General Relativity

Priv.-Doz. Dr. Matteo Maturi

Summer Semester 2023

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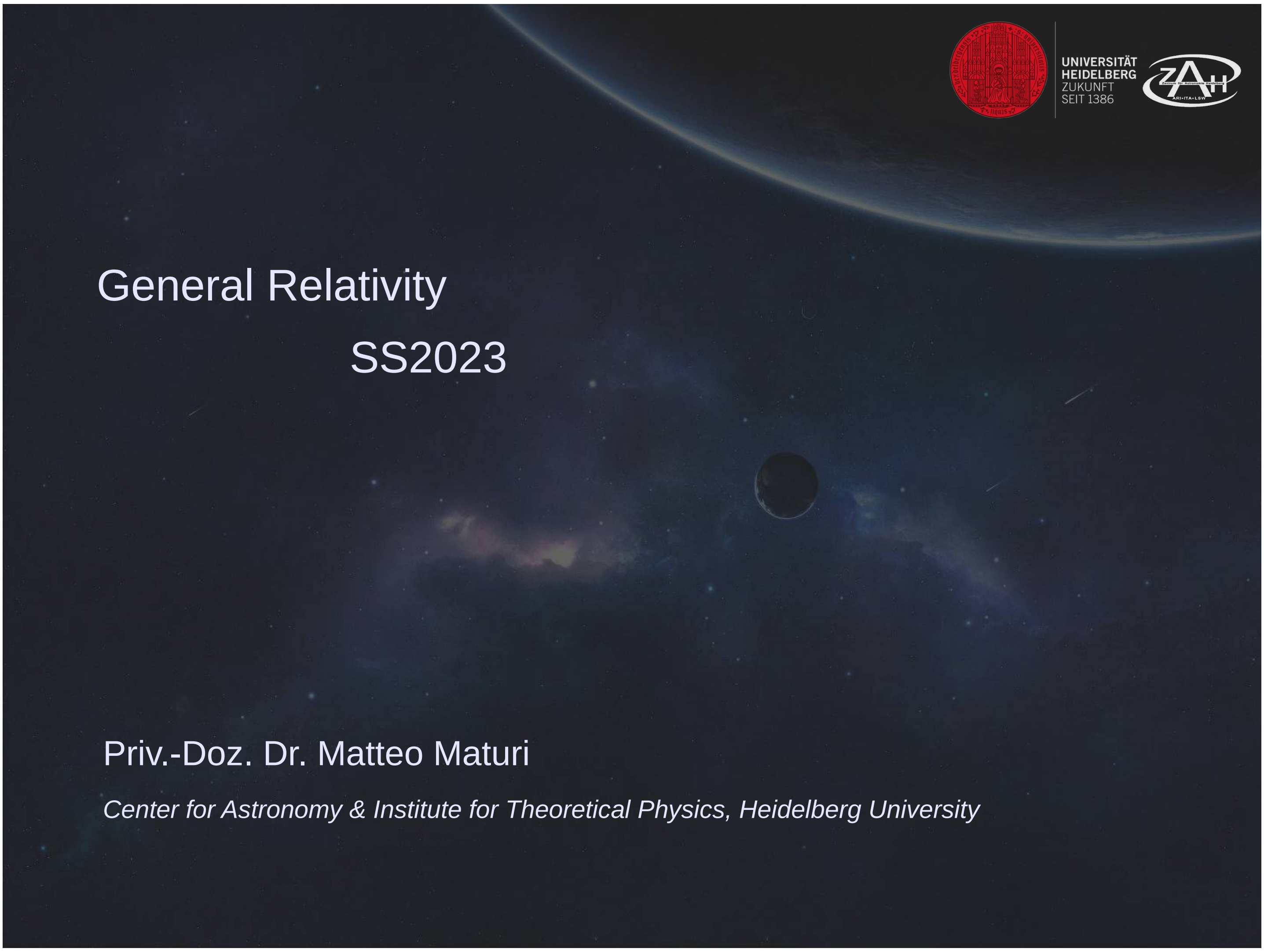
Part I

Introduction and outlook



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General Relativity

SS2023

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Welcome!

Nice to meet you:
Matteo Maturi, Madonna di Campiglio (Dolomites)
Center for Astronomy & Institute for Theoretical Physics
Cosmology, gravitational lensing, galaxy clusters
Involved in Euclid, KiDS, DESC-LSST, J-PAS

Contact:
maturi@uni-heidelberg.de (Matteo)
nussbaumer_n@thphys.uni-heidelberg.de> (Nadine, head tutor)

Lectures:
From April 17th to June 21st
Monday 09:15-11:00 (INF308/HS 2)
Wednesday 09:15-11:00 (INF308/HS 2)

Website, Uebungen:
https://uebungen.physik.uni-heidelberg.de/vorlesung/20231/1666

- Literature
- Lecture notes
- Additional material (slides, pdf files,…)
- Tutorials / Exercises
- dates of exam and all…
- answers to possible questions

Tutorials:
- Subscribe!
- First tutorial next week!
- The exercises will not be corrected and no mark will be given
- Ask the tutors and comment your solutions with them!
- It is possible to hand in exercises to get a feedback

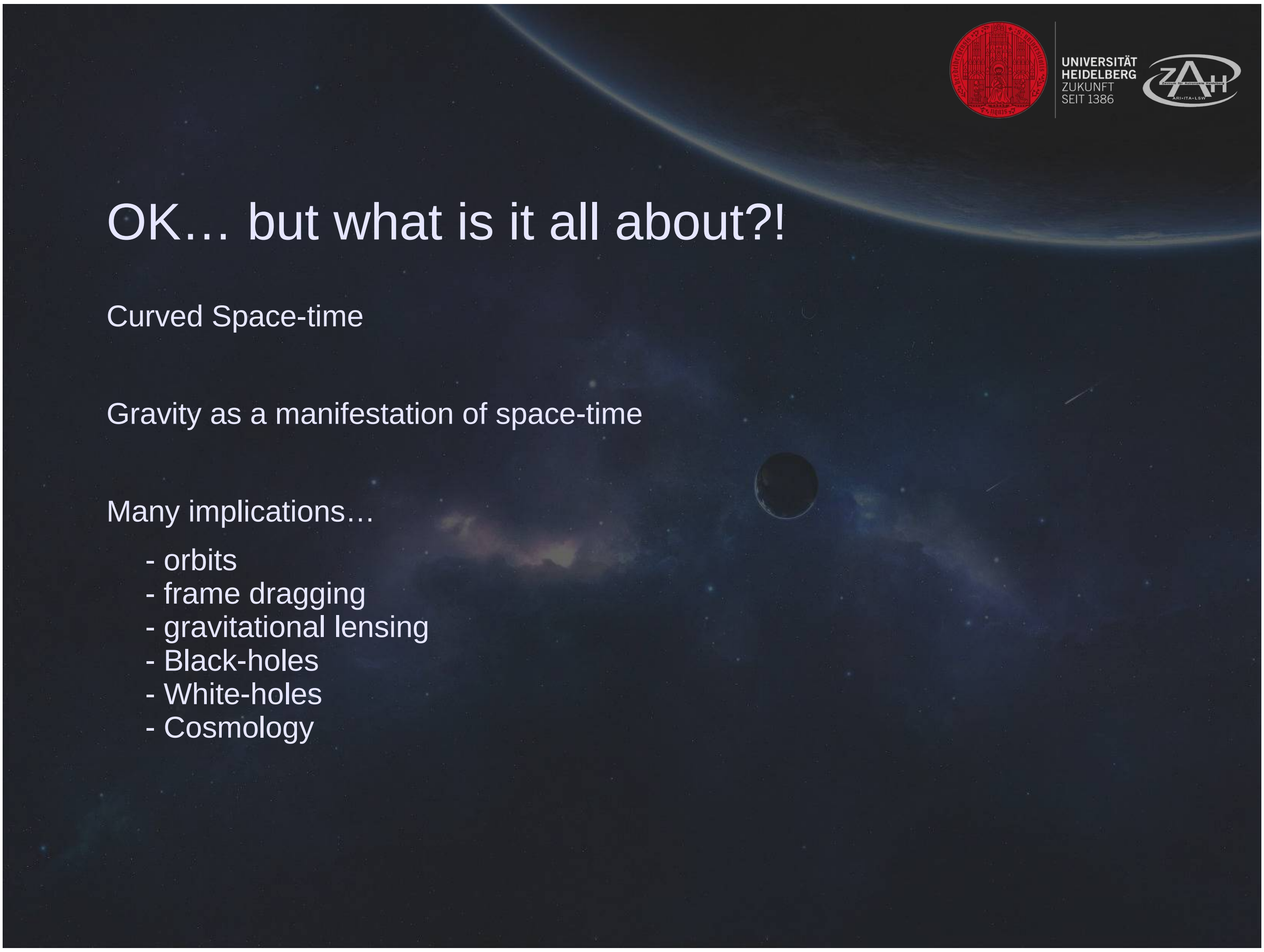
Exam:
Written
Same style of exercises

Admission to the exam: (50% attendance + (3 points)

1) attend at least 50% of the tutorials (your presence will be registered). If attendance < 50%, it is required to hand in 3 full exercise sheets that will be graded.

and

2) gain 3 points by:
- 1 point: present at the black board at least 1/3 of a sheet.
- 1 point: actively participating in the discussion during the tutorials



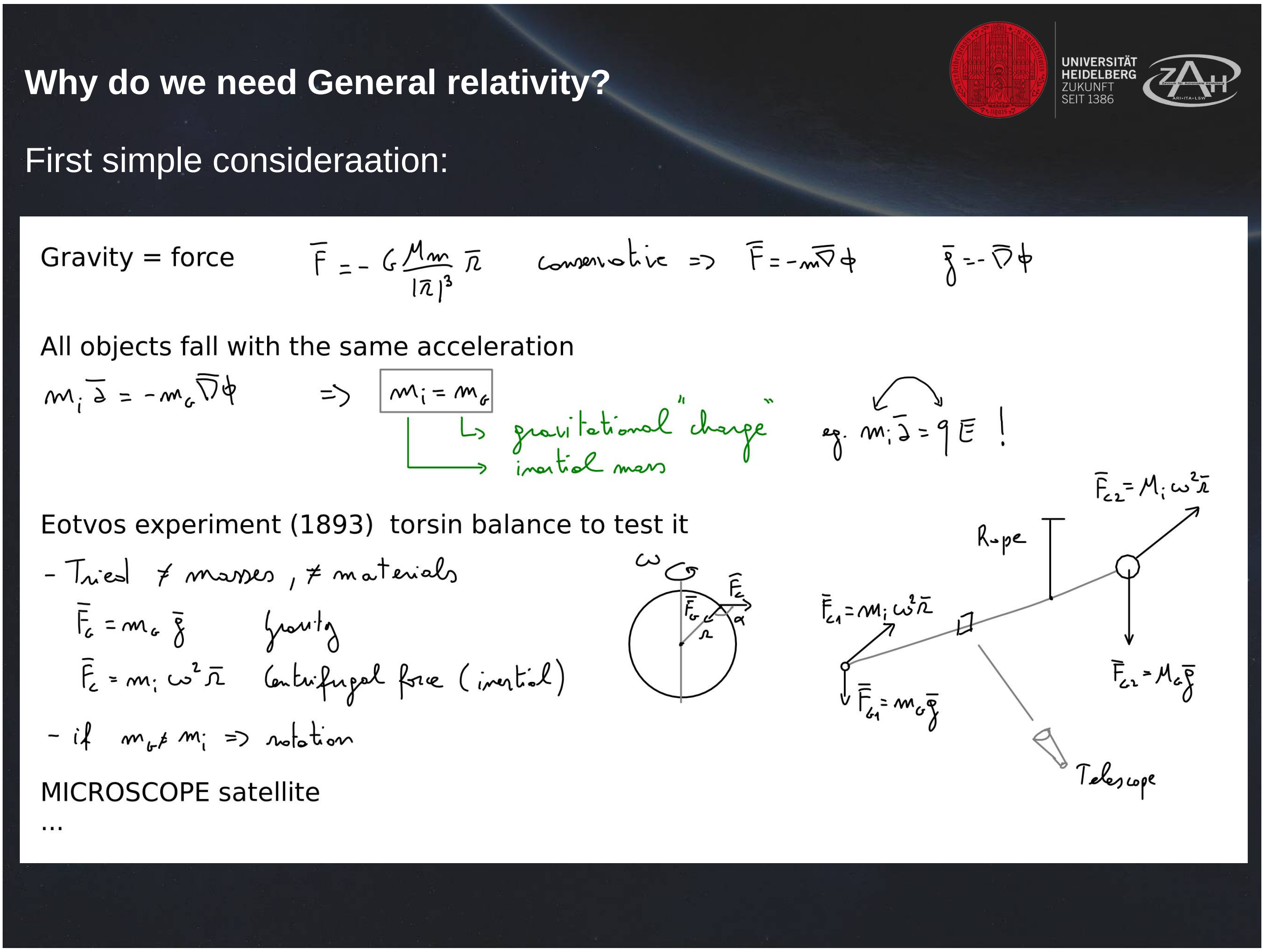
OK… but what is it all about?!

Curved Space-time

Gravity as a manifestation of space-time

Many implications…

- orbits
- frame dragging
- gravitational lensing
- Black-holes
- White-holes
- Cosmology



Why do we need General relativity?

First simple consideraation:

Gravity = force $\bar{F} = -G\frac{Mm}{|\bar{r}|^3}\bar{r}$ conservative => $\bar{F} = -m\bar{\nabla}\phi$ $\bar{g} = -\bar{\nabla}\phi$

All objects fall with the same acceleration

$m_i\bar{a} = -m_G\bar{\nabla}\phi$ => $\boxed{m_i = m_G}$

↳ gravitational "charge"

→ inertial mass

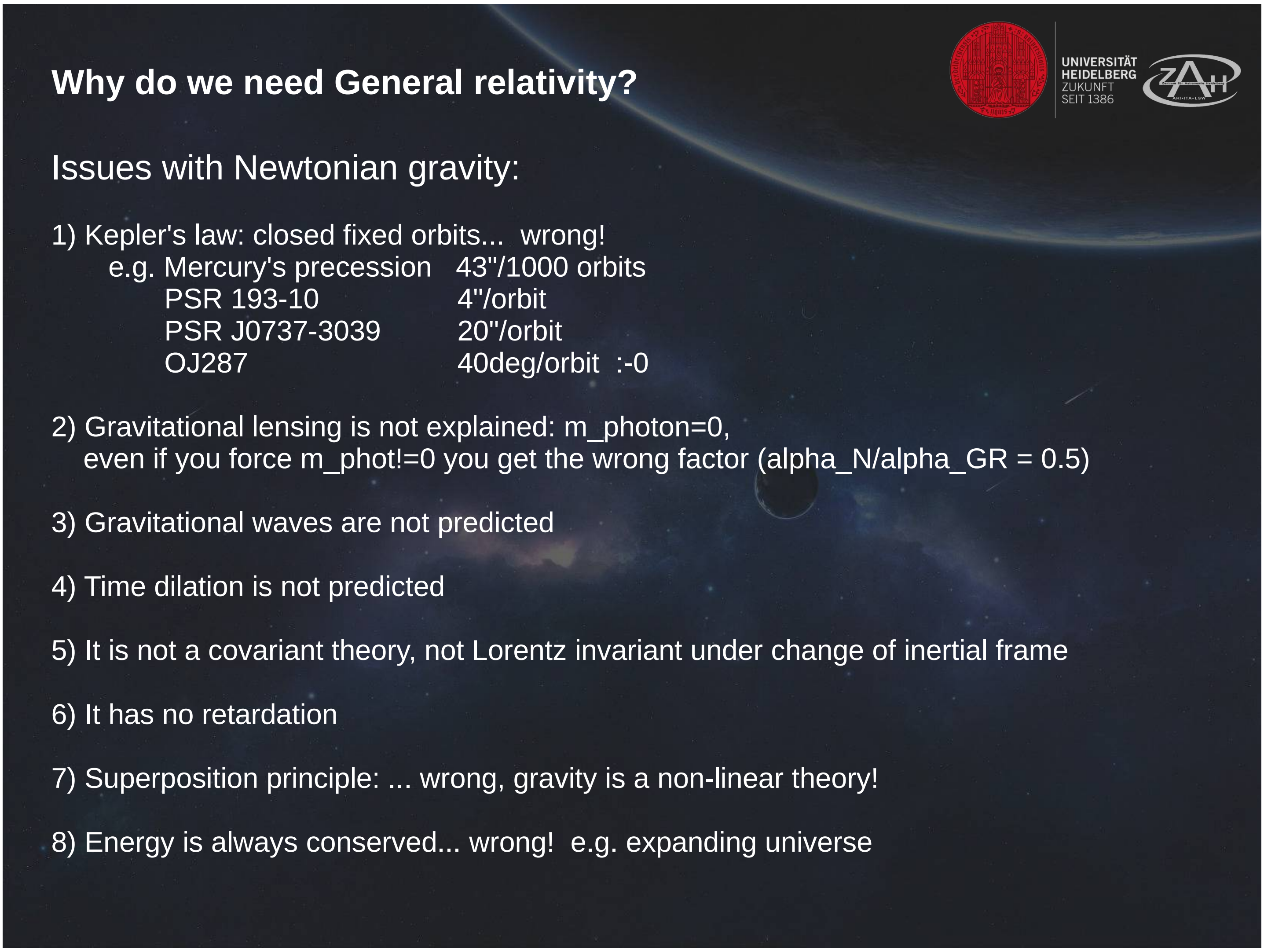
eg. $m_i\bar{a} = q\bar{E}$!

Eotvos experiment (1893) torsin balance to test it

- Tried ≠ masses, ≠ materials
  - $\bar{F}_G = m_G\,\bar{g}$ gravity
  - $\bar{F}_c = m_i\,\omega^2\bar{r}$ Centrifugal force (inertial)
- if $m_G \neq m_i$ => rotation

MICROSCOPE satellite

...



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Why do we need General relativity?

Issues with Newtonian gravity:

1) Kepler's law: closed fixed orbits... wrong!

e.g. Mercury's precession	43''/1000 orbits
PSR 193-10	4''/orbit
PSR J0737-3039	20''/orbit
OJ287	40deg/orbit :-0

2) Gravitational lensing is not explained: m_photon=0,
even if you force m_phot!=0 you get the wrong factor (alpha_N/alpha_GR = 0.5)

3) Gravitational waves are not predicted

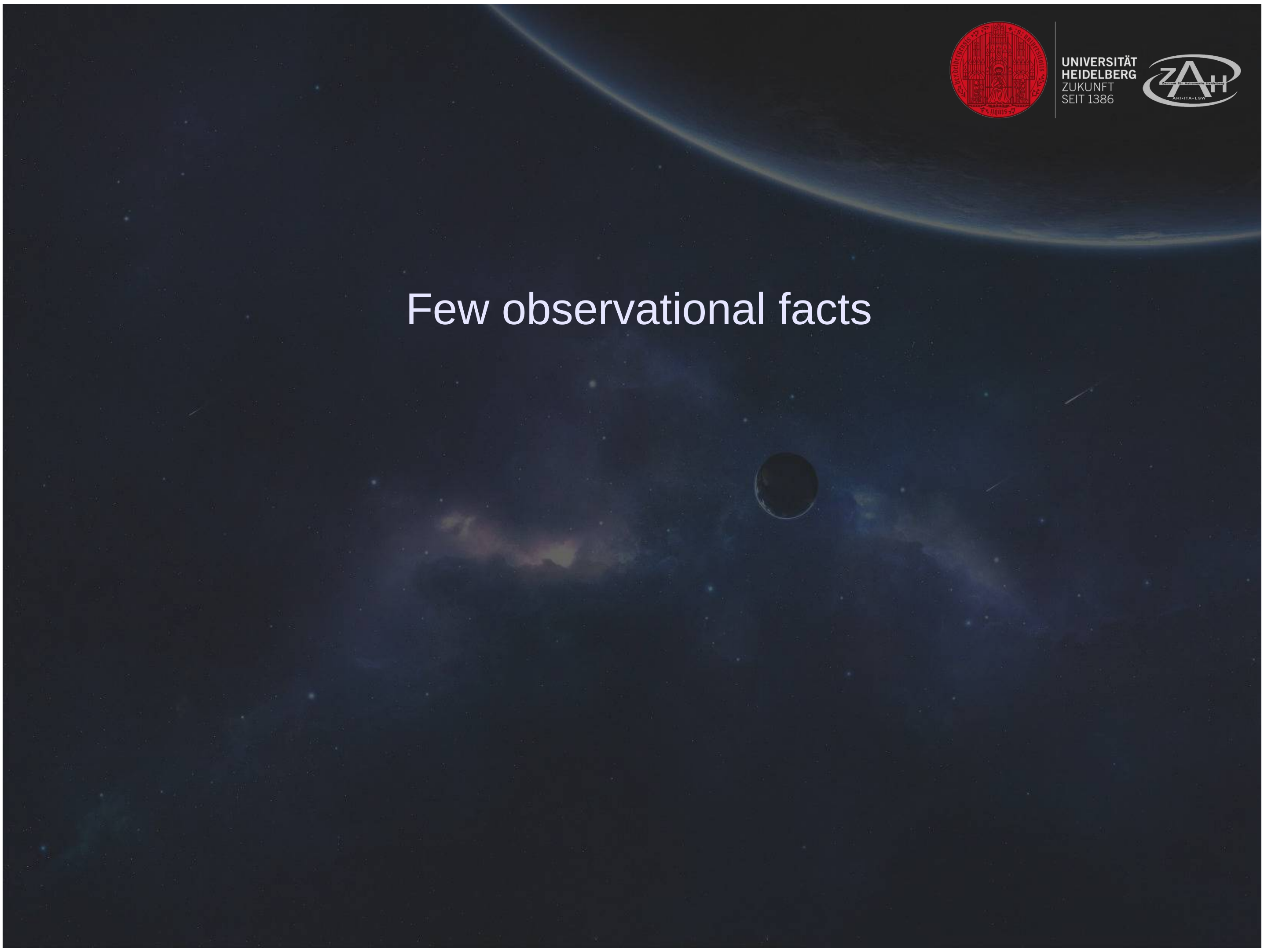
4) Time dilation is not predicted

5) It is not a covariant theory, not Lorentz invariant under change of inertial frame

