k) + 2\int_{\mathbf{q}} Z_2^2(\mathbf{q},\mathbf{k}-\mathbf{q})P_{\rm lin}(z,|\mathbf{k}-\mathbf{q}|)P_{\rm lin}(z,q) + 6Z_1(\mathbf{k})P_{\rm lin}(z,k)\int_{\mathbf{q}} Z_3(\mathbf{q},-\mathbf{q},\mathbf{k})P_{\rm lin}(z,q)$$

bias ↘ ↙ redshift distortion bias ↓ extra kernel ↓

$$Z_2(\mathbf{k}_a,\mathbf{k}_b) = b(k)\{F_2(\mathbf{k}_a,\mathbf{k}_b) + \beta(k)\mu^2 G_2(\mathbf{k}_a,\mathbf{k}_b) + \frac{\beta(k)\mu k}{2}\left[\frac{\mu_{az}}{k_a}b(k_b)(1+\beta(k_b)\mu_{bz}^2) + \frac{\mu_{bz}}{k_b}b(k_a)(1+\beta(k_a)\mu_{az}^2)\right]\} + \frac{b_2(k)}{2} + b_\Gamma(k)S_1(\mathbf{k}_a,\mathbf{k}_b)$$

original density kernel ↗ original velocity kernel ↑

$$\begin{aligned} Z_3(\mathbf{k}_a,\mathbf{k}_b,\mathbf{k}_c) = b\{&F_3(\mathbf{k}_a,\mathbf{k}_b,\mathbf{k}_c) + \beta\mu^2 G_3(\mathbf{k}_a,\mathbf{k}_b,\mathbf{k}_c) + \beta\mu k b_{ab}[F_2(\mathbf{k}_a,\mathbf{k}_b) + \beta_{ab}\mu_{abz}^2 G_2(\mathbf{k}_a,\mathbf{k}_b)]\frac{\mu_{cz}}{k_c} \\ &+ \beta\mu k b_a(1+\beta_a\mu_{az}^2)\frac{\mu_{bcz}}{k_{bc}}G_2(\mathbf{k}_b,\mathbf{k}_c) + \frac{(\beta\mu k)^2}{2}b b_a(1+\beta_a\mu_{az}^2)\frac{\mu_{bz}}{k_b}\frac{\mu_{cz}}{k_c}\} \\ &+ 2b_G S_1(\mathbf{k}_a,\mathbf{k}_b+\mathbf{k}_c)F_2(\mathbf{k}_b,\mathbf{k}_c) + b_G b\beta\mu k\frac{\mu_{az}}{k_a}S_1(\mathbf{k}_b,\mathbf{k}_c) \\ &+ 2b_\Gamma S_1(\mathbf{k}_a,\mathbf{k}_b+\mathbf{k}_c)(F_2(\mathbf{k}_b,\mathbf{k}_c) + G_2(\mathbf{k}_b,\mathbf{k}_c)) \end{aligned}$$

$$\mu_{bz} = \frac{\mathbf{k}_b\mathbf{z}}{k_b} \qquad S_1(\mathbf{k}_a,\mathbf{k}_b) = \frac{(\mathbf{k}_a\cdot\mathbf{k}_b)^2}{k_a^2k_b^2} - 1$$