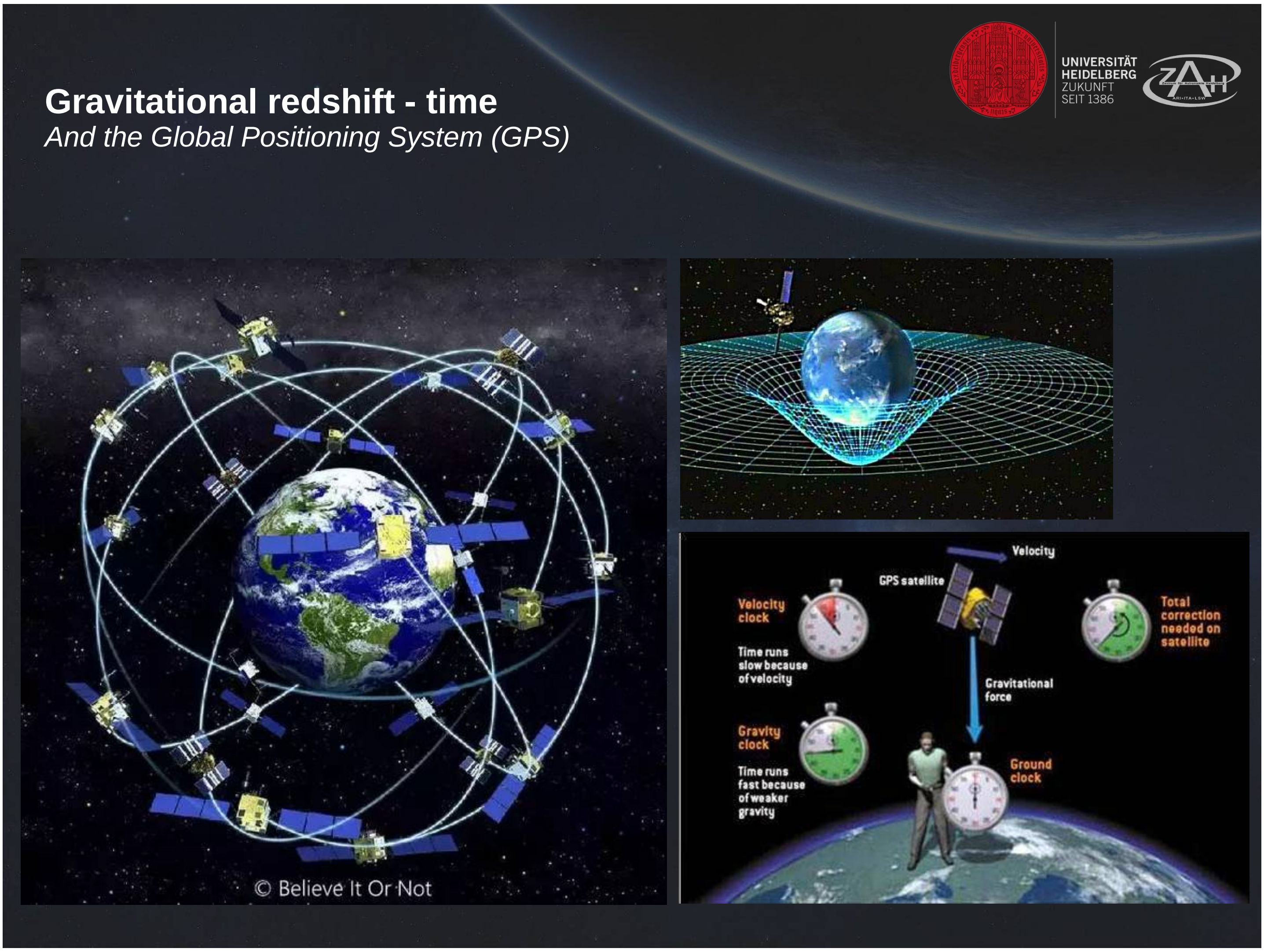
6) It has no retardation

7) Superposition principle: ... wrong, gravity is a non-linear theory!

8) Energy is always conserved... wrong! e.g. expanding universe



Few observational facts



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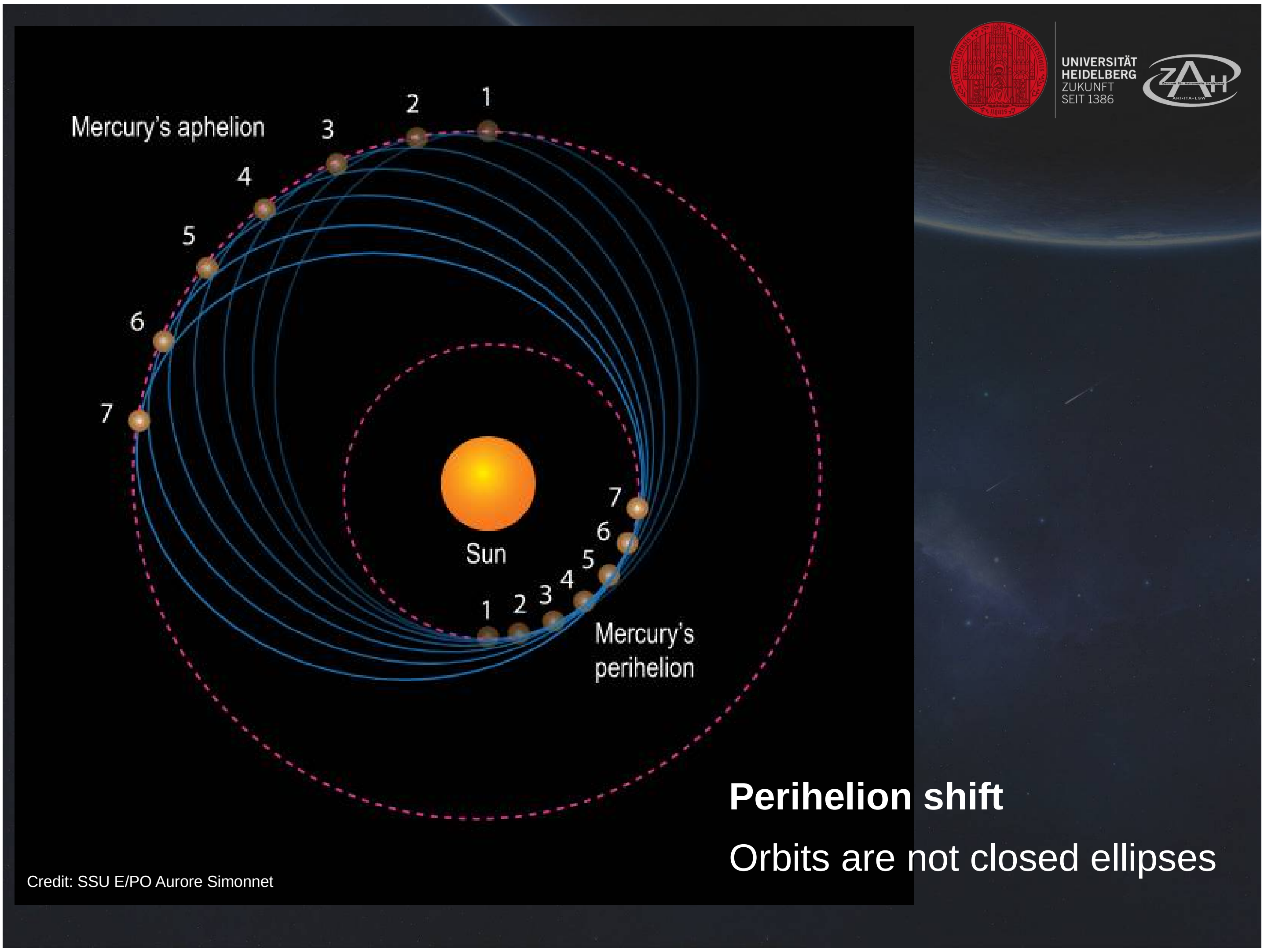
Gravitational redshift - time

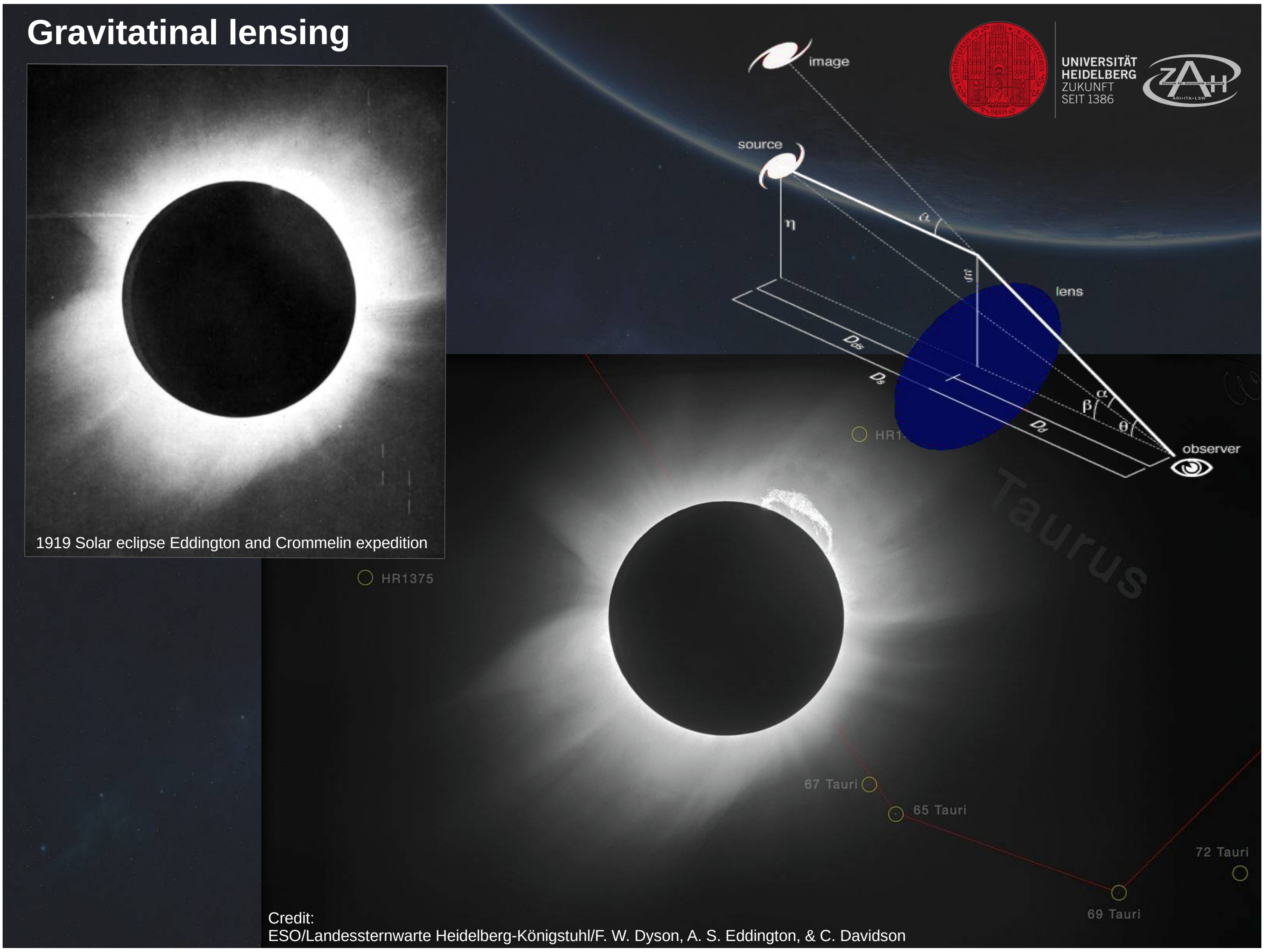
And the Global Positioning System (GPS)



Perihelion shift

Orbits are not closed ellipses

Credit: SSU E/PO Aurore Simonnet



Gravitatinal lensing

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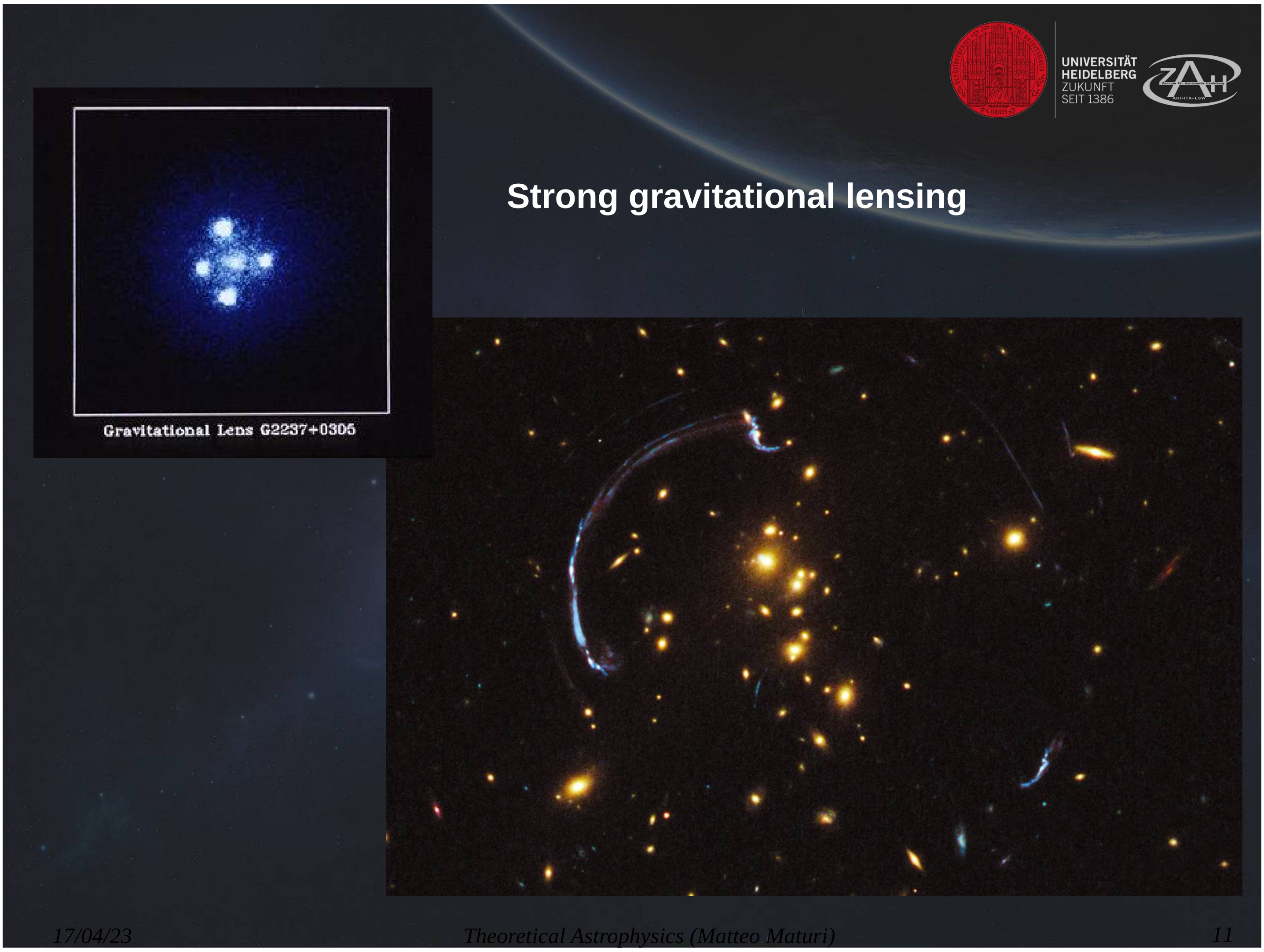
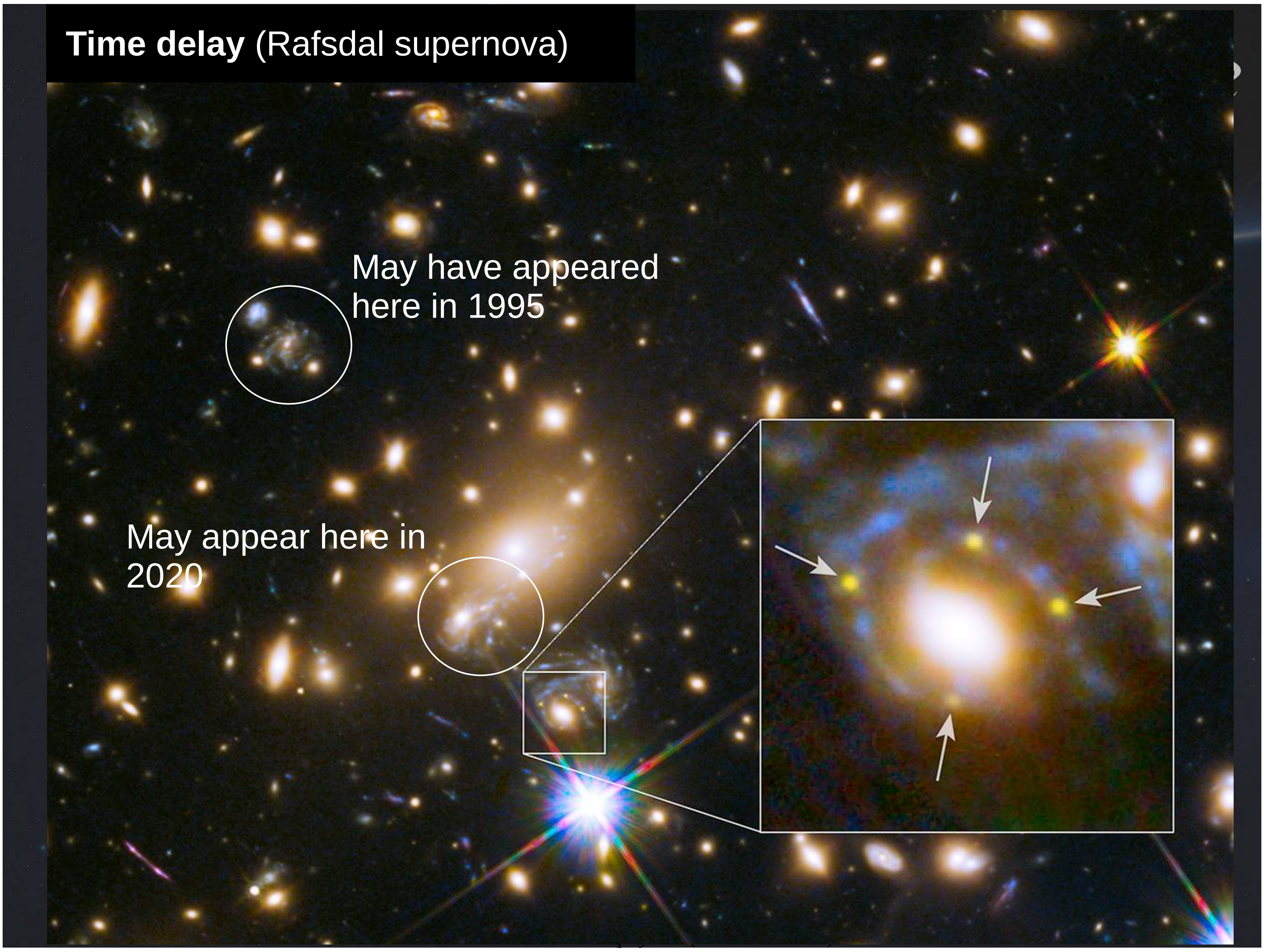
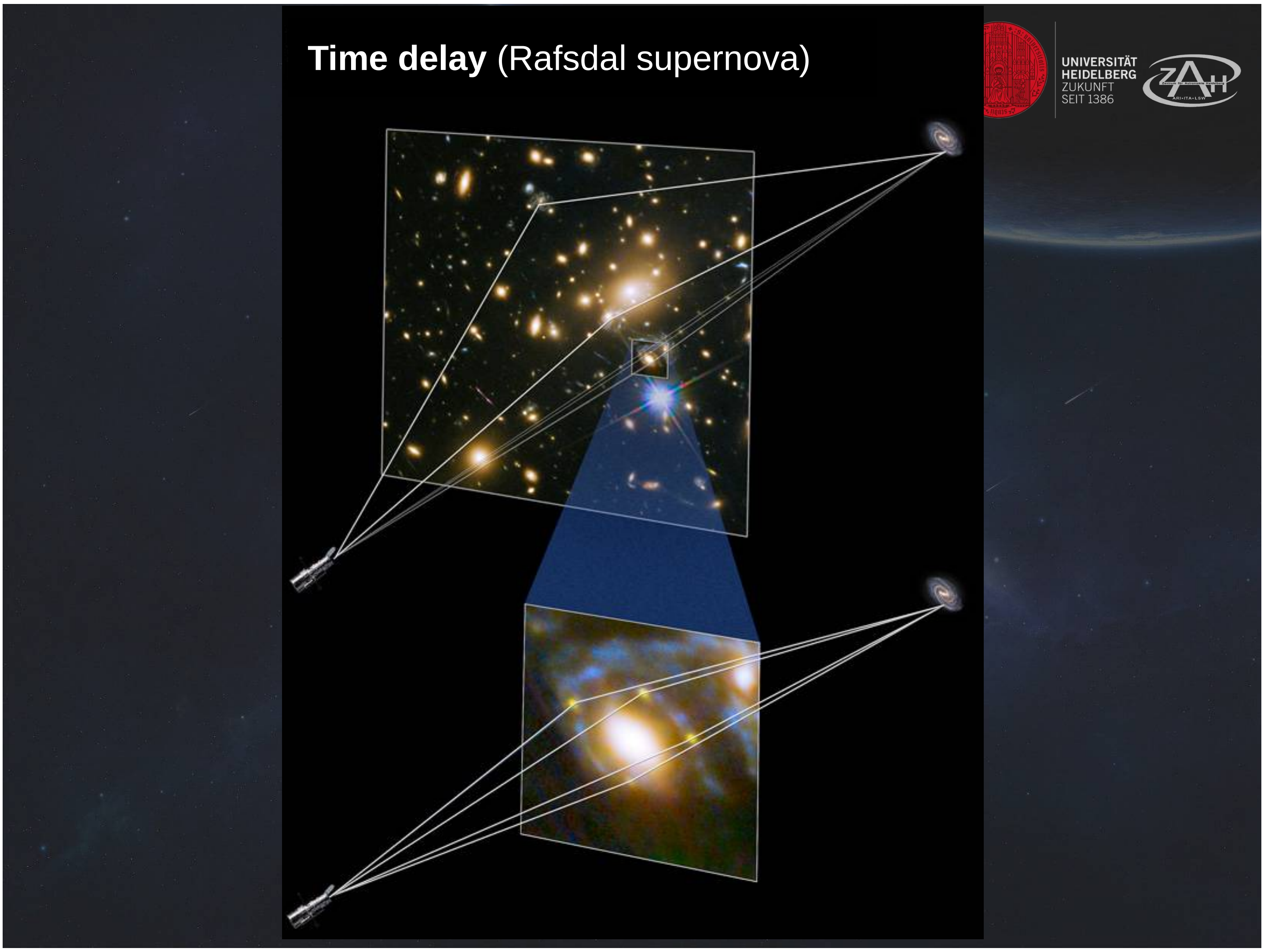
1919 Solar eclipse Eddington and Crommelin expedition

Credit:
ESO/Landessternwarte Heidelberg-Königstuhl/F. W. Dyson, A. S. Eddington, & C. Davidson



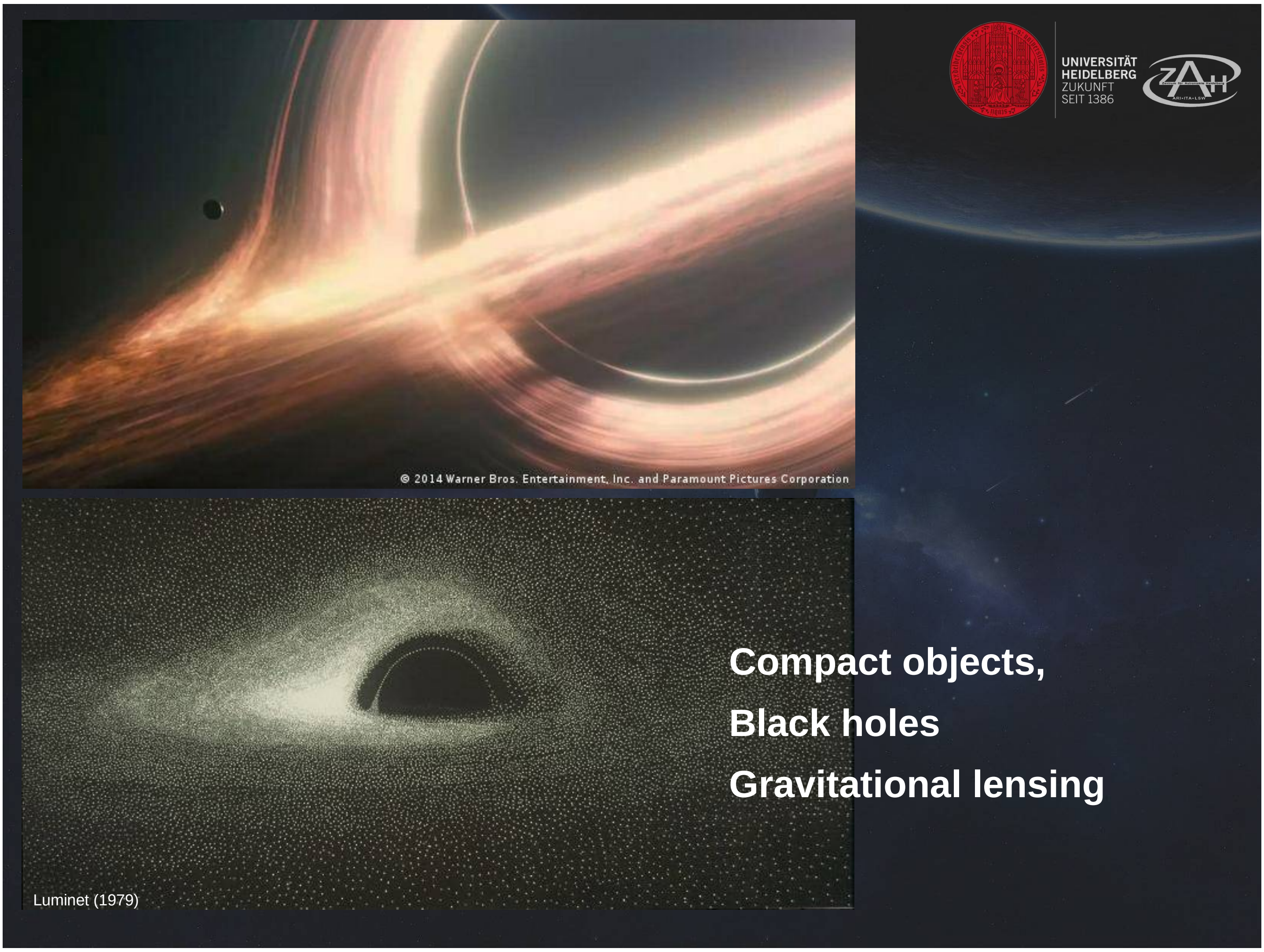
Strong gravitational lensing


Time delay (Rafsdal supernova)
May have appeared here in 1995
May appear here in 2020



Time delay (Rafsdal supernova)

Weak gravitational lensing: Mapping dark matter



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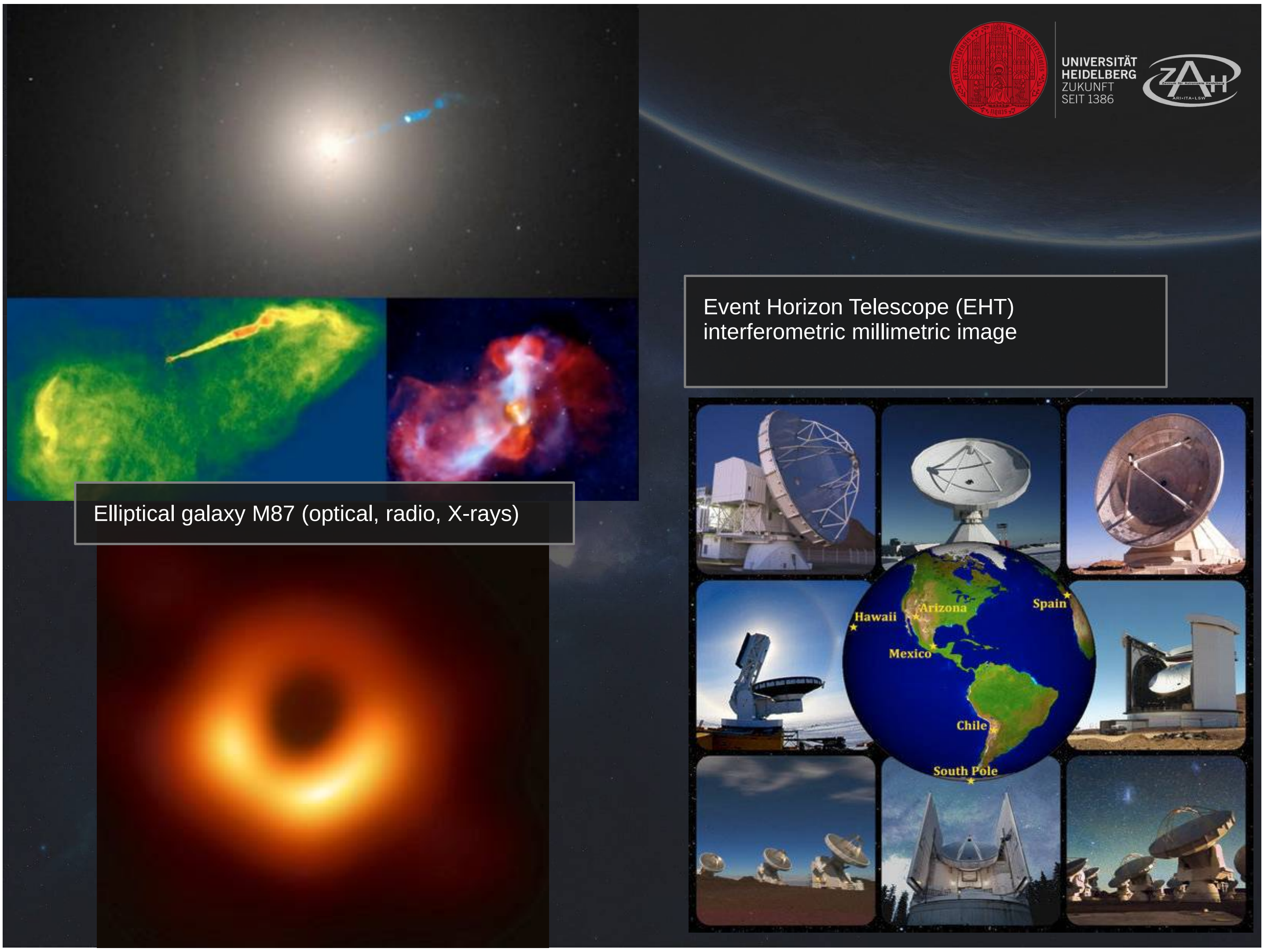
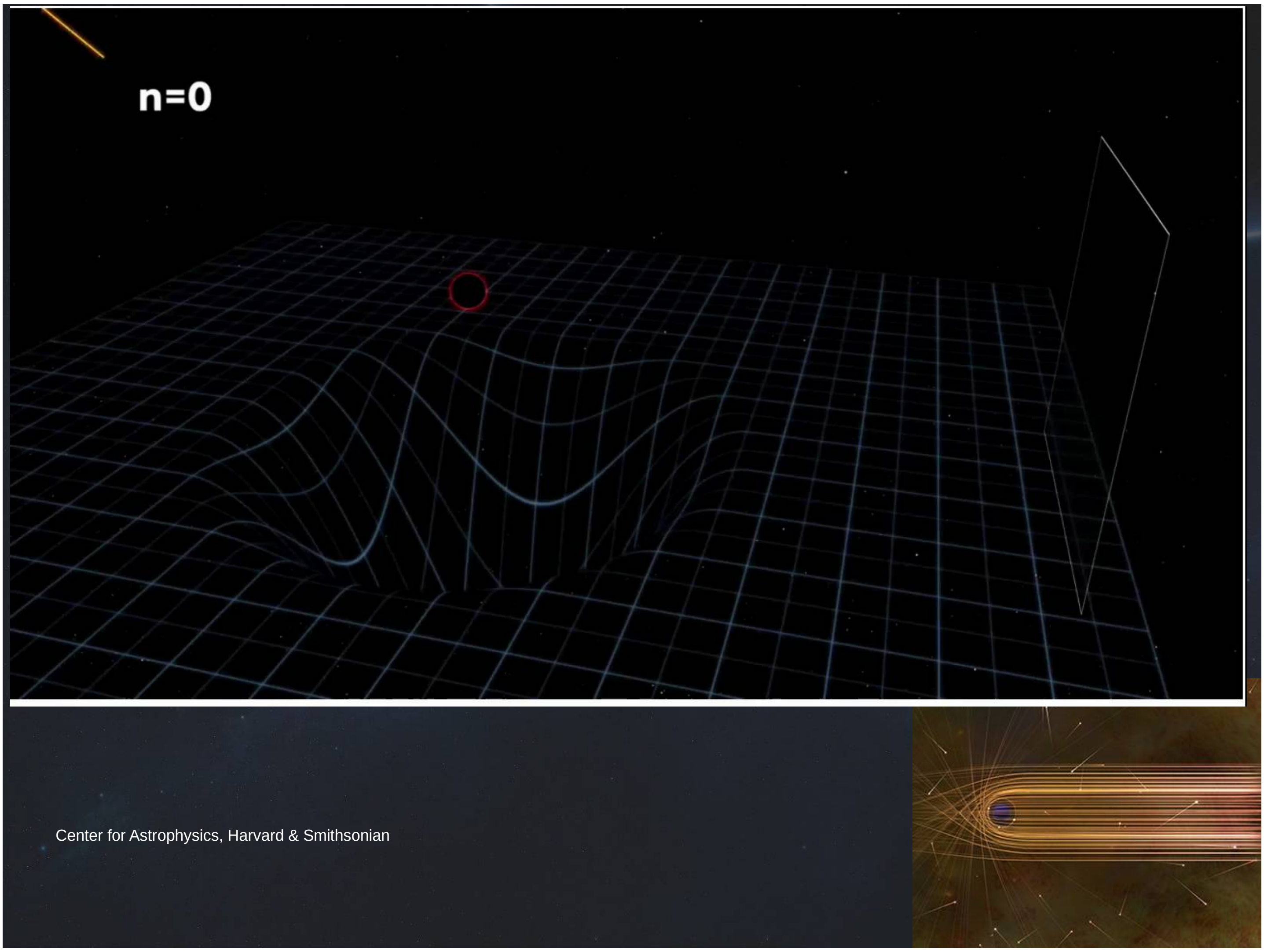
Luminet (1979)

Compact objects,
Black holes
Gravitational lensing

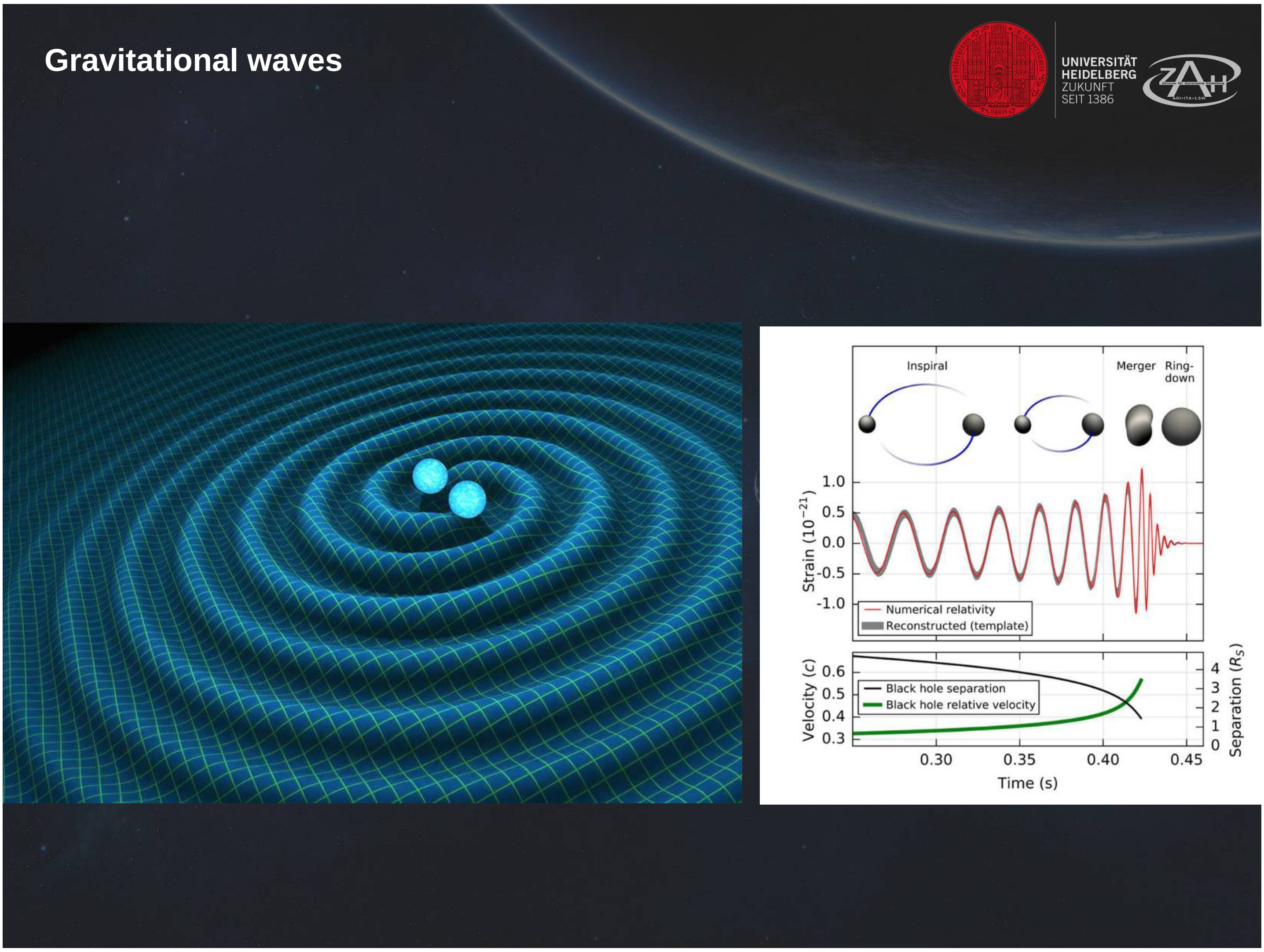


Event Horizon Telescope (EHT) interferometric millimetric image

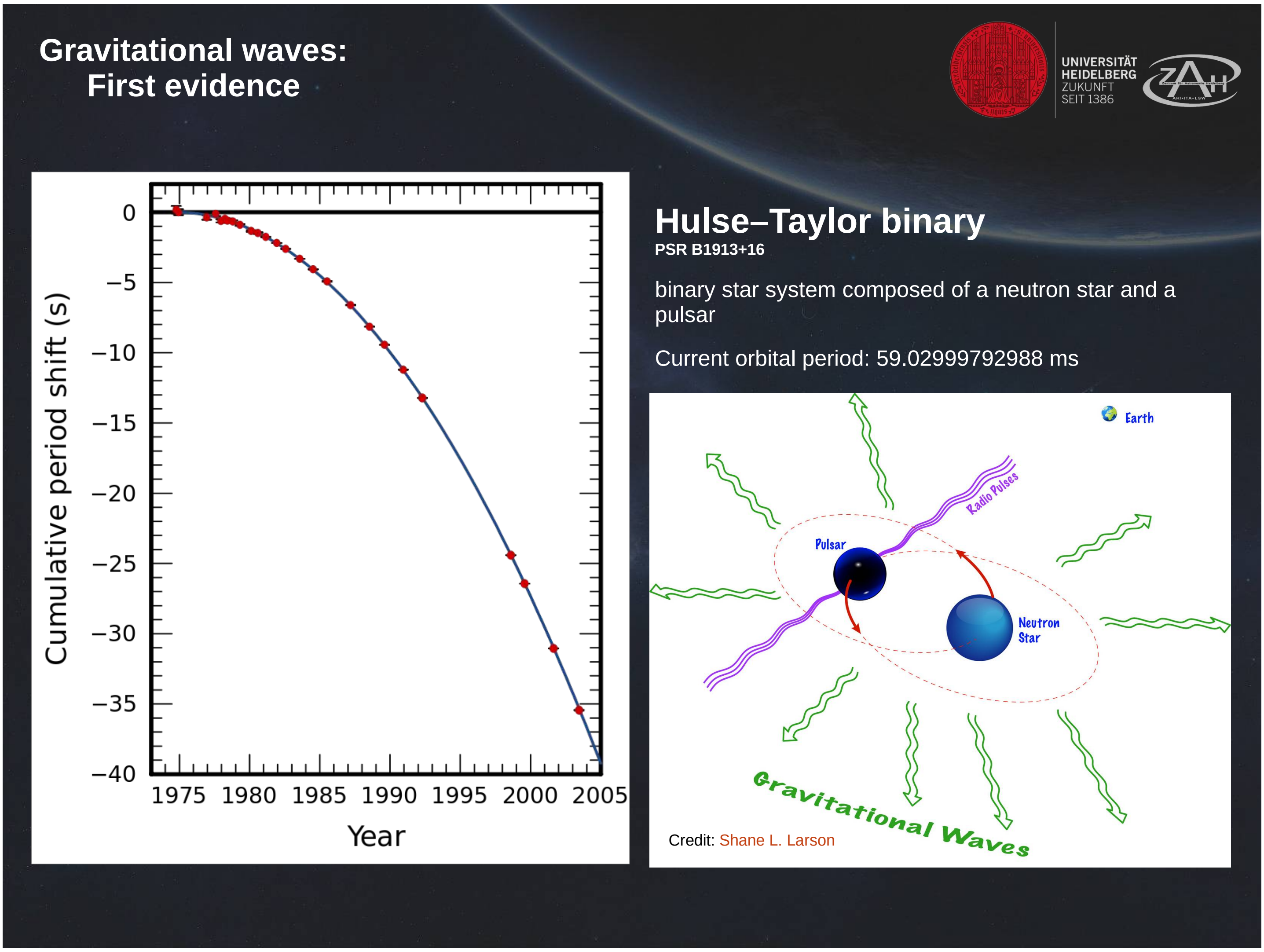
Elliptical galaxy M87 (optical, radio, X-rays)