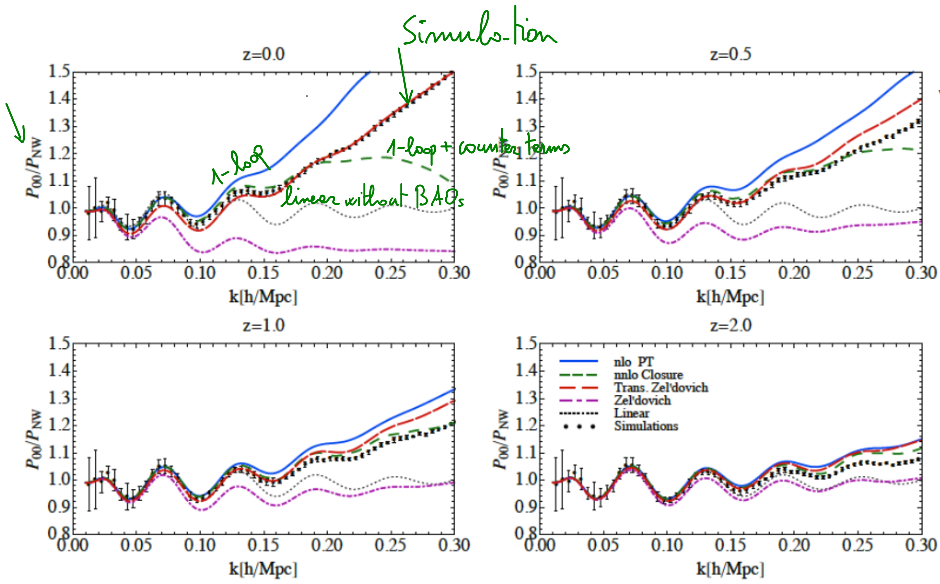


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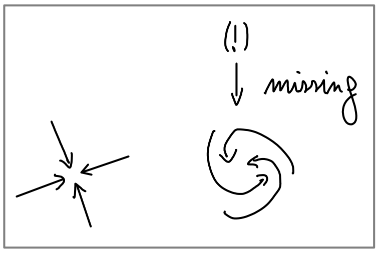
Figure 1. $P_{00}(k)$ power spectrum term is plotted at four redshifts $z = 0.0,\ 0.5,\ 1.0$ and 2.0. We show linear result (black, dotted), one loop PT (blue, solid), two loop closure (green, dashed), corrected Zel'dovich (red, long-dashed) of [33] , simple Zel'dovich (magenta, dot-dashed) and simulation measurements (black dots). The error bars show the variance among realizations in simulations. The power spectrum is divided by no-wiggle fitting formula from [34], to reduce the dynamic range.

Counter terms

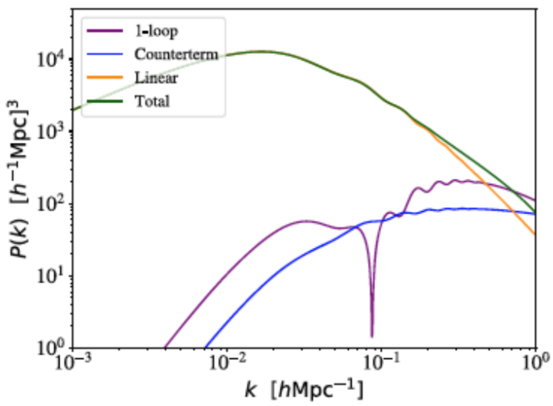
Causes of discrepancies:

1) We included 1-loop only (2° order)
2) We consider irrotational velocities only (!!!!) more important



To account for (2): people included the "counter terms"
Not from analitical evaluations
just some analytical motivated relations to be fitted to data : other free parameters

$$P_{\mathrm{ctr,RSD},\nabla^2\delta}(z,k,\mu) = -2\underline{\tilde{c}_0(z)}k^2P_{\mathrm{lin}}(z,k) - 2\underline{\tilde{c}_2(z)}f(z)\mu^2k^2P_{\mathrm{lin}}(z,k) - 2\underline{\tilde{c}_4(z)}f^2(z)\mu^4k^2P_{\mathrm{lin}}(z,k)$$

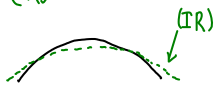


IR resummation

- Position of the peaks is improved by also including Infrared resummation
- The kernels account for mode coupling but not entirely, all modes should be considered
- low modes (Infrared) are also coupled to high modes where you have the wiggles
- How to? (1) Split the power spectrum : P_{nw} smooth part (no wiggles)
  P_w wiggle contribution

  (2) Add IR contribution to P_w infrared resummation (analytical)

$$
\begin{aligned}
P_{\rm gg}(z,k,\mu) = {} & (b_1(z)+f(z)\mu^2)^2\Big(P_{\rm nw}(z,k)+\overbrace{\mathrm{e}^{-k^2\Sigma^2_{\rm tot}(z,\mu)}}^{(\mathrm{IR})}P_{\rm w}(z,k)(1+k^2\Sigma^2_{\rm tot}(z,\mu))\Big)\\
& + P_{\rm gg,nw,RSD,1\text{-}loop}(z,k,\mu)+\underbrace{\mathrm{e}^{-k^2\Sigma^2_{\rm tot}(z,\mu)}}_{(\mathrm{IR})}P_{\rm gg,w,RSD,1\text{-}loop}(z,k,\mu),
\end{aligned}
$$

- it smooths the peaks in the weaggly part: 

- When you add the final P_w to P_{nw} ⇒ the final position of the peaks is shifted

(!) →