Center for Astrophysics, Harvard & Smithsonian



Gravitational waves



Gravitational waves: First evidence

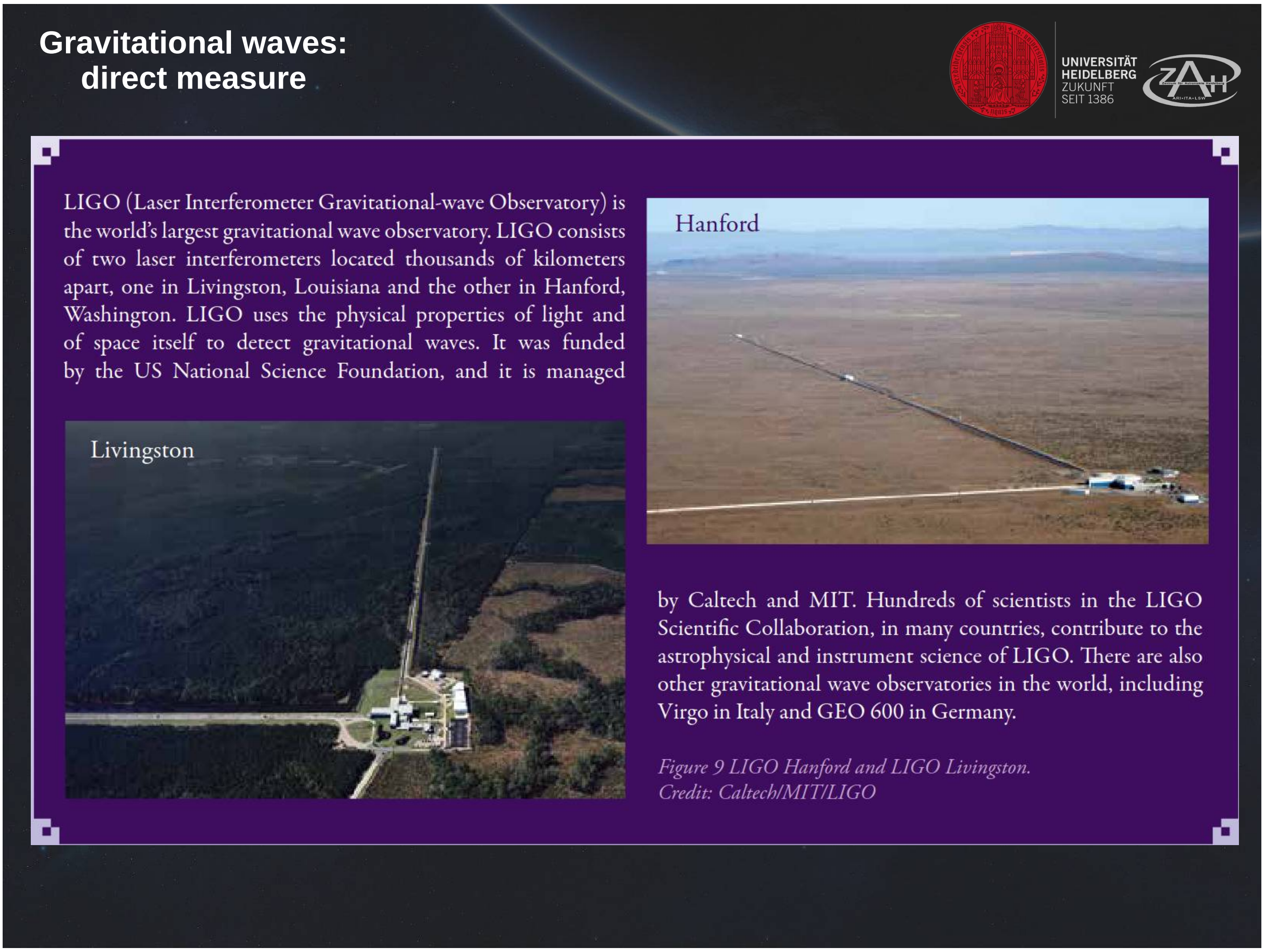
Hulse–Taylor binary

PSR B1913+16

binary star system composed of a neutron star and a pulsar

Current orbital period: 59.02999792988 ms

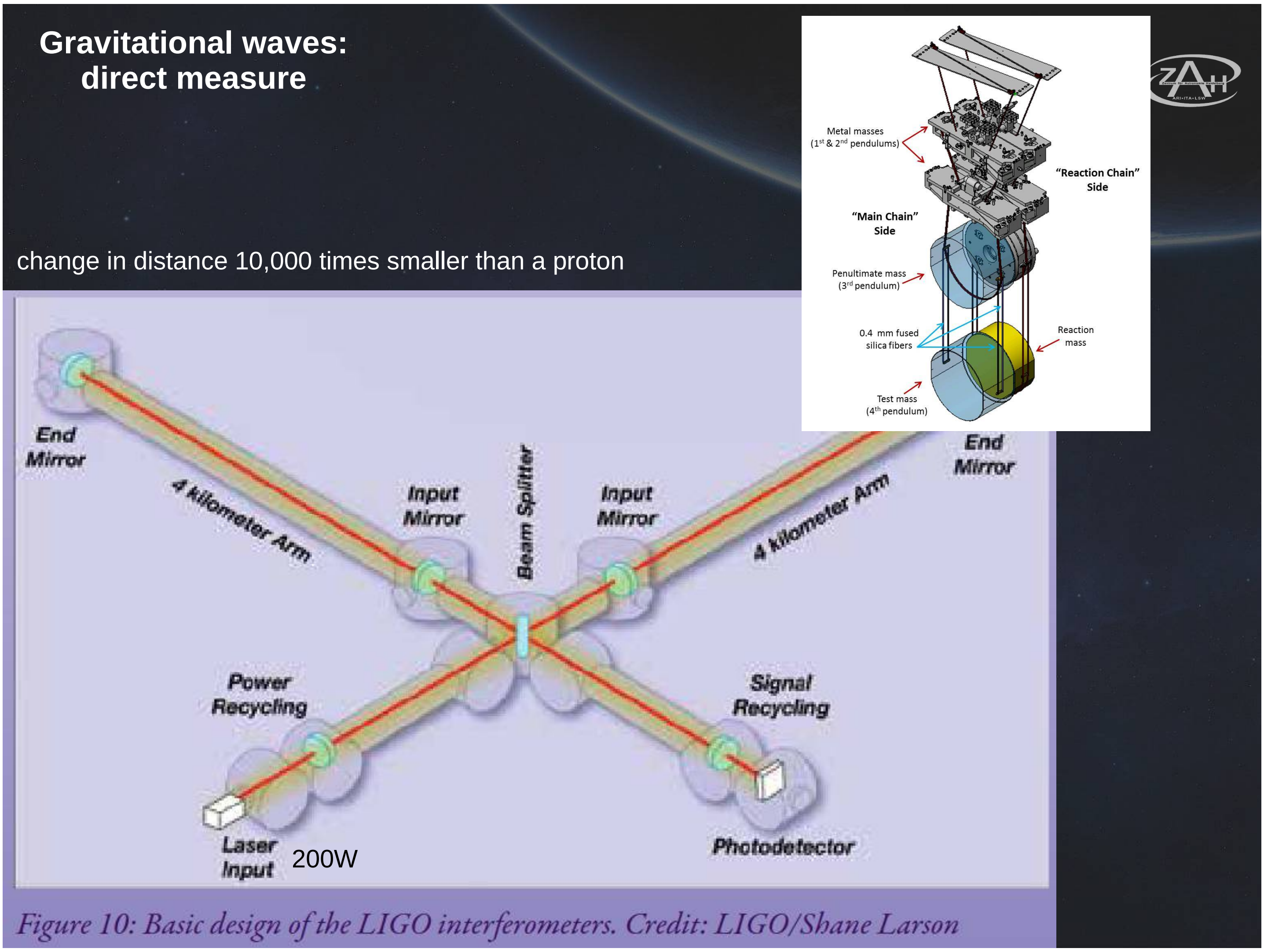
Credit: Shane L. Larson



Gravitational waves: direct measure

LIGO (Laser Interferometer Gravitational-wave Observatory) is the world's largest gravitational wave observatory. LIGO consists of two laser interferometers located thousands of kilometers apart, one in Livingston, Louisiana and the other in Hanford, Washington. LIGO uses the physical properties of light and of space itself to detect gravitational waves. It was funded by the US National Science Foundation, and it is managed by Caltech and MIT. Hundreds of scientists in the LIGO Scientific Collaboration, in many countries, contribute to the astrophysical and instrument science of LIGO. There are also other gravitational wave observatories in the world, including Virgo in Italy and GEO 600 in Germany.

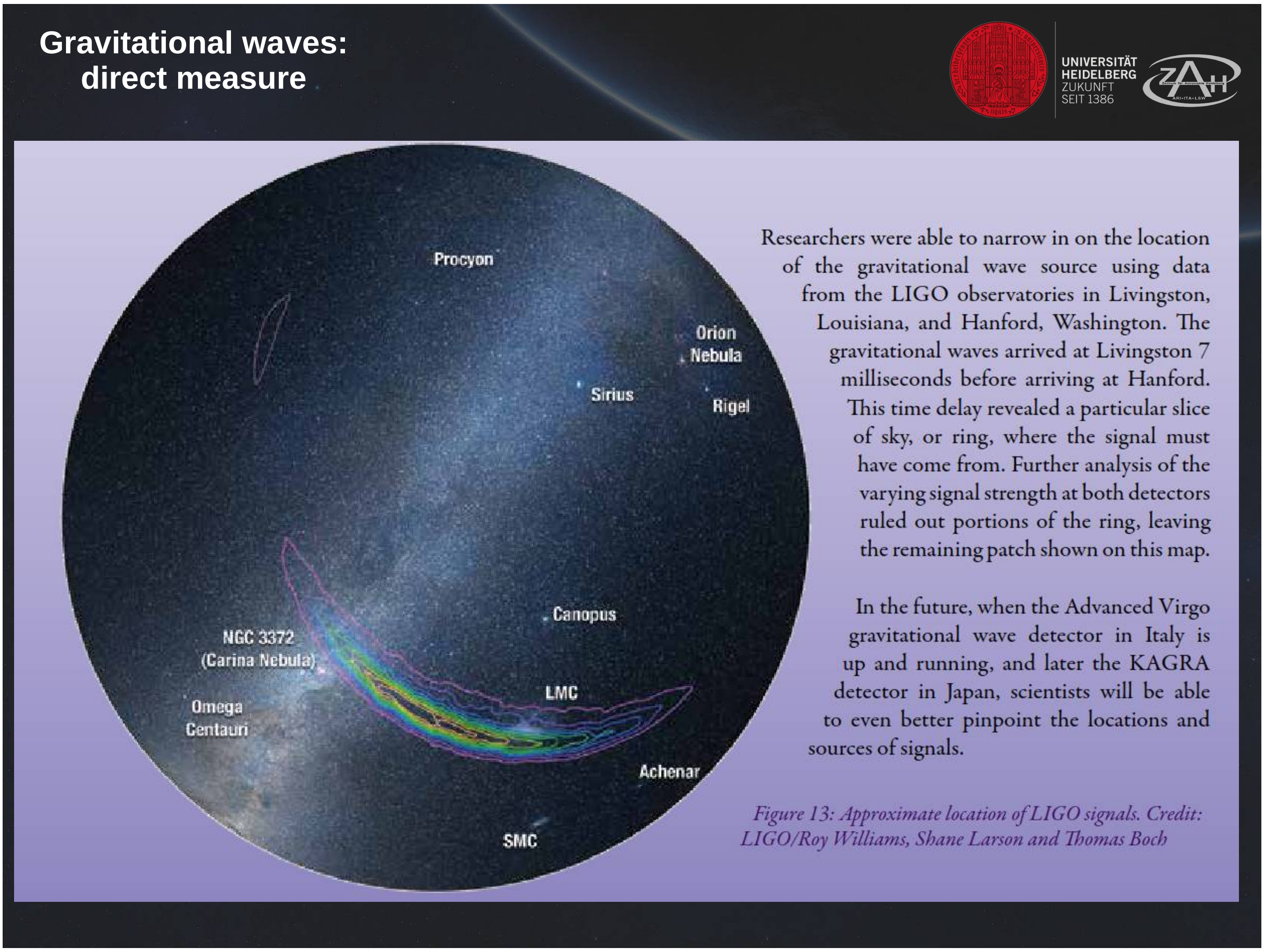
Figure 9 LIGO Hanford and LIGO Livingston.
Credit: Caltech/MIT/LIGO



Gravitational waves: direct measure

change in distance 10,000 times smaller than a proton

Figure 10: Basic design of the LIGO interferometers. Credit: LIGO/Shane Larson



Gravitational waves: direct measure

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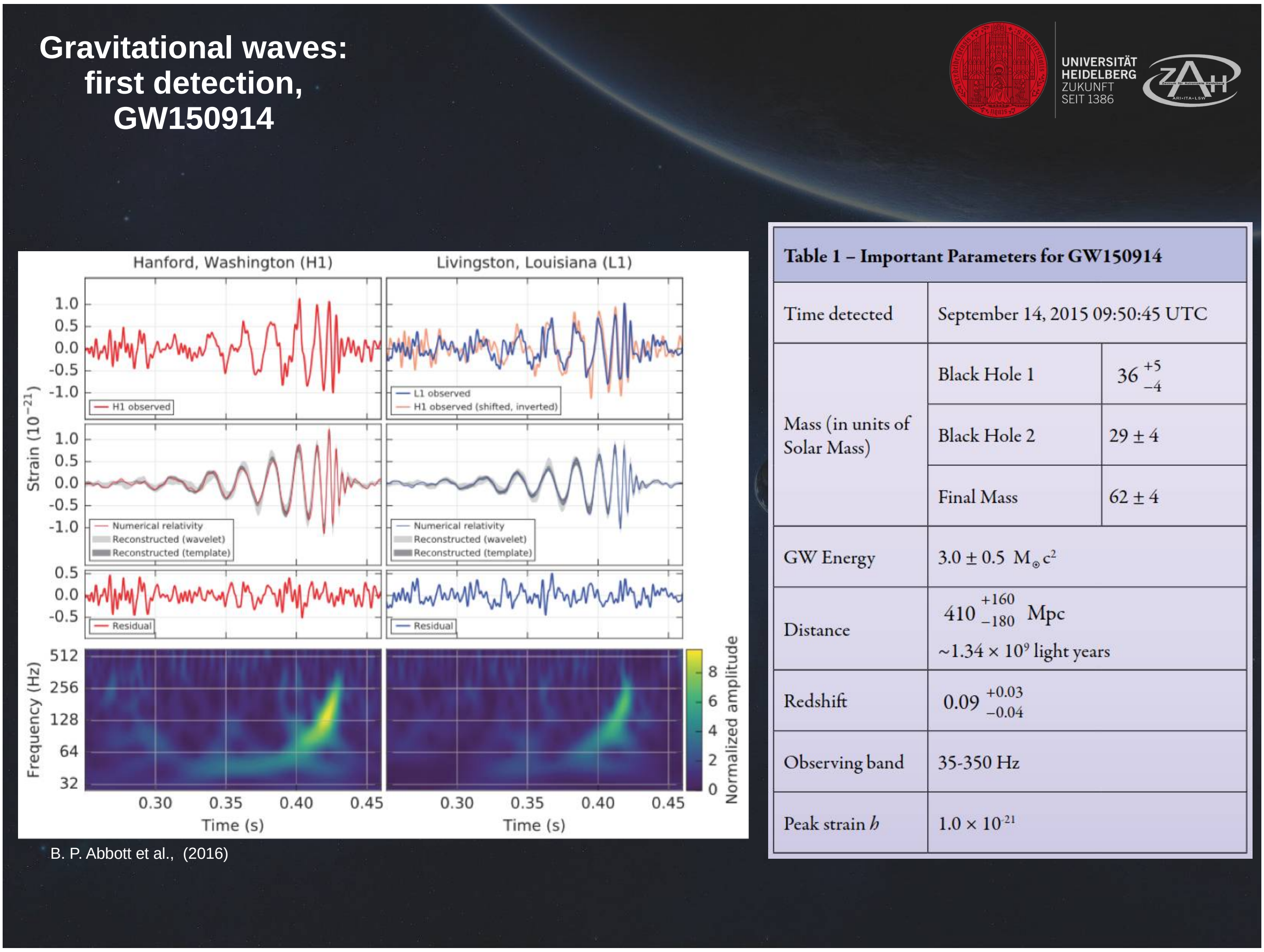
Researchers were able to narrow in on the location of the gravitational wave source using data from the LIGO observatories in Livingston, Louisiana, and Hanford, Washington. The gravitational waves arrived at Livingston 7 milliseconds before arriving at Hanford. This time delay revealed a particular slice of sky, or ring, where the signal must have come from. Further analysis of the varying signal strength at both detectors ruled out portions of the ring, leaving the remaining patch shown on this map.

In the future, when the Advanced Virgo gravitational wave detector in Italy is up and running, and later the KAGRA detector in Japan, scientists will be able to even better pinpoint the locations and sources of signals.

Figure 13: Approximate location of LIGO signals. Credit: LIGO/Roy Williams, Shane Larson and Thomas Boch

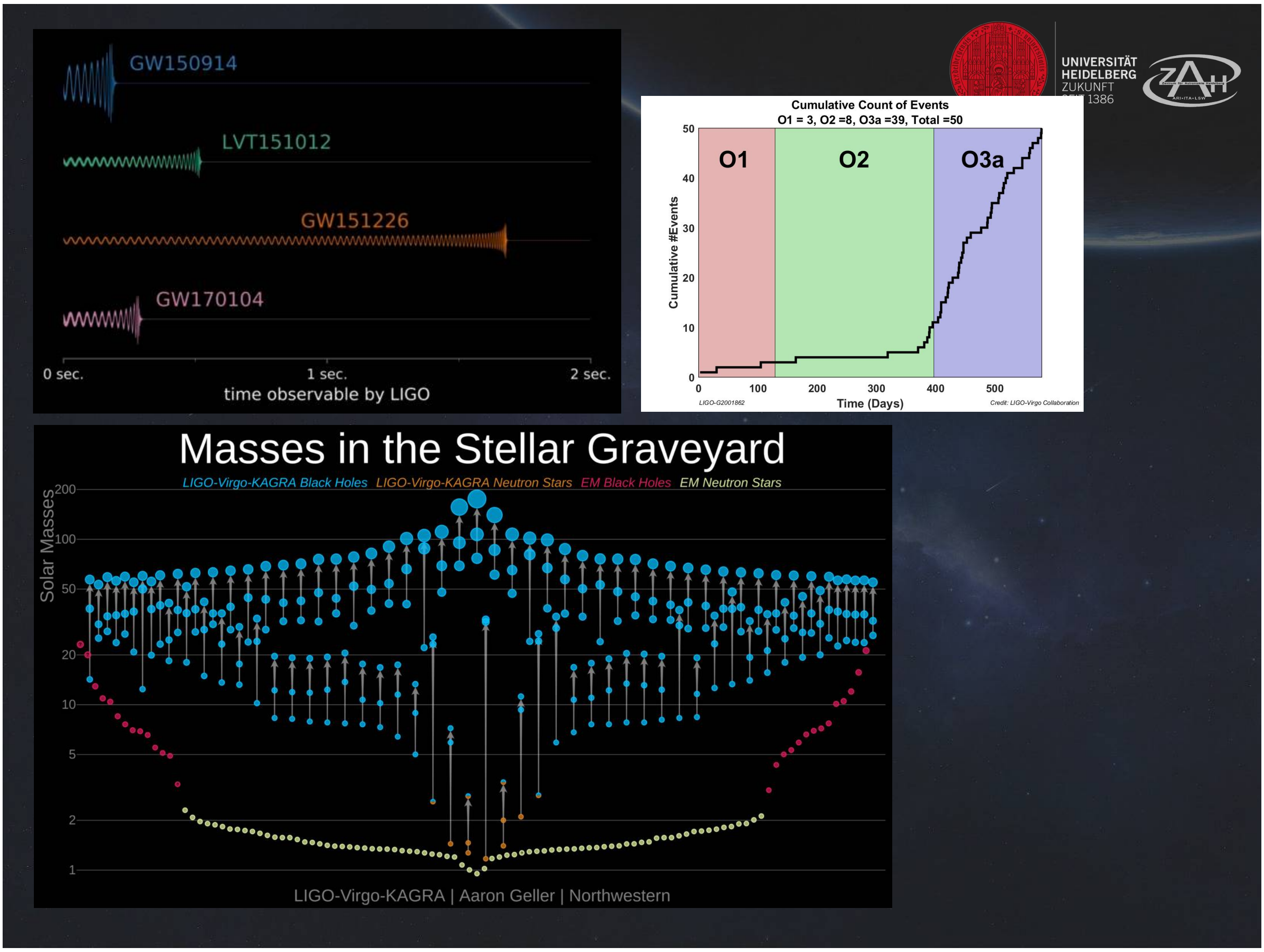


Gravitational waves: first detection, GW150914

Table 1 – Important Parameters for GW150914

Time detected	September 14, 2015 09:50:45 UTC	
Mass (in units of Solar Mass)	Black Hole 1	36^{+5}_{-4}
	Black Hole 2	29 ± 4
	Final Mass	62 ± 4
GW Energy	$3.0 \pm 0.5\ M_{\odot} c^2$	
Distance	410^{+160}_{-180} Mpc ~1.34×10^9 light years	
Redshift	$0.09^{+0.03}_{-0.04}$	
Observing band	35-350 Hz	
Peak strain *h*	1.0×10^{-21}	

B. P. Abbott et al., (2016)


GW150914
LVT151012
GW151226
GW170104
0 sec.
1 sec.
2 sec.
time observable by LIGO
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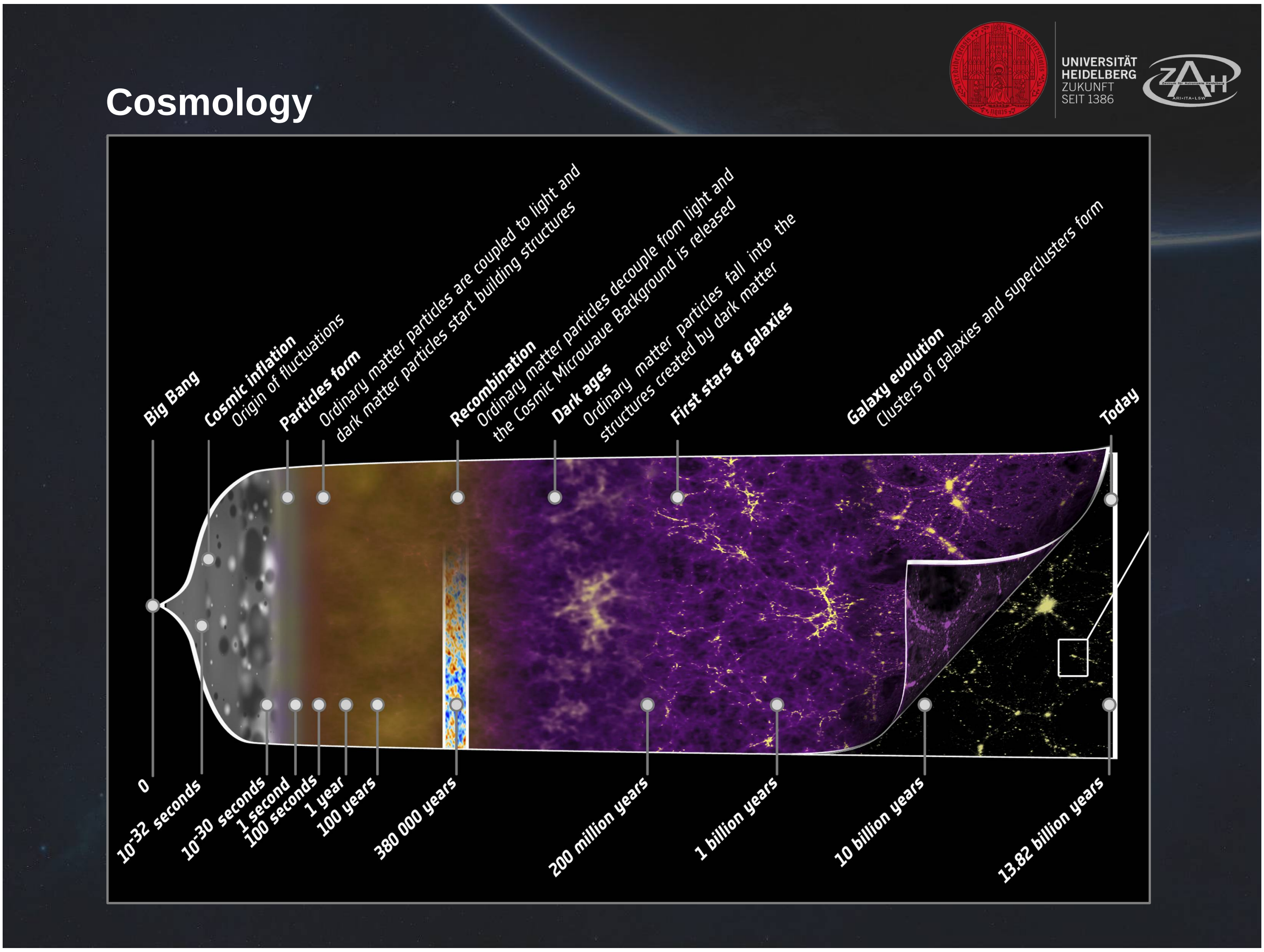
ZAH
Cumulative Count of Events
O1 = 3, O2 =8, O3a =39, Total =50
O1
O2
O3a
Cumulative #Events
0
10
20
30
40
50
0
100
200
300
400
500
Time (Days)
LIGO-G2001862
Credit: LIGO-Virgo Collaboration
Masses in the Stellar Graveyard
LIGO-Virgo-KAGRA Black Holes
LIGO-Virgo-KAGRA Neutron Stars
EM Black Holes
EM Neutron Stars
Solar Masses
200
100
50
20
10
5
2
1
LIGO-Virgo-KAGRA | Aaron Geller | Northwestern



Cosmology

Big Bang

Cosmic inflation
Origin of fluctuations

Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures

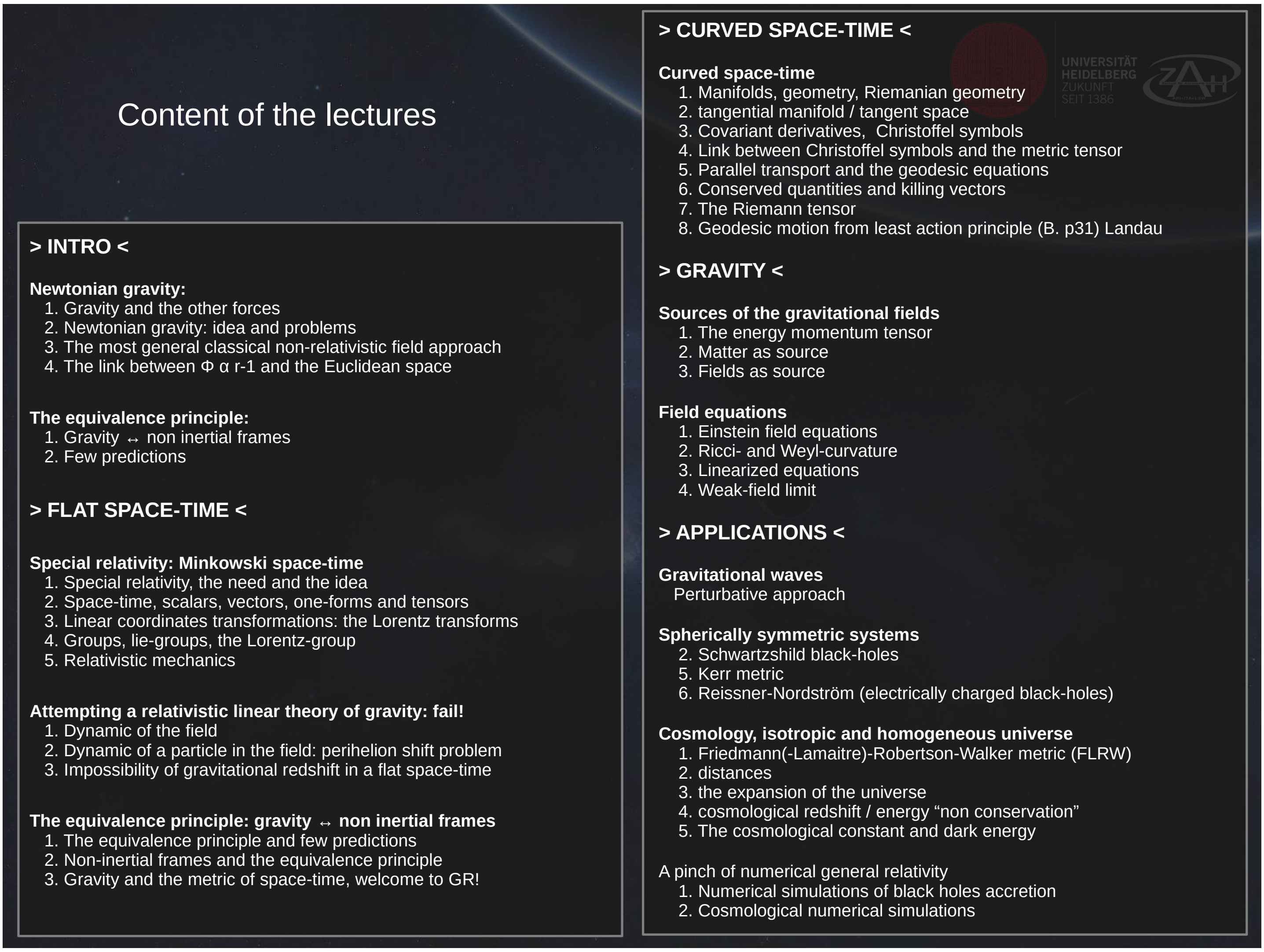
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwave Background is released

Dark ages
Ordinary matter particles fall into the structures created by dark matter

First stars & galaxies

Galaxy evolution
Clusters of galaxies and superclusters form

Today



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Content of the lectures

> INTRO <

Newtonian gravity:
1. Gravity and the other forces
2. Newtonian gravity: idea and problems
3. The most general classical non-relativistic field approach
4. The link between Φ α r-1 and the Euclidean space

The equivalence principle:
1. Gravity ↔ non inertial frames
2. Few predictions

> FLAT SPACE-TIME <

Special relativity: Minkowski space-time
1. Special relativity, the need and the idea
2. Space-time, scalars, vectors, one-forms and tensors
3. Linear coordinates transformations: the Lorentz transforms
4. Groups, lie-groups, the Lorentz-group
5. Relativistic mechanics

Attempting a relativistic linear theory of gravity: fail!
1. Dynamic of the field
2. Dynamic of a particle in the field: perihelion shift problem
3. Impossibility of gravitational redshift in a flat space-time

The equivalence principle: gravity ↔ non inertial frames
1. The equivalence principle and few predictions
2. Non-inertial frames and the equivalence principle
3. Gravity and the metric of space-time, welcome to GR!

> CURVED SPACE-TIME <

Curved space-time
1. Manifolds, geometry, Riemanian geometry
2. tangential manifold / tangent space
3. Covariant derivatives, Christoffel symbols
4. Link between Christoffel symbols and the metric tensor
5. Parallel transport and the geodesic equations
6. Conserved quantities and killing vectors
7. The Riemann tensor
8. Geodesic motion from least action principle (B. p31) Landau

> GRAVITY <

Sources of the gravitational fields
1. The energy momentum tensor
2. Matter as source
3. Fields as source

Field equations
1. Einstein field equations
2. Ricci- and Weyl-curvature
3. Linearized equations
4. Weak-field limit

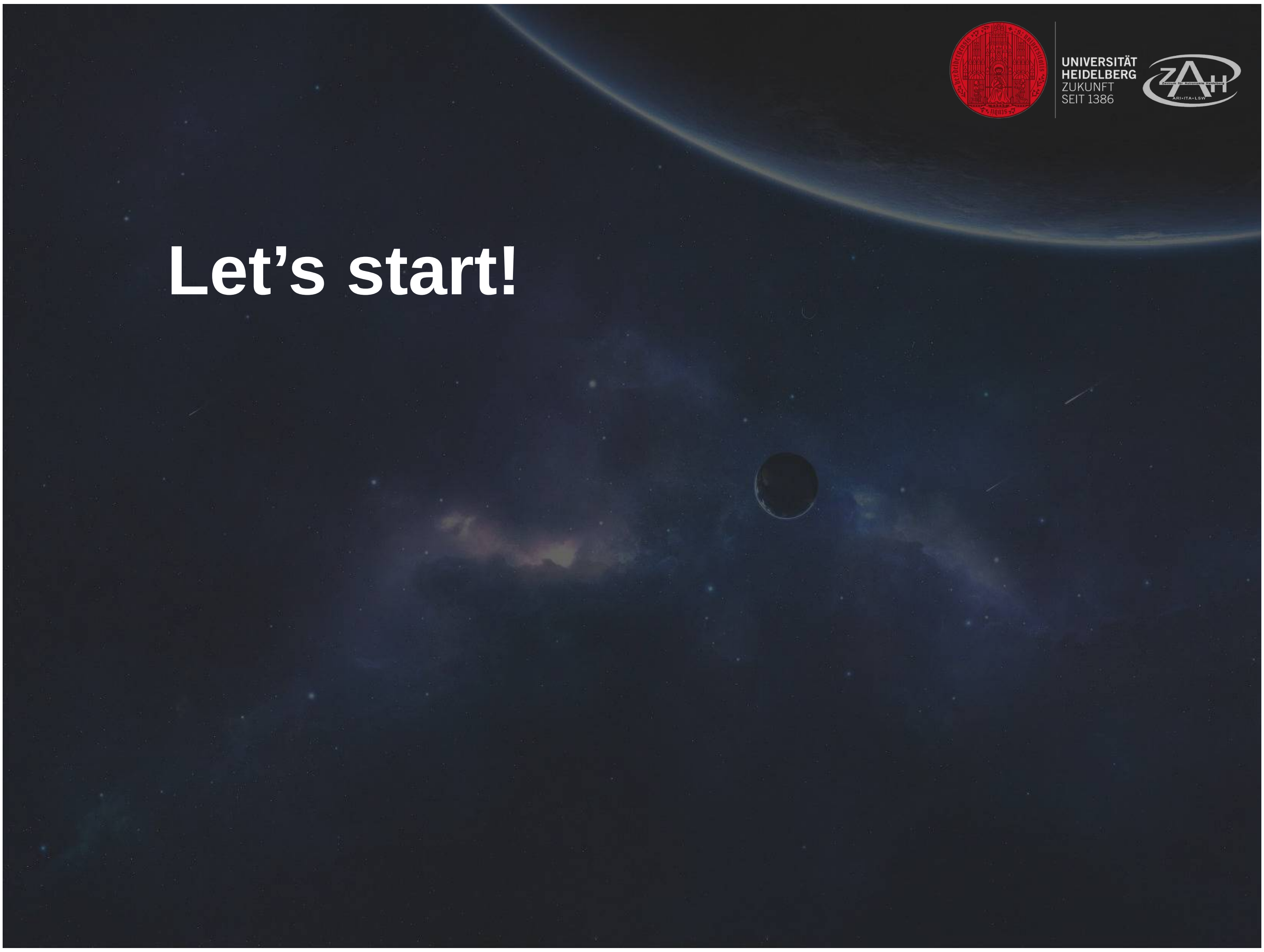
> APPLICATIONS <

Gravitational waves
Perturbative approach

Spherically symmetric systems
2. Schwartzshild black-holes
5. Kerr metric
6. Reissner-Nordström (electrically charged black-holes)

Cosmology, isotropic and homogeneous universe
1. Friedmann(-Lamaitre)-Robertson-Walker metric (FLRW)
2. distances
3. the expansion of the universe
4. cosmological redshift / energy “non conservation”
5. The cosmological constant and dark energy

A pinch of numerical general relativity
1. Numerical simulations of black holes accretion
2. Cosmological numerical simulations



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Let’s start!

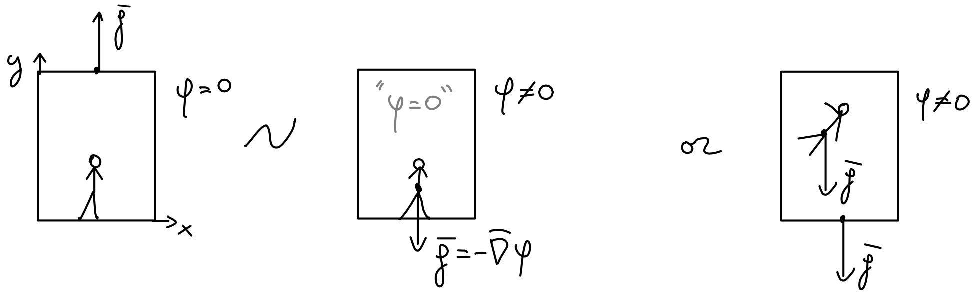
The equivalence principle

- All objects move in a gravitational field in the same way regardless their mass (given the same initial conditions)

$m_i \cdot \bar{a}^2 = -m_G \bar{\nabla}\varphi$ $m_i = m_G$! inertial mass = "gravitational charge"

=> m_G is sort of associated to some inertial phenomenon

- Motion in a gravitational field is analogous to a motion in a non-inertial frame



in $\varphi \neq 0$
free falling observer,
feels no gravity (locally)

=> Equivalence principle:

an accelerated frame (non inertial) is equivalent to a gravitational field

=> Gravity is relative

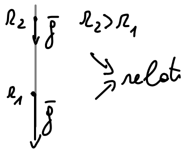
- Be careful here:

- Analogy, not "reality"
- Same motion (locally) but very different in nature!

↳ "fields" associated to non-inertial frames are not identical to real ones

↳ Real "fields" can not be eliminated everywhere and simultaneously by choosing a proper frame. You "can remove it" only locally (in the elevator)

↳ This works only locally! On large scales you have tydal effects →

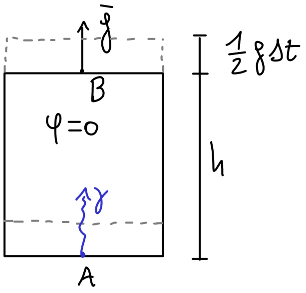


$r_2 > r_1$ ⇉ relative acceleration!

But not arbitrarily small scales: on small scales quantum effects (Heisemberg indetermination) where you need quantum gravity!

Equivalence principle: gravitational redshift

- Elevator: small enough such that if $\varphi \neq 0$, $g = \text{const}$ through it



- Equivalence principle: set $\begin{cases} \varphi = 0 & \text{no gravitational field} \\ \bar{g}\uparrow & \text{elevator accelerated upward} \end{cases}$

  "gravity in elevator"

- at $t=0$ photon emitted at A upward reaches B after $\Delta t = \frac{h}{c}$ (in elevator frame)

  $c \Delta t = h + \frac{1}{2} g \Delta t^2 \Rightarrow$ quadratic eq., solve for Δt $\quad \frac{1}{2} g \Delta t^2 - c \Delta t + h = 0$

$$\Delta t_{\pm} = \frac{1}{g}\left(c \pm \sqrt{c^2 - 2gh}\right) = \frac{c}{g}\left(1 \pm \sqrt{1 - \frac{2gh}{c^2}}\right) \simeq \frac{c}{g}\left[1 \pm \left(1 - \frac{gh}{c^2}\right)\right] = \begin{cases} - & h/c \\ + & \cancel{\frac{c}{g}\left(2 - \frac{gh}{c^2}\right)} \end{cases}$$

not physical
$g \to 0 \Rightarrow \Delta t \to \infty$

- When photon reaches detector in B, the latter has velocity Δv

  $\Delta v = g \Delta t \simeq \frac{gh}{c}$ with respect to velocity at emission

  $\Rightarrow$ Doppler shift

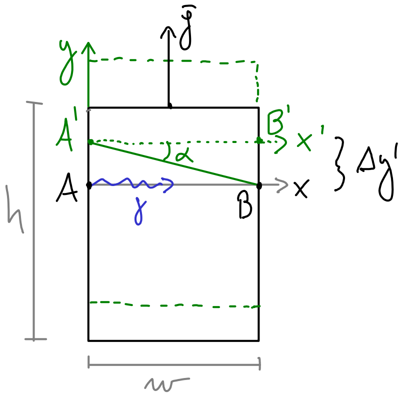
$$\lambda' = \left(1 + \frac{\Delta v}{c}\right)\lambda = \left(1 + \frac{gh}{c^2}\right)\lambda = \left(1 + \frac{|\bar{\nabla}\varphi| h}{c^2}\right)\lambda = \left(1 + \frac{\Delta\varphi}{c^2}\right)\lambda$$

equivalence principle
$g = |\bar{\nabla}\varphi|$

$$\boxed{z \equiv \frac{\lambda' - \lambda}{\lambda} = \frac{\Delta\varphi}{c^2}}$$ redshift

Equivalence principle: gravitational lensing

- Elevator : small enough such that if $\varphi \neq 0$, g = const through it
- Equivalence principle: set $\varphi = 0$, $\vec{g} \uparrow$



in O, photon goes straight along x=0 arrives in B

in O' (elevator) photon arrives in a point below B: $\Delta y' < 0$!

=> in rest frame of elevator photons move along a bent trajectory

$\Delta t = \frac{w}{c}$ arrival time

$\Delta v = g \Delta t = g \frac{w}{c}$ y component of velocity in B

$\boxed{\alpha} = \frac{\Delta v}{c} = \frac{g w}{c^2} = \boxed{\frac{|\nabla \varphi| w}{c^2}}$ deflection angle

equivalence $g \overset{!}{=} |\nabla \varphi|$

Summary:

=> in O, $\varphi = 0$ => photons go straight } same photon!
in O', $g \neq 0$ => " " along bent trajectories
↑
$|\nabla \varphi|$ for equivalence

↓
different perceived trajectory for different non-inertial frames
⇓
<u>relativity of gravity</u>

- P.S.

Possible to construct theories obeying equivalence principle but with no gravitational redshift

↳ not interesting: gravitational lensing is observed!

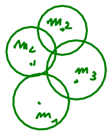
Gravity: the most general non-relativistic linear theory

- We now construct the theory as an extention of Newtonian gravity
- Use the least action principle ($\partial S = 0$) to derive the field equations $S =$ action

  i.e. derive the eq. of motion of the field

  $\partial S = 0 \longrightarrow$ Euler-Lagrange eq. $\longrightarrow$ eq. of motion
- $S = \int \mathcal{L}\, d^3x$ what should the lagrangian density $\mathcal{L}$ be?

- Define the lagrangian density, we want:



Tot: φ

$\varphi = \sum_i \varphi_i$

1) linear theory to have superposition principle, i.e. total field = sum of fields

$\Downarrow$

$\mathcal{L}(\varphi, \partial_i\varphi)$ lagrangian density with no higher order derivatives $\cancel{\partial_i\partial_j\varphi}$

$\Rightarrow$ at most squares of the potential in $\mathcal{L}$: φ^2

(∂S lowers the order $\Rightarrow$ linear "$\partial\varphi^2 = 2\varphi\partial\varphi$"
see Ostrogradski theorem)

2) Isotropic field around a spherically symmetric matter distribution

$\Rightarrow$ $\mathcal{L}$ must contain $\vec{\nabla}^2\varphi \equiv \partial_i\partial^i\varphi$ $\partial_i \equiv \frac{\partial}{\partial x^i}$ $\partial_i\partial^i = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} = \sum_i \frac{\partial^2}{\partial x^{i\,2}}$ $\varphi_{,i} \equiv \partial_i\varphi$

↳ scalar product between two vectors ($\vec{\nabla}$) is invariant under rotations

$$\vdash\Rightarrow \boxed{\mathcal{L}(\varphi, \partial_i\varphi) = \frac{1}{2}\partial_i\varphi\,\partial^i\varphi + 4\pi G\rho\varphi + \lambda\varphi + \frac{\alpha^2}{2}\varphi^2}$$

3 constants: G, λ, α!

Self interaction (kinetic term)

Interaction matter-field

for λ, α more considerations are needed

- Field equation

$-\ \partial S = 0 \Rightarrow$ Euler-Lagrange eq. $\underbrace{\frac{\partial \mathcal{L}}{\partial\varphi}}_{(1)} - \partial_i \underbrace{\frac{\partial \mathcal{L}}{\partial\varphi_{,i}}}_{(2)} = 0$ $\mathcal{L}(\varphi, \varphi_{,i})$

$$\left.\begin{array}{ll} (1) & \frac{\partial\mathcal{L}}{\partial\varphi} = 4\pi G\rho + \lambda + \alpha^2\varphi \\ (2) & \frac{\partial\mathcal{L}}{\partial\varphi_{,i}} = \partial^i\varphi \end{array}\right\} \quad 4\pi G\rho + \lambda + \alpha^2\varphi - \vec{\nabla}^2\varphi = 0 \quad \Rightarrow \quad \boxed{(\vec{\nabla}^2 - \alpha^2)\varphi = 4\pi G\rho + \lambda}$$

$\vec{\nabla}^2 \equiv \Delta$

Meanng of G, λ, α :

- **G?** set $\lambda = 0 = \alpha$ $\Rightarrow$ $\boxed{\Delta\varphi = 4\pi G\rho}$ solve it in polar coordinates, find φ

$\rho = \text{const}$ $\downarrow$

$$\cancel{\frac{1}{r^2}}\frac{\delta}{\delta r}\left(r^2\frac{\delta\varphi}{\delta r}\right) = 4\pi G\rho\, r^2 \qquad r^2\frac{\delta\varphi}{\delta r} = 4\pi G\rho\frac{r^3}{3} + A = GM_r + A \qquad M_r \equiv \frac{4}{3}\pi\rho r^3$$

integration const. A acts as a const. matter distribution, set $A = 0$

$$\frac{\delta\varphi}{\delta r} = \frac{GM_r}{r^2} \quad \text{for } r > R \quad (M_R = \text{const}) \quad \Rightarrow \quad \boxed{\varphi = -\frac{GM_R}{r}}$$

$\Rightarrow$ Attractive force and G "sets the strength"

What if $\rho = 0$? $\Rightarrow$ $\boxed{\Delta\varphi = \cancel{\frac{1}{r^2}}\frac{\delta}{\delta r}\left(r^2\frac{\delta\varphi}{\delta r}\right) = 0}$ $\Rightarrow$ $\cancel{r^2}\frac{\delta\varphi}{\delta r} = A\, r^{-2}$ $\quad \varphi = -A r^{-1} + B \quad r \to \infty\ \varphi = 0 \Rightarrow B \overset{!}{=} 0$

$\boxed{\varphi \propto r^{-1}}$ Solution in vacuum, ϕ exists $\neq 0$ without matter

- **λ?** set $G = 0 = \alpha$ $\quad \boxed{\Delta\varphi = \lambda}$ $\Rightarrow$ effect related to empty space (ρ is not there)

$$\cancel{\frac{1}{r^2}}\frac{\delta}{\delta r}\left(r^2\frac{\delta}{\delta r}\varphi\right) = \lambda\, r^2 \qquad \cancel{r^2}\frac{\delta\varphi}{\delta r} = \lambda\frac{r^{\cancel{3}\,1}}{3} \qquad \boxed{\varphi = \frac{\lambda}{6}r^2}$$

note: $\frac{\delta\varphi}{\delta r} = -f_r = \lambda\frac{r}{3}$ $\Rightarrow$ acceleration ↑ if r ↑

gravity is repulsive on large scales! if $\lambda \neq 0$

yes! cosmic accelerated expansion on large scales!
↳ acts as a cosmological constant

- **α>?** set $G = 0 = \lambda$ $\quad \boxed{(\Delta - \alpha^2)\varphi = 0}$

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta}{\delta r}\varphi\right) - \alpha^2\varphi = \frac{1}{r^2}\left(2r\frac{\delta\varphi}{\delta r} + r^2\frac{\delta^2\varphi}{\delta r}\right) - \alpha^2\varphi = 0 \qquad \frac{\delta\varphi}{\delta r} \equiv \dot\varphi$$

$$2r\dot\varphi + r^2\ddot\varphi - \alpha^2\varphi r^2 = 0 \qquad \boxed{\varphi \propto r^{-1}\exp(-\alpha r)}$$ Yukawa potential

α = screening constant (shielding)

currently not observed i.e. currently $\alpha = 0$

- Conclusion:

$$\boxed{(\Delta - \alpha^2)\phi = 4\pi G\rho + \lambda}$$

$\uparrow$ shielding $\quad \uparrow$ attraction $\quad \uparrow$ repulsion

$\phi\ \forall$ also in vacuum ("not associated to matter")
↓
it exists in itself

Can this theory solve some problems of "standard" Newtonian gravity?

Bertrand's theorem stil holds (mechanics) => orbits are stil closed ellipses in this theory because $\varphi \propto r^{-1}$

Nor λ, nor α can explain perihelion shift => this theory fails!

(+ this theory is not covariant!) => we need a new theory!

First, let's investigate the connection of $\varphi \propto r^{-1}$ to space: why r^{-1} ?!

$\varphi \propto r^{-1}$ and the Gauss theorem

Poisson eq.: $\overset{(1)}{\bar\nabla^2 \varphi} = \overset{(2)}{4\pi G \rho}$ integrate over volume

$$(1): \int_V \bar\nabla^2 \varphi \, d^3x = -\int_V \bar\nabla \overset{\bar g = -\bar\nabla\varphi}{\bar g} \, d^3x = \int_{\delta V} \bar g \, d\bar A$$

Gauss Theorem => $\bar g$ flux of $\bar\nabla^2 \varphi$

$$(2): \int_V 4\pi G \rho \, d^3x = 4\pi G \overbrace{\int_V \rho \, d^3x}^{M_V} = 4\pi G M_V = \text{const}$$

M_V const if mass is all within $R<r$

$$\Downarrow$$

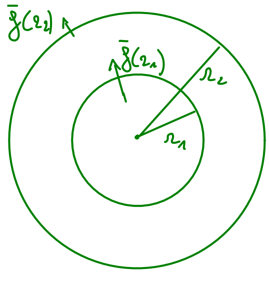
$$\boxed{\int_{\delta V} \bar g \, d\bar A = \text{const}}$$

isotropic case (spherical integral)

flux across the surface $\delta V = 4\pi r^2$ (Euclidean space!)

=> $\varphi \propto r^{-1}$ depends on how surfaces change with radius $dA(r)$

$\neq$ geometry → different $dA(r)$ → different $\varphi(r)$ dependency !!



$A(r) = 4\pi r^2$ yes... but in Euclidean geometry!

Equation of motion of a particle in a gravitational field

- Lagrangian

$$L(x,\dot{x}^i) = \overset{T}{\tfrac{1}{2} m \dot{x}_i \dot{x}^i} - \overset{V}{m\varphi} = m\left(\tfrac{1}{2}\dot{x}_i\dot{x}^i - \varphi\right) \quad \Rightarrow \quad m \text{ is irrelevant} \quad (m_{\text{grav.}} = m_{\text{inertial}}!)$$

note: $L \to L' = aL + b \quad a, b \in \mathbb{R}$ give the same eq. of motion

$$\delta S = \partial \int L\, dt = \int \partial L\, dt = 0$$

$$\delta S = \partial \int L'\, dt = \int \partial(aL+b)\, dt = \underset{\substack{\uparrow \\ \text{irrelevant}}}{a} \int \partial L\, dt = 0 = \int \partial L\, dl$$

- Euler-Lagrange eq. → eq. of motion

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{x}^i} - \frac{\delta \mathcal{L}}{\delta x^i} = 0 \qquad \frac{\delta L}{\delta \dot{x}^i} = \dot{x}^i \qquad \frac{\delta L}{\delta x^i} = -\frac{\delta \varphi}{\delta x^i} \quad \Rightarrow \quad \ddot{x}^i + \delta_i \varphi = 0 \qquad \boxed{\bar{\ddot{x}} = -\bar{\nabla}\varphi}$$

Part II

The flat space-time

Special Relativity: the concept

- $P=(t,x,y,z)$ <u>Event</u> : something happening in P
- S, S' <u>Coordinate frames</u> (in standard configuration)
- <u>Generic linear transformation</u> (preserve homogeneity!)

$$x' = At + Bx \qquad y' = y$$
$$t' = Dt + Ex \qquad z' = z$$

Find coefficients:

$$\begin{cases} x=0 \Rightarrow & x' = At,\ t' = Dt \qquad \frac{x'}{t'} \equiv v' = \frac{A}{D} \qquad \boxed{A = Dv'} \\ x'=0 \Rightarrow & 0 = At + B\underset{\overset{\uparrow}{x=vt}}{x} = At + Bvt \qquad A = -Bv \end{cases} \searrow\nearrow \quad D\overset{\overset{-v}{\downarrow}}{v'} = -Bv \Rightarrow \boxed{D=B}$$

$$\boxed{x'} = Dv't + Dx = D(v't+x) = \boxed{D(x-vt)} \qquad y'=y \qquad \circledast \qquad D = ? \quad E = ?$$
$$\boxed{t' = Dt + Ex} \qquad z'=z$$

- <u>Postulate : $t \overset{!}{=} t'$</u> (Absolute time, Galilean frame)

$$t' = Dt + Ex \quad \forall t,x \iff D=1 \quad E=0 \quad \Rightarrow \quad \boxed{\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned}}$$

- <u>Postulate : $c \overset{!}{=}$ const $\forall$ frame</u> (observations)

- $c=$ const $\Rightarrow$ it make sense to investigate light
- Consider a photon in $2 \neq$ frames

$$\begin{aligned} c^2t^2 &= x^2+y^2+z^2 \\ c^2t'^2 &= x'^2+y'^2+z'^2 \end{aligned} \Rightarrow \begin{aligned} 0 &= -c^2t^2+x^2+y^2+z^2 \\ 0 &= -c^2t'^2+x'^2+y'^2+z'^2 \end{aligned} \Rightarrow \boxed{-c^2t^2+x^2+y^2+z^2 = -c^2t'^2+x'^2+y'^2+z'^2}$$

in a cartesian frame !

Plug $\circledast$

$$\Rightarrow \boxed{\begin{aligned} ct' &= \gamma(ct - \beta x) \qquad & y' &= y \\ x' &= \gamma(x - \beta ct) \qquad & z' &= z \end{aligned}} \qquad \gamma = (1-\beta^2)^{-1/2} \qquad \beta = \frac{v}{c}$$

<u>Lorentz transformations</u>

↳ Space and time are connected "Not like the frames of a movie"

↳ $\boxed{\text{The Space-Time}}$! $\Rightarrow$ event $P \equiv (x^\mu) \equiv (ct, x, y, z)$

ct to have units of lengths
x^μ elements of a 4-vector

$$\text{4-interval}: d\bar{x}^2 = -(\underbrace{cdt}_{dx^0})^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \textstyle\sum_i \sum_j \eta_{ij}\, dx^i dx^j \qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Minkowski metric</u>
(!)

Some abmiguity is left

For photons $ds^2 = 0 = ds'^2$ also $\phi(v)\, ds^2 = 0$ $\phi \in \mathbb{R}$ $ds'^2 = \phi(v)\, ds^2$

$$ds^2 = \phi(v) \overset{v}{d}s'^2 = \phi^2(v)\, ds^2$$

$\left(\text{e.g. } -c^2 dt^2 + d\bar{r}^2 = 0 \text{ or } c^2 dt^2 - d\bar{r}^2 = 0 \right)$ $\Rightarrow$ $\boxed{\phi(v) = \pm 1}$ signature

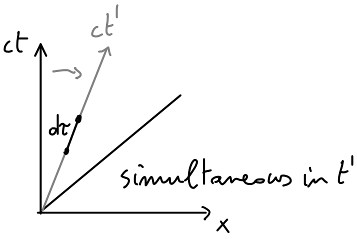
one can choose

Here we will use $(-,+,+,+)$ signature

The proper time

Time as measured with a clock at rest in a frame

$$-c^2dt^2+dx^2+dy^2+dz^2=-c^2dt'^2+\underbrace{dx'^2+dy'^2+dz'^2}_{\text{center such that }\overset{!}{=}0}$$



we call $dt' \equiv d\tau$ proper time

$$-c^2d\tau^2=-c^2dt^2+dx^2+dy^2+dz^2=ds^2 \quad \Rightarrow \quad \boxed{d\tau^2=-\frac{ds^2}{c^2}} \quad (1)$$

$$=-c^2dt^2\underbrace{\left(1-\frac{dx^2+dy^2+dz^2}{c^2dt}\right)}_{\left(1-\frac{\vec{v}^2}{c^2}\right)=\left(1-\vec{\beta}^2\right)=\gamma^{-2}} \quad \Rightarrow \quad \boxed{d\tau=\gamma^{-1}dt} \quad (2)$$

$$dt=\gamma\, d\tau$$
↑
time dilation

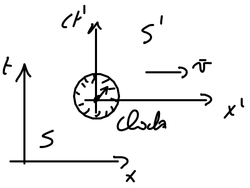
Time interval

$$\Delta\tau=\int_{w} d\tau=\int_{w}\gamma^{-1}dt \quad \leftarrow (2)$$

$$=\int_{w}\frac{1}{c}\sqrt{-ds^2} \quad \leftarrow (1)$$

$$=\int_{w}\frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \quad \leftarrow \text{The metric is "hidden" there!}$$

↑
(!)



in G.R $\underline{\eta_{\mu\nu}\rightarrow g_{\mu\nu}}$ given by local properties of (curved) space-time

Implication of the Lorentz transforms

- $ct' = \gamma(ct - \frac{v}{c}x)$ $x' = \gamma(x - vt)$ $y' = y$ $z' = z$ (Standard configuration of S, S')

- Time dilation

$t_0 \equiv t_B - t_A$ (proper time)

$t = t_0(1-\beta^2)^{-1/2} = t_0\gamma \Rightarrow v\uparrow \Rightarrow t\uparrow$

- Length contraction

$l_0 \equiv x'_0 - x'_A$ (proper length)

$l = l_0(1-\beta^2)^{1/2} = l_0\gamma^{-1}$ $(v\uparrow \Rightarrow l\downarrow)$

what if $v = c$? $\Rightarrow l \to 0$

$\Rightarrow \rho = \frac{m}{l^3} = \infty!$

$\Leftrightarrow v = c$ not possible for massive particles

- Velocity transformation

$$\boxed{u'_x} \equiv \frac{\overset{\downarrow}{dx'}}{\underset{\uparrow}{dt'}} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \boxed{\frac{u_x - v}{1 - \frac{u_x v}{c^2}}}$$

(in lab frame: $u_x = \frac{u'_x + v}{1 + \frac{u_x v}{c^2}}$)

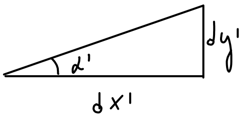
$$\boxed{u'_y \equiv \frac{dy'}{\underset{\nwarrow}{dt'}} = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})}}\ (!) \qquad \boxed{u'_z \equiv \frac{dz'}{\underset{\nwarrow}{dt'}} = \frac{u_z}{\gamma(1 - \frac{u_x v}{c^2})}}\ (!)$$

because the time is affected

- Accelerations

$a'_x \equiv \frac{\overset{\searrow}{du'_x}}{\underset{\nwarrow}{dt'}}$ ....

- Angles

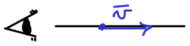


$dy' = dx'\,tg\,\alpha' \Rightarrow$

$$\boxed{tg\,\alpha'} = \frac{\overset{\downarrow}{dy'}}{\underset{\nwarrow}{dx'}} = \frac{dy}{\gamma(dx - v\,dt)} = \frac{dy}{\gamma\,dt(\frac{dx}{dt} - v)} = \boxed{\frac{u_y}{\gamma(u_x - v)}}$$

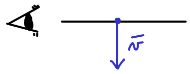
- Doppler effect

↳ longitudinal :



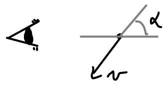
$\frac{\nu}{\nu'} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$

↳ transverse :



$\nu = \nu'\gamma^{-1}$ $(t = t'\gamma)$ purely relativistic effect

↳ Arbitrary direction :

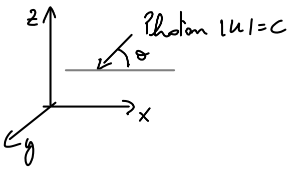


$\frac{\nu}{\nu'} = [\gamma(1 + \beta\cos\theta)]^{-1}$ $\frac{\nu}{\nu'} = \gamma(1 - \beta\cos\theta')$

angle seen by the observer

angle in rest frame of moving source

- Aberration of light



Polar coordinates: $u'_x = c\cos\theta'$ $u'_y = 0$ $u'_z = c\sin\theta'$

$u_x = c\cos\theta$ $u_y = 0$ $u_z = c\sin\theta$

Lorentz transf. of velocities → $u'_x = \dfrac{u_x - v}{1 - \frac{u_x v}{c^2}} = \dfrac{c\cos\theta - v}{1 - \beta\cos\theta}$ ⇒ $\boxed{\cos\theta' = \dfrac{\cos\theta - \beta}{1 - \beta\cos\theta}}$

Photon

in lab frame: $\cos\theta = \dfrac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$

- Beaming of light : (solid angles)

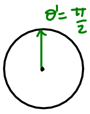
$d\Omega' = d\phi' d\cos\theta'$

$d\Omega = d\phi\, d\cos\theta$: $d\phi = d\phi'$ no change because ⊥ to velocity

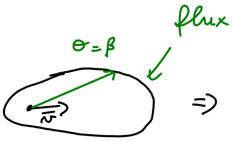
$d\cos\theta = d\left(\dfrac{\cos\theta' + \beta}{1 + \beta\cos\theta'}\right) = \dfrac{d\cos\theta'}{\gamma^2(1 + \beta\cos\theta')^2}$

$= \dfrac{d\phi\, d\cos\theta'}{\gamma^2(1 + \beta\cos\theta')}$

Super relevant for emission processes :



isotropic emission in source rest frame



anisotropic emission for observer

⇒ priviledge direction of emission

eg. $\theta' = \frac{\pi}{2}$: $\cos\theta' = 0$ ⇒ $\cos\theta = \dfrac{\cos\theta' + \beta}{1 + \beta\cos\theta'} = \beta$

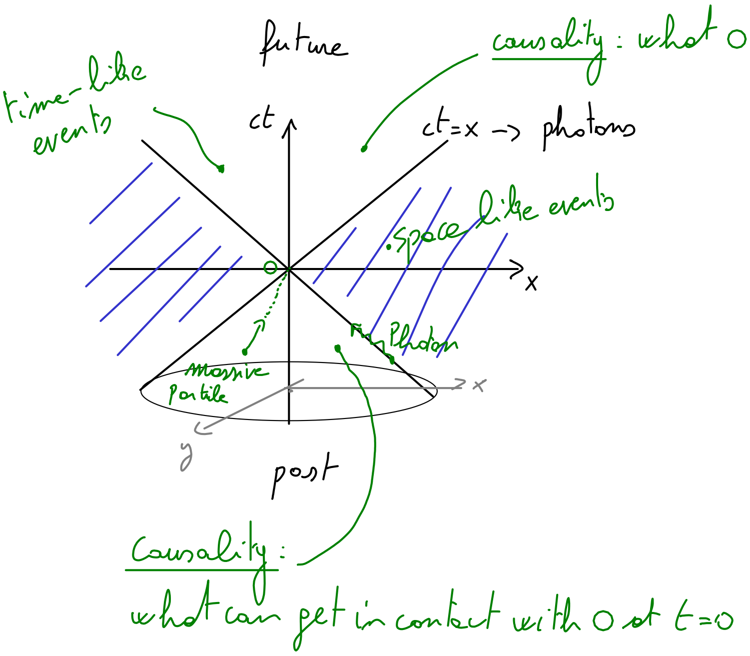
$\beta \ll 1$ ⇒ $\cos\theta \sim 0$ $\theta \sim \pi/2$

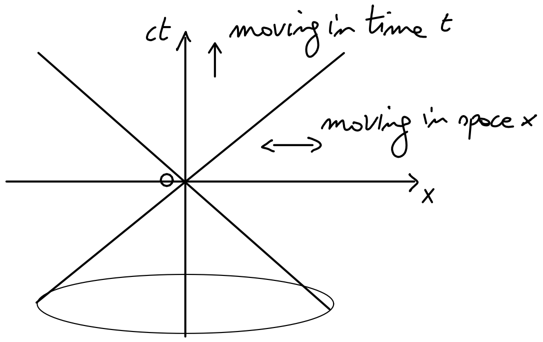
$\beta \sim 1$ ⇒ $\cos\theta \sim 1$ $\theta \sim 0$ (beamed forward)

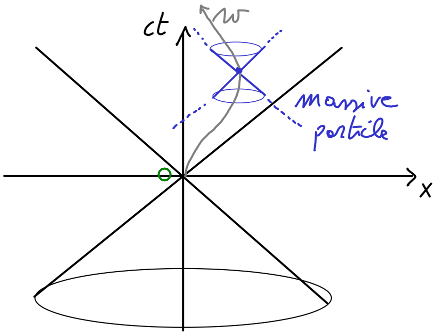
- Cross section

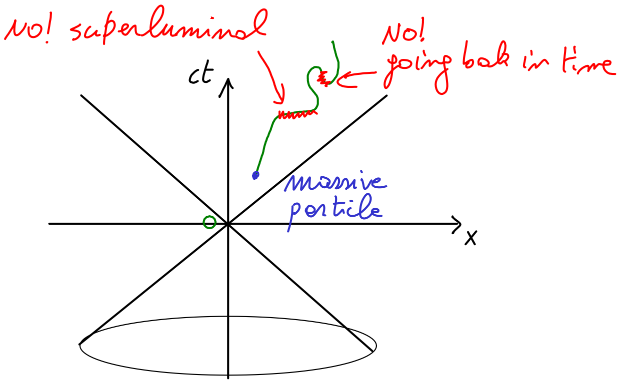
Solid angles are affected ⇒ cross sections transform as well

Space-time diagrams

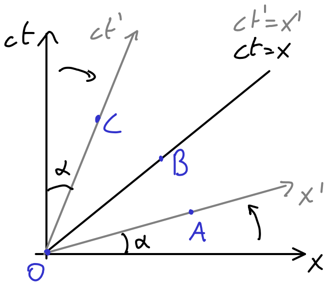








Note: ds^2 is not positive definite



OA: Space like event $ds>0$ (frame in which $\Delta t'=0$)

OB: null $ds=0$

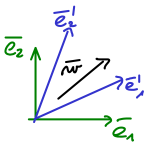
OC: Time like event $ds<0$ (frame in which $\Delta x'=0$)

Lorentz transf. are "rotations" (boosts) $\frac{dx}{c\,dt}=\frac{v}{c} \Rightarrow \alpha = \mathrm{atan}\left(\frac{v}{c}\right)$

Because of postulate:
- null line do not change under Lorentz. transf.
- ds^2 is invariant under Lorentz. transf. (i.e. is the same ∀ frame)

↓

"The norm of a vector does not depend on the basis"



Lorentz geometry

Geometric interpretation of the ct,x plane

- Define a vector space
- $\bar{x} = (x^\mu) = (x^0, x^1, x^2, x^3)$ 4-vector $x^0 = ct$ $\bar{x} \in \mathbb{M}$ $\mathbb{M}$ Minkowski space
- Separations in the space-time are expressed by the 4-interval $d\bar{s}$

  ⇒ Distances are measured in termes of a metric ⇒ metric vector space

$$d\bar{s}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

4-interval ⇒ vector space $d\bar{s} \in \mathbb{M}$ vector in Minkowski space

$$= \sum_{\nu=0}^{3}\sum_{\mu=0}^{3} \eta_{\mu\nu}\, dx^\mu dx^\nu$$

Minkowski metric $\eta = \mathrm{diag}(-1,1,1,1)$ $\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu$$

← Einstein notation

$$= dx_\nu\, dx^\nu$$

← and index lowering

Based on a frame ⇒ components

Frameless representation

$$= d\bar{x}^T \eta\, d\bar{x}$$

matrix notation

$$= \eta(d\bar{x}, d\bar{y})$$

$\eta : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ $\eta(\bar{u}, \bar{v}) = a \in \mathbb{R}$ $\bar{u}, \bar{v} \in \mathbb{M}$ bilinear map identifying the scalar product

$$= \langle d\bar{x}, d\bar{x} \rangle$$

scalar product

$$= \tilde{dx}(d\bar{x})$$

$\eta(\bar{v}, -) \equiv \tilde{v}$ $\tilde{v} : \mathbb{M} \to \mathbb{R}$ $\tilde{v}(\bar{u}) = a \in \mathbb{R}$ $(v_\nu) \equiv \bar{v}^* \equiv \tilde{v} \in \mathbb{M}^*$ $\mathbb{M}^*$ Dual space of $\mathbb{M}$

1-form, dual vector, tangent vector....

$$= \|d\bar{s}\|^2 \in \mathbb{R}$$

norm of $d\bar{s} \in \mathbb{M}$ ⇒ invariant, it can be negative!

Here we used η but this formalism is valid for any arbitrary metric g! in a grav. field $g \neq \eta$!

The metric is "hidden" in many places!

- Proper time $\tau = \frac{1}{c}\sqrt{-ds^2} = \frac{1}{c}\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} = \frac{1}{c}\sqrt{-\eta(d\bar{x}, d\bar{x})}$

  ↑

- Do you see?! The metric is hidden in crucial places!

- Another example? It is even in the dear old Kinetic energy...

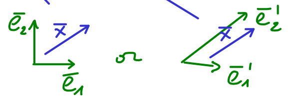
$$T = \frac{1}{2} m \bar{v}^2 = \frac{1}{2} m \bar{v}\bar{v} = \frac{1}{2} m\, \delta_{ij} v^i v^j \qquad \bar{v} \in \mathbb{R}^3$$

↑ $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$ (Euclidean 3D space + cartesian system)

Note: above we have choosen a specific basis set (cartesian)

$\bar{x} = x^{\mu} \bar{e}_{\mu} = x^{\mu'} \bar{e}_{\mu'}$ 4-vectors as linear combination of basis

e.g.



different $\{\bar{e}_\mu\}$ but $\bar{x}$ is $\bar{x}$

$\{\bar{e}_\mu\}$ basis set $\quad \eta_{\mu\nu} = \eta(\bar{e}_\mu, \bar{e}_\nu) = \bar{e}_\mu \bar{e}_\nu \quad$ η defines the scalar product

↑ some linearly independent vector

$\bar{e}_0 \bar{e}_0 = -1 \quad \bar{e}_i \bar{e}_i = 1 \quad \bar{e}_i \bar{e}_j = 0 \quad i \neq j$ ← Cartesian frame

$\bar{e}_0 = (1,0,0,0)^T, \; \bar{e}_1 = (0,1,0,0)^T \ldots$

$d\bar{s}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$\bar{e}_\mu \; \bar{e}_{\mu'}$ basis of S and S' frame

Minkowski metric η can also be not diagonal. Example with ≠ basis:

$$ds^2 = \underbrace{-c^2dt^2 + dx^2 + dy^2 + dz^2}_{\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} = (\underbrace{-cdt + dx}_{\equiv du})(\underbrace{cdt + dx}_{\equiv dv}) + dy^2 + dz^2 \equiv \underbrace{du\,dv + dy^2 + dz^2}_{\eta = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Other coordinate frames: polar, Kruskal, Eddington-Finkenstein, ...

This is just linear algebra!

Frame transformations

- We are dealing with inertial frames

$$\frac{dx^\mu}{d\tau} = \text{const.} \qquad \frac{dx^{\mu'}}{d\tau} = \text{const.'} \qquad (\text{shift of coordinates through a point})$$

$$\frac{d^2x^\mu}{d\tau^2} = 0 \qquad \frac{d^2x^{\mu'}}{d\tau^2} = 0 \qquad (\text{no acceleration})$$

here, we parameterize $x^\mu(\tau)$ with proper time τ, but any affine parameter can be used $\lambda = a + \tau b$

- Relation between frames S, S'

$$x^{\mu'} = x^{\mu'}(x^\mu(\tau))$$

transformations law

transformed component

$$\frac{dx^{\mu'}}{d\tau} = \frac{\delta x^{\mu'}}{\delta x^\mu} \frac{dx^{\tilde{\mu}}}{d\tau}$$

$$\frac{\delta x^{\mu'}}{\delta x^\mu} = \text{Jacobian of the transformation}$$

$$\underset{\overset{!}{=}0}{\frac{d^2x^{\mu'}}{d\tau^2}} = \frac{d}{d\tau}\left(\frac{dx^{\mu'}}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\delta x^{\mu'}}{\delta x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu} \overbrace{\frac{dx^\nu}{d\tau}}^{\text{constant}} \frac{dx^\mu}{d\tau} + \frac{\delta x^{\mu'}}{\delta x^\mu} \underset{\overset{!}{=}0 \ (\text{inertial frame})}{\frac{d^2x^\mu}{d\tau^2}} \overset{!}{=} 0$$

$$\Downarrow$$

$\Rightarrow$ constrain for inertial frames : $\boxed{\frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu} \overset{!}{=} 0}$ solve for $x^{\mu'}$ $\Rightarrow$ $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu + a^{\mu'}$ (see next page : *)

- This implies a linear transfrmation of vectors between frames

$$\boxed{x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu + \cancel{a^{\mu'}}}$$

not relevant (origin shift)

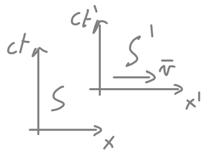
→ preserves homogeneity ($\bar{x}' = \Lambda \bar{x}$ frameless)

→ " metric

→ i.e. preserves the scalar product $\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}', \bar{v}' \rangle$

- Here we want c invariant $\Rightarrow$ Lorentz transforms as we have seen in previous lectures

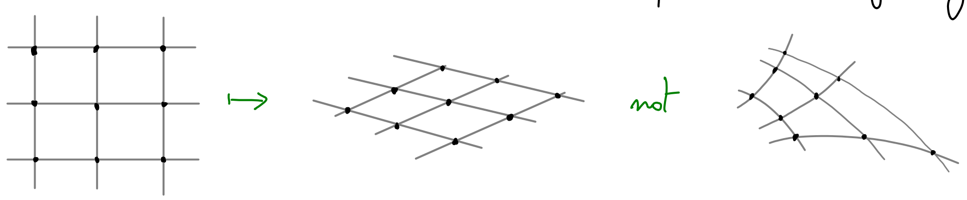
$$(\Lambda^{\mu'}{}_\nu) = \left(\frac{\delta x^{\mu'}}{\delta x^\nu}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ for "normal configuration"



$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix}$$

Another way to define the needed transformation

- The 4-interval must be invariant to preserve homogeneity



distance between points invariant under coord. transformation

$\Rightarrow$ Need a linear transformation $\boxed{x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} + a^{\mu'}}$ $\Lambda^{\mu'}{}_{\nu} = \frac{\delta x^{\mu'}}{\delta x^{\nu}}$

to keep the scalar product invariant, i.e. $\boxed{\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}', \bar{v}' \rangle}$

equivalent to say Λ must preserve the metric, i.e. $\boxed{\eta = \Lambda^T \eta \Lambda}$ same η in all frames

(Recall: same η on booth sides, eg. cartesian frame
$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2$)

in fact $d\bar{s}^2 = d\bar{s}'^2$ with $\eta_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'}$

$$d\bar{s}^2 = d\bar{x}^T \eta\, d\bar{x}$$
$$(!) \; = d\bar{x}'^T \eta\, d\bar{x}'$$
$$= \underbrace{d\bar{x}^T \Lambda^T}\, \eta\, \underbrace{\Lambda d\bar{x}}$$

$$\overset{(1)}{\boxed{\eta = \Lambda^T \eta \Lambda}} \leftrightarrow \overset{(1)}{\boxed{\eta_{\mu\nu} = \Lambda^{\alpha'}{}_{\mu} \Lambda^{\beta'}{}_{\nu}\, \eta_{\alpha'\beta'}}} \quad \text{or} \quad \overset{(1)}{\eta_{\alpha'\beta'} = \Lambda^{\mu}{}_{\alpha'} \Lambda^{\nu}{}_{\beta'}\, \eta_{\mu\nu}}$$

! the same $\Rightarrow$ Λ such that η invariant under such transf.

- This is the necessary and sufficient condition that a transformation $\Lambda^{\mu}{}_{\nu'} \equiv \frac{\delta x^{\mu}}{\delta x^{\nu'}}$ is a Lorentz transform (being $\eta \overset{!}{=} \eta'$)

⊛

Find transformation: solve for $x^{\mu'}$

integrate by part

$$0 = \int \frac{\delta^2 x^{\mu'}}{\delta x^{\nu} \delta x^{\mu}} x^{\mu} dx^{\nu} = \frac{\delta x^{\mu'}}{\delta x^{\mu}} x^{\mu} - \int \overbrace{\overbrace{\frac{\delta x^{\mu'}}{\delta x^{\mu}}}^{\Lambda^{\mu'}{}_{\mu}} \overbrace{\frac{\delta x^{\mu}}{\delta x^{\nu}}}^{\Lambda^{\mu}{}_{\nu}}}^{\Lambda^{\mu'}{}_{\nu}} dx^{\nu} + a^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} - \overbrace{\int dx^{\mu'}}^{x^{\mu'}} + a^{\mu'}$$

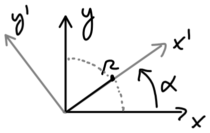
$\Rightarrow x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} + a^{\mu'}$ i.e. linear transformation !

Lorentz invariant quantities

Why are there quantities that do not change under coordinate transformations?

A familiar example: rotation in 3D

i.e. same values ∀ frame



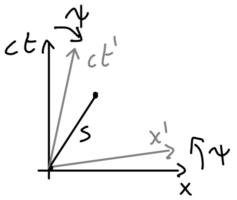
e.g. in 2D $r^2 = x^2+y^2 = \delta_{ij}x^ix^j$ Invariant under rotations

↳ corresponds to the statement $\cos^2\alpha+\sin^2\alpha=1 \quad \forall\alpha\in\mathbb{R}$

$= (r\cos\alpha)^2+(r\sin\alpha)^2$ $\quad r^2\geq 0$

Likewise, for Lorentz transforms: define a "rotation angle" $\alpha \to \psi$

- ψ = rapidity parameter (Sort of polar coord.)



$\cosh\psi \equiv \gamma \quad \in [1,\infty]$

$\sinh\psi \equiv \beta\gamma \quad \in [-\infty,\infty]$

$\tanh\psi = \frac{\beta\gamma}{\gamma} = \beta$

$\underline{\cosh^2\psi-\sinh^2\psi} = \gamma^2-\beta^2\gamma^2=\gamma^2(1-\beta^2)=\underline{1}$

$s^2 \equiv -(dx^0)^2+(dx^1)^2 = \eta_{\mu\nu}dx^\mu dx^\nu$

$= -(s\sinh\psi)^2+(s\cosh\psi)^2 = s^2\underbrace{(\cosh^2\psi-\sinh^2\psi)}_{=1}$

- Lorentz transformations $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$ ~ hyperbolic rotation : boost

- Λ form the so called Lorentz group (see next page)
  the invariance is an intrinsic property of this group
  not just a property of sinh, cosh !

The Lorentz group

- What is a groups?

Set of elements $\{G_i\}$ with a connection $*$ between elements $(G,*)$ with these properties:

$*$	connection between elements
$a*b=c$	addition, $\forall a,b \in G \Rightarrow c \in G$
$(a*b)*c = a*(b*c)$	associativity, $\forall a,b,c \in G$
$e*a = a*e = a$	$\exists$ of identity element e, $\forall a \in G$
$a*b = b*a = e$	$\exists$ of inverse, $\forall a,b \in G$
extra $a*b = b*a$	commutativity, $\forall a,b \in G \Rightarrow$ Abelian group, e.g. $GL(n)$ is not abelian Rotations

: e.g. $(a \in \mathbb{R}, \cdot)$ $1_G = 1$
$(a \in \mathbb{R}, +)$ $1_G = 0$
$(a \in \mathbb{R}^n \times \mathbb{R}^n, \cdot)$ $1_G = I^n$

- What is a Lie group?

Continuous group $\{G(\alpha_1, \dots, \alpha_m)\}$ i.e. Elements are "functions" of α_j continuous parameters

Have finite dimensional differentiable (i.e. smooth) manifold

- What is the Lorentz group?

A non Abelian Lie group which elements are the Lorentz transformations

$$SO(3,1,\mathbb{R}) = \{\Lambda \in \mathcal{M}(4,\mathbb{R}) \mid \langle \bar u, \bar v \rangle = \langle \overset{\bar u'}{\Lambda \bar u}, \overset{\bar v'}{\Lambda \bar v} \rangle \ \forall \bar u, \bar v \in \mathbb{M}\}$$

↑ orthogonal $\Lambda^T \Lambda = I_4$ ↑ 4x4 real matrixes

↑ Special : $\det(\Lambda) = 1$

- Continuous group $\{\Lambda\}$
- Elements (Λ) are "functions" of continuous parameters : v or ψ
- Finite dimensional differentiable (i.e. smooth) manifold

} Lie group

- $\dim O(n) = \frac{n(n-1)}{2}$ $n=4$
  - 6 free parameters (1) : 3 boosts (x^i, x^0 planes), 3 rotations (x^i, x^j $i \neq j$ plans)
  - 4 translations (2) : shift in time and time

$$x^{\mu'} = \overset{(1)}{\Lambda^{\mu'}{}_{\nu}} x^{\nu} + \overset{(2)}{a^{\mu'}}$$

$\Rightarrow$ 10 parameters (Poincaré group)

- Non abelian group (Λ along different directions do not commute)

Inverse transformation $(v \to -v)$ $x^{\mu'} \to x^{\mu}$

$\bar{x} \to \bar{x}'$: $x^{\mu'} = \Lambda^{\mu'}{}_{\alpha} x^{\alpha}$

$\bar{x}' \to \bar{x}$: ? look for it: $x^{\mu} = \Lambda^{\mu}{}_{\alpha'} \Lambda^{\alpha'}{}_{\nu} x^{\nu} \Rightarrow \Lambda^{\mu}{}_{\alpha'} \Lambda^{\alpha'}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$$\Lambda_{\mu'}{}^{\nu} = \frac{\delta x^{\nu}}{\delta x^{\mu'}}$$

$\Rightarrow (\Lambda^{\mu}{}_{\alpha'})$ is the inverse of $(\Lambda^{\alpha'}{}_{\nu})$

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$x^{\mu'} = \Lambda^{\mu'}{}_{\alpha} x^{\alpha}$ $\quad \underbrace{\Lambda^{\beta}{}_{\mu'} x^{\mu'}}_{= x^{\alpha}} = \Lambda^{\beta}{}_{\mu'} \Lambda^{\mu'}{}_{\alpha} x^{\alpha} \Rightarrow \Lambda^{\beta}{}_{\mu'} \Lambda^{\mu'}{}_{\alpha} = \delta^{\beta}_{\alpha}$?

$\Lambda^{T} \Lambda = I_4$

$(\Lambda^{-1})^{\nu'}{}_{\mu} \equiv \Lambda_{\nu'}{}^{\mu} \Rightarrow \Lambda_{\nu'}{}^{\alpha} \Lambda^{\mu'}{}_{\alpha} = \delta^{\mu'}_{\nu'}$ or $\Lambda_{\alpha'}{}^{\mu} \Lambda^{\alpha'}{}_{\nu} = \delta^{\mu}_{\nu}$

cool

Groups

(Lorentz group. → Special relat.) rapidity parameter

Set of elements $(G,*)$ $\{G_i\}$

$*$ connection between elements : e.g. $(+, 1_G=0)$, $(\cdot, 1_G=1)$, $(\cdot, I_{d_m})$

ℝ

$GL(m)$

additive group

$a*b=c$ addition $\forall a,b \in G \Rightarrow c \in G$

$(a*b)*c = a*(b*c)$ associativity $\forall a,b,c \in G$

$e*a = a*e = a$ $\exists$ of identity element $\forall a \in G$ $e := 1_G$

$a*b = b*a = e$ $\exists$ of inverse $\forall a \in G$ $\Rightarrow$ imply division $xa=b$ $xaa^{-1}=ba^{-1}$ $x=ba^{-1}$

"extra"

$a*b=b*a$ Commutativity $\forall a,b \in G$ $\Rightarrow$ Abelian group , eg. $GL(m)$ is not. Rotations are

Homomorphism : imaging between groups

it preserves the group structure

$f: G \to H$ $(G,*)$ $(H,\circledast)$

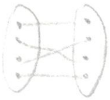
"linear"

$g,h \in G$ $f(g*h) = f(g) \circledast f(h)$ $f(g), f(h) \in H$

Isomorphism : $\exists$ 2 homomorphism such that $f: G \to H$ $s: H \to G$ (inverse)

$f(s(h)) = h$ $s(f(g)) = g$

G, H contain the same information



Bijective

General linear group

$GL(m,K) := \{A \in M(m \times m, K) \mid \det A \neq 0\}$ invertible matrix

$* =$ matrix multiplication , $1_G = I_{d_m}$

Orthonormal group

$O(m,K) := \{A \in GL(m,K) \mid A^T A = A A^T = I_{d_m}\}$ orthogonal matrix $A^T = A^{-1}$

distance preserving homomorphism

$\Rightarrow$ Rotations and reflections $\dim O(n) = \frac{n(n-1)}{2}$ (independent elements)

Special orthogonal group

$SO(m,K) := \{A \in O(m,K) \mid \det(A) = 1\}$

$\Rightarrow$ Rotations. $SO(2) \to$ around one point (no reflections)

$SO(3) \to$ " " line

- Lie group ($GL(n,\mathbb{R})$, $O(n,\mathbb{R})$, $SO(n,\mathbb{R})$ are Lie groups)
  - has a finite dimensional differential smooth manifold (*)
  - with a group structure such that $f: G \cdot G \to G$ and $g \to g^{-1}: G \to G$ } smooth maps

$G, H \in$ Lie group $\Rightarrow f: G \to H$ smooth isomorphism

- Lorentz group $\Rightarrow$ Lie group $O(n,\mathbb{R}) \to O(1,3,\mathbb{R})$

manifold : Minkowski space $-+++$ (signatu $\rightarrow$ (3-1)

$n=4 \Rightarrow \dim O(4) = \frac{n(n-1)}{2} = 6$ independent values

Ortocronous L.g. or proper L.g. $SO^{+}(1,3,\mathbb{R})$

$\rightarrow$ preserves orientation of time ($\rightarrow$ future)

- Matrix representation of a group

is given by the connection *

represent $g \in G$ as elements of a vector space $g \in V^n$

$f(g): V^n \to V^n$ $f(g) = Dg$

$D(g)D(h) = D(gh)$

as any mapping, it depends on the basis of V^n

(1.5) Rapitity parameter

irrelevant for now

connected (there is always one inbetween

x label of the

$g(x) \in G \quad x \in S$ $S=$ connected subset of $\mathbb{R}$ $\left(x < z < y \quad x, y \in \mathbb{R} \Rightarrow z \in \mathbb{R} \right)$

element of the group(!) not

$x=$ lables the group members

1 parameter group $\rightarrow$ (Lorentz transf: v) (!)

* connection of the group

Connection

- Composition product: $g(x), g(y) \in G, \; x, y \in S \Rightarrow g(x) \cdot g(y) := g(x \circ y) \in G$

(closure) $\underbrace{x \circ y}_{} = z \in S$

$\circ$ is generic, not $g \circ f(x) = g(f($

(a) Prove that it is a group

- Unity element $\Longrightarrow$ $g(x) \cdot 1_G = 1_G \cdot g(x) = g(x)$

depends on the neutral element of S

$$g(x) \cdot g(1_S) = g(x \circ 1_S) = g(x)$$

$$g(1_S) g(x) = g(1_S \circ x) = g(x) \quad \models\!\!\!> \quad 1_G = g(1_S)$$

- Associativity $\Longrightarrow$ $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$

because S is associative

$$(g(x) g(y)) g(z) = g(x \circ y) \cdot g(z) = g(x \circ y \circ z) = g(x) \cdot g(y \circ z) = g(x) \left(g(y \circ z) \right)$$

- $\exists$ of inverse $:\Longrightarrow$ $g(x) \cdot g(y) = 1_g$

$g(x)^{-1} = g(x^{-1})$

$$g(x) \cdot g(y) = g(x \circ y) \underset{\underset{y = x^{-1}}{\uparrow}}{=} g(x \circ x^{-1}) = g(1_S) \quad \forall x \in S \quad \models\!\!\!> \quad g(x)^{-1} = g(x^{-1})$$

(b) Condition on $\circ$ #1

strictly monotonic $\Rightarrow$ Bijective $\Longleftarrow$

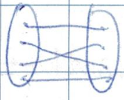
$x \circ y = f_x(y)$, smooth $\Rightarrow f'_x(1_S) \neq 0$ ① ? show: $f'_{x \circ y}(1_S) = f'_x(y) f'_y(1_S) \Rightarrow f'_x(y) \neq 0 \; \forall y \in S$

$$f_{x \circ y}(z) = x \circ (y \circ z) = f_x(y \circ z) = f_x[f_y(z)] \overset{①}{\Longrightarrow} f'_{x \circ y}(z) = f'_x[f_y(z)] \cdot f'_y(z)$$

$z = 1_S$

$$\underbrace{f'_{x \circ y}(1_S)}_{\neq 0 \text{ for } ①} = f'_x[f_y(1_S)] \cdot f'_y(1_S) = f'_x[y \circ 1_S] \cdot f'_y(1_S) = f'_x(y) \cdot \underbrace{f'_y(1_S)}_{\neq 0 \text{ for } 1} \quad \models\!\!\!> \quad f'_x(y) \neq 0 \;\; \forall y \in S$$

$\models\!\!\!>$ $f_x(y)$ strictly monotonic $\Rightarrow$ Bijective



$\downarrow$

f' or is always + or is always −

c) Set new lables such that $\lambda(x \circ y) := \lambda(x) + \lambda(y)$

$\uparrow$

we want additive connection between two lables

$$\lambda(x\circ y) = \lambda(f_x(y)) := \lambda(x) + \lambda(y) \quad \Big\downarrow \frac{\delta}{\delta y}$$

$$\lambda'(x\circ y) = \lambda'(f_x(y)) \cdot f_x'(y) = \lambda'(y) \quad \Big\downarrow y = 1_s$$

you can because $f_x'(y) \neq 0 \ \forall y \in$

$$\lambda'(\underbrace{f_x(1_s)}_{\underbrace{x \circ 1_s}_{x}}) \cdot f_x'(1_s) = \underbrace{\lambda'(1_s)}_{\substack{\text{const}=1 \\ \text{multiplicative} \\ \text{constant}}}$$

$$\Rightarrow \lambda'(x) = f_x'(1_s)^{-1} \quad \Rightarrow \int \lambda'(x)\,dx = \lambda(x) + \text{const}$$

$$\int \lambda'(x)\,dx = \underline{\underline{\lambda(x)}} + \text{const} = \int_{1_s}^{x} f_t'(1_s)^{-1}\,dt$$

Uniqueness : $X = \lambda(x)$ $Y = \lambda(y)$ are additive $\Rightarrow \mu(X+Y) = \mu(X) + \mu(Y)$

$\Rightarrow \mu$ ~~must be~~ is linear $\Rightarrow$ unique up to an additive constant (equivalent) !

d) Specify for Lorentz $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a^2 - b^2 = 1 \Rightarrow b = \sqrt{1-a^2} \in \mathbb{R} \Rightarrow a \leq 1$ det(M)=

$GL(2,\cdot)$ $\Rightarrow$ 1 parameter group $a, b \in \mathbb{R}$ $S = \mathbb{R}$

$\Rightarrow \cdot =$ matrix multiplication , $1_G = I_{d2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$g(b)\cdot g(b') = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ b' & a' \end{pmatrix} = \begin{pmatrix} aa' + bb' & ab' + ba' \\ ab' + ba' & bb' + aa' \end{pmatrix} = \begin{pmatrix} a'' & b'' \\ b'' & a'' \end{pmatrix} = g(b \circ b')$$

$$\Rightarrow b \circ b' = ab' + ba' = \sqrt{1+b^2}\cdot b' + b\sqrt{1+b'^2} \Rightarrow f_b'(b') = \sqrt{1+b^2} + 2b'\frac{1}{2}(1+b'^2)^{-1/2}$$

$$f_b'(1_s) = f_b'(0) = \sqrt{1+b^2}$$

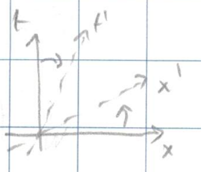
$\sinh^2(x) + \cosh^2(x) = 1$

$$\Rightarrow x = \lambda(b) = \int_0^b \frac{dt}{\sqrt{1+t^2}} = \sinh^{-1}(b) \Rightarrow \boxed{\begin{array}{c} b = \sinh(x) \\ \hline a = \cosh(x) \end{array}} \quad a^2 = 1 + \sinh^2(x) = \cosh^2(x$$

$$\Rightarrow M = \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}$$

$\rightarrow$ the sinh(x) cosh(x) do not rotate but "squeeze" (rotate the axes one toward the other

Rapidity parameter b !



(1.6) **Space-time** (from generic case by imposing physical requirements/conditions)

1• Homogeneity ⇒ linear transformation (no deformations ⇒ dogs in v. lines)

2• Group structure

3• Space inversion : what about time? or the common/proper group??

① ⇒ $\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} a(\varphi) & b(\varphi) \\ c(\varphi) & d(\varphi) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = M(\varphi) \begin{pmatrix} t \\ x \end{pmatrix}$ Linear with respect to φ

additive (Rapidity parameter)

Fully general

② (exercise 5)
(compat.) closure → $M(\varphi+\varphi') = M(\varphi)M(\varphi')$
inverse → $M(-\varphi) = M^{-1}(\varphi)$
identity → $M(0) = I_d$

a) Invert space coordinates ③ ⇒ $\hat{M}(\varphi) = \begin{pmatrix} a(\varphi) & -b(\varphi) \\ -c(\varphi) & d(\varphi) \end{pmatrix} \leftarrow \begin{pmatrix} t \\ -x \end{pmatrix} = \overset{\Lambda}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \begin{pmatrix} t \\ x \end{pmatrix} \Rightarrow \hat{M}(\varphi) = \Lambda^T M \Lambda$

③ ⇒ $\hat{M}(\varphi)$ isomorphic $M(\varphi)$ $f: G \to H,\ s: H \to G \Rightarrow f(s(h)) = h,\ s(f(g)) = g$

find the isomorphism $\hat{M} = M(\overset{\text{Bijective}}{\underbrace{\chi(\varphi)}_{\hat{\varphi}}})$ such and $\chi(\varphi) = K\varphi$ $\chi: G \to G$

$$M(\hat{\varphi}) = M(\chi(\overbrace{\varphi_1+\varphi_2}^{\varphi})) = \hat{M}(\varphi_1+\varphi_2) \overset{\text{closure}}{=} \hat{M}(\varphi_1)\cdot\hat{M}(\varphi_2) = M(\chi(\varphi_1))\cdot M(\chi(\varphi_2))$$
$$= M(\chi(\varphi_1)+\chi(\varphi_2))$$

⇒ $\chi(\varphi)$ is linear ⇒ $\chi(\varphi) = K\varphi$ ✓

invert axes twice ⇒ $\chi(\chi(\varphi)) = K(K\varphi) = K^2\varphi$ must be the same ⇒ $K^2 = 1$ ✓ ⇒ $K=1$, $K=-1$

b) Set $K=1$ ⇒ $\hat{M}(\varphi) = M(\varphi)$ ⇒ $b = c = 0$ ⇒ no mixing between x and t
↳ only coordinates rescaling

$$\begin{pmatrix} a(\varphi) & 0 \\ 0 & d(\varphi) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} at \\ dx \end{pmatrix}$$

(c) K=-1 ... $a(-\varphi)$ $b(-\varphi)$ ... $a(\varphi)+b(\varphi)$

isomorphism

ⓒ **Set $K=-1$** $\Rightarrow \hat{M}(\varphi)=M(-\varphi)$

$$\underbrace{\begin{pmatrix} a(-\varphi) & b(-\varphi) \\ c(-\varphi) & d(-\varphi) \end{pmatrix}}_{M(-\varphi)} = \underbrace{\begin{pmatrix} a(\varphi) & -b(\varphi) \\ -c(\varphi) & d(\varphi) \end{pmatrix}}_{\hat{M}(-\varphi)} \Rightarrow \begin{array}{l} \underset{M(-\varphi)}{} \quad \underset{\hat{M}(\varphi)}{} \\ a(-\varphi)=a(\varphi) \\ b(-\varphi)=-b(\varphi) \\ c(-\varphi)=-c(\varphi) \\ d(-\varphi)=d(\varphi) \end{array}$$

$$\Rightarrow \det M(-\varphi) = a(-\varphi)\,d(-\varphi) - b(-\varphi)\,c(-\varphi)$$

$$\det \hat{M}(\varphi) = \overset{\shortparallel}{a(\varphi)}\,\overset{\shortparallel}{d(\varphi)} - \overset{\shortparallel(-1)}{b(\varphi)}\,\overset{\shortparallel(-1)}{c(\varphi)}$$

$$\det M(\varphi) = \quad '' \quad\quad ''$$

$\det \hat{M}(\varphi)=\det M(\varphi)$ for the isomorphism

? $\det \hat{M}(\varphi)=\det M(\varphi)$

$$\Rightarrow \det M(-\varphi) \overset{M(\varphi)^{-1}}{=} (\det M(\varphi))^{-1} \Rightarrow \det M(\varphi)\cdot\det(M(\varphi)) = 1$$

$$\Rightarrow \det M(\varphi) = \pm 1 \Rightarrow$$

$\det M(\varphi)=1$

$\Rightarrow$ Kramer $c\,M^{-1}=\hat{M}$

$$\pm 1 \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

not -1 or $b=-b$ and $c=-c$

$$M^{-1}(\varphi)=M(-\varphi) \Rightarrow a=d$$

$$\Rightarrow \det M = a^2 - bc = 1 \qquad M(\varphi)=\begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

ⓓ **~~Use~~ Multiplication law** : from $M(\varphi)\cdot M(\varphi') = M(\varphi+\varphi')$

(group theory)

$$a(\varphi+\varphi') = a(\varphi)\,a(\varphi') + b(\varphi)\,c(\varphi')$$

$$a(\varphi'+\varphi) = a(\varphi')\,a(\varphi) + b(\varphi')\,c(\varphi)$$

commutativity

$\frac{c}{b}=C$

$$\frac{b(\varphi)}{c(\varphi)} = \frac{b(\varphi')}{c(\varphi')} \overset{\forall \varphi,\varphi'}{=} C \quad \Leftarrow \quad b(\varphi)\,c(\varphi') = b(\varphi')\,c(\varphi)$$

you can set $C=+/-1$ because it would be just a rescaling

ⓔ **Set $C=1$** $\Rightarrow M(\varphi)=\begin{pmatrix} a(\varphi) & b(\varphi) \\ b(\varphi) & a(\varphi) \end{pmatrix}$ → The Lorentz group

Set $C=1$ $\Rightarrow M(\varphi)=\begin{pmatrix} a & -b \\ +b & a \end{pmatrix}$ → Rotation group

$b=c=0$ $\Rightarrow M(\varphi)=\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ → Galilei group

Lie-groups

- Continuous groups $\{X(\alpha_1, \ldots, \alpha_m)\}$
- Elements are "functions" of α_j continuous parameters
- Have finite dimensional differentiable (i.e. smooth) manifold

e.g. collection of phase factors $e^{i\alpha}$, α = phase $U(1) = \{e^{i\alpha} ; \alpha \in \mathbb{R}\}$

" " rotations $R(\alpha)$ angle α (in a plane), $R(\alpha, \beta, \gamma)$ in 3D

" " boosts $\Lambda(\psi)$ rapidity ψ (in a plane)

Lie-algebra

- Define an element of the group with a set of "basis" : generators of the group M_k
- Elements of a Lie group X can be written as

$$X = \exp\left(i \sum \alpha_k M_k\right) \qquad \alpha_k \in \mathbb{R} \text{ parameters} \qquad k = 1, \ldots, r$$

$$= e^{i\alpha_1 M_1} \ldots e^{i\alpha_r M_r} \qquad M_k = -i \left(\frac{\delta X}{\delta \alpha_k}\right)_{\{\alpha_k = 0\}} \text{ generator}$$

$\uparrow$

X satisfies a Lie algebra if $[M_k, M_\ell] = i f_{k\ell m} M_m$ ($[a,b] \equiv ab - ba$ commutator)

$f_{k\ell m}$ = structure constant of the group (completely anti-symmetric on $k\ell m$)

it defines the connection between the elements X

- if set of generators $\{M_k\}$ $M_k \to M'_k = S M_k S'$ S $n \times n$ invertible matrix

- We have $X = e^Y \Rightarrow \det X = e^{\mathrm{Tr}\, Y}$

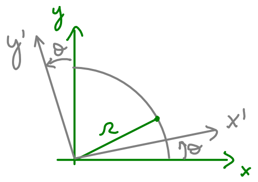
for $\mathrm{Tr}\, Y = 0 \Rightarrow \det X = 1$ you have conserved quantities

$\uparrow$

$\mathrm{Tr}\left(\sum_k \alpha_k M_k\right) = 0$

eg. $X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ Rotations

$\det(X) = \cos^2\theta + \sin^2\theta = 1$



$r = r'$

eg. Lorentz boosts Λ : $ds^2 = ds'^2$

The Lorentz group is a Lie group

- Here, consider rotation in the x^0, x^1 plane $\Rightarrow$ only 1 parameter $\alpha = \psi$

$$X = \exp\left(i \sum \alpha_K \mathcal{M}_K\right) \qquad K=1 \quad \alpha_1 = \psi \quad \mathcal{M}_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\sigma^{(3)}$$

as generators we can use a Pauli matrix

$$\Lambda = \exp\left(\psi\sigma^{(3)}\right)$$

Taylor expansion

$$= \sum_n \frac{1}{n!}\left(\psi\sigma^{(3)}\right)^n$$

$$\hookrightarrow \left(\sigma^{(3)}\right)^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{(0)}$$

$$\left(\sigma^{(3)}\right)^1 = \sigma^{(3)}$$

$$\left(\sigma^{(3)}\right)^2 = \sigma^{(3)}\sigma^{(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{(0)}$$

$$\left(\sigma^{(3)}\right)^3 = \sigma^{(0)}\sigma^{(3)} = \sigma^{(3)}$$

......

$\Rightarrow$

$n=0$ even : $\sigma^{(0)}$

$n=1$ odd : $\sigma^{(3)}$

$n=2$ even : $\sigma^{(0)}$

$n=3$ odd : $\sigma^{(3)}$

$$= \sigma^{(0)}\sum_n \frac{1}{(2n)!}\psi^{2n} + \sigma^{(3)}\sum_n \frac{1}{(2n+1)!}\psi^{2n+1}$$

Taylor expansion of sinh, cosh

$$= \sigma^{(0)}\cosh(\psi) + \sigma^{(3)}\sinh(\psi)$$

$$= \begin{pmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{pmatrix}$$

for $x^{\mu'} \to x^{\mu}$; inverse transf. $x^{\mu} \to x^{\mu'}$ $\Rightarrow$ $\psi \to -\psi$ $\quad \Lambda = \begin{pmatrix} \cosh(\psi) & -\sinh(\psi) \\ -\sinh(\psi) & \cosh(\psi) \end{pmatrix}$

- we could have been using $\sigma^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as well (i.e. there are different possibilities for $\mathcal{M}_K$)

Invariants under boosts (Lorentz invariance)

- $\Lambda = e^{(\psi\sigma^{(3)})}$: $\det\Lambda = e^{\mathrm{Tr}(\psi\sigma^{(3)})} = e^0 = 1 \Rightarrow \boxed{\det\Lambda = 1}$

because $\sigma^{(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ It provides invariant quantities $ds^2 = ds'^2$!

$\Rightarrow$ This is a property of the group rather than of sinh, cosh

$\det(\Lambda) = \cosh^2\psi - \sinh^2\psi = 1$ as we have seen before

Combination of boosts in one plane

- here the structure constant of the group $f = \varepsilon$ Levi-Civita symbol

$$\Lambda(\phi)\Lambda(\psi) = \exp(\phi\sigma^{(3)})\exp(\psi\sigma^{(3)}) = \exp[(\phi+\psi)\sigma^{(3)}] = \Lambda(\phi+\psi) \quad (1)\ (2)$$

$$= \Lambda(\psi+\phi) = \Lambda(\psi)\Lambda(\phi) \quad (3)$$

$\Rightarrow$

(1) Two boosts in the same plane is given by one boost with $\psi + \phi = \psi_{tot}$ (additive in ψ not v!)
(as in a Galilean transformation with velocities $v_1 + v_2 = v_{Tot}$)

(2) $\exists$ of inverse Λ^{-1}: $\Lambda(\psi)\Lambda(-\psi) = \Lambda(0) = I$ $\quad \Lambda(\psi)^{-1} = \Lambda(-\psi)$

(3) In one plane, boosts Λ are commutative $\quad [\Lambda(\phi), \Lambda(\psi)] = 0$

Λ is orthogonal with respect to η not δ!

Conbination of boosts in different planes

- A, B basis generators elements generating boosts around $2 \neq$ axis

1° apply A then apply B, i.e. $\exp(A)\exp(B) \neq \exp(A+B)$ because

→ Final boost given by Baker-Hausdorf-Campbell formula

$$\exp(A)\exp(B) = \exp(A+B)\exp\left(-\tfrac{1}{2}[A,B]\right)$$

$[A,B] \equiv AB - BA \neq 0$ As from the f of the group these generators do not commute

Rotation group $O(3,K)$

$K \to K$
$I \to L$

$$\boxed{R_{\bar{m}}(\delta\phi) = I_{d_m} + \delta\phi\, i\, \bar{m}\bar{J} + O((\delta\phi)^2)}$$

infinitesimal rotation around $\bar{m}$ (unity vec

$\bar{J}$ = infinitesimal generators

$$\boxed{J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \Rightarrow (J_k)_{em} = -i\,\varepsilon_{kem} \quad \text{Generators}$$

↳ Rotations around the 3 axes

$$[J_k, J_e] = i\,\varepsilon_{kem} J_m$$ algebra for (e.g.) angular momentum $\hbar \bar{J}$

$= \varepsilon_{ke1}J_1 + \varepsilon_{ke2}J_2 + \varepsilon_{ke3}J_3$

→ $[J_k, J_e] = J_k J_e - J_e J_k$

$$\Rightarrow R(\phi) = \exp(i\phi\bar{m}\bar{J}) = \sum_{k=0}^{\infty} \frac{1}{k!}(i\phi\bar{m}\bar{J})^k = I_d + i\phi\bar{m}\bar{J} - \frac{(\phi\bar{m}\bar{J})^2}{2} + i\frac{1}{3}\big(\overset{\phi\bar{m}\bar{J}}{\phi\bar{m}\bar{J}}\big)^3 \ldots \quad ?$$

$$= I_d + i\sin(\theta)\,\bar{m}\bar{J} + (\cos\phi - 1)(\bar{m}\cdot\bar{J})^2$$

- You could transform J_1, J_2, J_3 as $J_i^S = S J_i S^{-1}$ to have spherical coordinates like in quantum mechanics $J_1^S = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $J_2^S = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $J_3^S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Lorentz group

$$R_{\bar{e}}(\delta\chi) = I_d + \delta\chi\, i\,\bar{e}\bar{K} + O((\delta\chi)^2)$$

Has the generators of the spatial rotations $\begin{pmatrix} 0 & 0\,0\,0 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & \boxed{J_k} \end{pmatrix}$

mixed t,x terms →

$$K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$ Generators for the boost

↑ pure space terms (J_i)

$$[J_k, J_e] = i\,\varepsilon_{kem}J_m \quad [J_k, K_e] = i\,\varepsilon_{kem}K_m \quad [K_k, K_e] = -i\,\varepsilon_{kem}J_m$$ Lie algebra for the Lorentz group

$$\Rightarrow \boxed{L(\chi) = \exp(i\chi\bar{e}\bar{K}) = I_d + i\sinh(\chi)\,\bar{e}\bar{K} + (\cosh(\chi) - 1)(\bar{e}\bar{K})^2}$$ Pure boost along $\bar{e}$

$$\cosh(\chi) = \gamma^{-1} = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

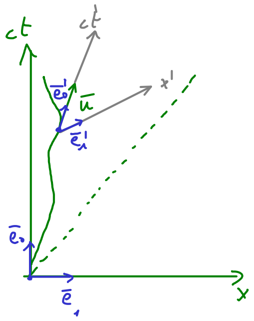
Relativistic mechanics

- Behavejour of paticles in the space-time

- 4-vector : $(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$ not invariant

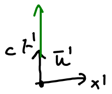
- World-line : $\boxed{x^\mu(\tau)}$ trajectory of particle in space-time τ = proper time

- 4-velocity : $\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt}}$ $(u^\mu) = \gamma\left(c\frac{dt}{dt}, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right) = \gamma(c, \bar{v}) = c\gamma(1, \bar{\beta})$



- note at which coordinates pass by an observer
- not invarient
- $\langle \bar{u}, \bar{u} \rangle = u_\mu u^\mu = \eta_{\mu\nu} u^\nu u^\mu = c^2\gamma^2 \overbrace{(-1+\bar{\beta}^2)}^{-\gamma^{-2}} = -c^2 < 0$ invariant

  (this is because of ds^2: $-c^2 d\tau = dx_\mu dx^\mu \Rightarrow -c^2 = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = u_\mu u^\mu$)
- time-like $\Rightarrow$ particles can only move inside the light cone
- $\bar{u}$ is tangent to the world line because it is a multiple of $d\bar{x}$: $\bar{u} = \frac{d\bar{x}}{d\tau}$ ← scalar
- In particle rest frame $\bar{u}' = (c, 0, 0, 0)^T \Rightarrow \bar{u}' = c\,\bar{e}_0$



- 4-acceleration : $\boxed{a^\mu \equiv \frac{du^\mu}{d\tau}}$ $(a^\mu) = \frac{d}{d\tau}[\gamma(c, \bar{v})] = \gamma\frac{d}{dt}[\overset{\gamma(v)}{\gamma}(c, \bar{v})] = \gamma(\dot{\gamma}c, \dot{\gamma}\bar{v} + \gamma\dot{\bar{v}})$

- note: $\frac{d}{d\tau}(\underbrace{u^\mu u_\mu}_{-c^2}) = \frac{du^\mu}{d\tau}u_\mu + u^\mu\frac{du_\mu}{d\tau} = \frac{du^\mu}{d\tau}u_\mu + u_\mu\frac{du^\mu}{d\tau} = 2u_\mu a^\mu = 0$

  $\Rightarrow$ $\bar{u} \perp \bar{a}$ always! (orthogonal in 4-D)
- In istantaneous rest frame : $\gamma = 1$ $\dot{\gamma} = 0$ $(a^\mu) = (0, \bar{a})$ ⇒

- 4-momentum : $\boxed{p^\mu \equiv m_0 u^\mu}$

$(p^\mu) = \gamma(\underset{(2)}{m_0 c}, \underset{(1)}{m_0\bar{v}})$

(1) $\bar{p} = \gamma m_0 \bar{v}$

(2) $\gamma m_0 c = m_0 c(1-\bar{\beta}^2)^{-1/2} \underset{\beta \ll 1}{\simeq} m_0 c(1 + \frac{1}{2}\bar{\beta}^2 + \dots) = m_0 c + \frac{1}{2} m_0 \cancel{c} \frac{\bar{v}^2}{c^{\cancel{2}}} = \frac{1}{c}(m_0 c^2 + \frac{1}{2}m_0\bar{v}^2) = \frac{E}{c}$

rest energy ⇐ kinetic energy

$\Rightarrow (p^\mu) = \gamma(\frac{E_0}{c}, m_0\bar{v}) = (\frac{E}{c}, \bar{p})$ ← Energy-momentum (one entity!)

- $\langle P, P \rangle = -\frac{E^2}{c^2} + \bar{p}^2$

  $\langle P, P \rangle = p_\mu p^\mu = m_0^2 u_\mu u^\mu = \underset{\text{invariant.}}{-m_0^2 c^2}$

  $\Bigg\}$ $-\frac{E^2}{c^2} + \bar{p}^2 = -m_0^2c^2$ $\boxed{E^2 = m_0^2c^4 + c^2\bar{p}^2}$ at rest $E = m_0c^2$

  m_0 = rest mass

- $\bar{p} = \gamma m_0 \bar{v}$ $\frac{E}{c} = \gamma m_0 c$ ↳ $\boxed{\bar{p} = \frac{E}{c^2}\bar{v}}$ ↲

Energy and momentum are components of the same entity

$\Rightarrow$ energy-momentum conservation!

Momentum of a photon

moves along null lines $\Rightarrow p_\mu p^\mu = 0$ $\bar{p} = (p^0, p^1, 0, 0)$ $\Rightarrow$ $p^0 = p^1$ i.e. $p = E/c$!

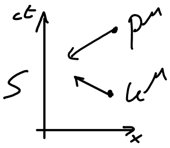
$\frac{p^1}{p_0} = 1$ alternative way to state $v = c$

Massive particles have 0<v<c

$$p_\mu p^\mu = -p^{0^2} + p^{1^2} = -m_0^2 c^2 \Rightarrow p^1 = (p^0 - m_0^2 c^2)^{1/2} \quad \frac{p^1}{p^0} = \left(1 - \frac{m_0^2 c^2}{p^{0^2}}\right)^{1/2} \to 1 \text{ (but not 1!) for } p^0 = \frac{E}{c} \to \infty$$

$\Rightarrow$ No matter how much energy we give to the particle, $v < c$!

A convenient expression: energy of a particle measured by a moving observer



p^μ of a particle in frame S (eg. a cosmic ray)

u^μ 4-velocity of an observer moving in S

$\Rightarrow \boxed{E' = -p_\mu u^\mu}$

$S : u = \gamma(c, \bar{v})^T$ $p = (\frac{E}{c}, \bar{p})^T$ (lab)

$S' : u = (c, \bar{0})^T$ $E' = ?$ (moving observer)

$$p_\mu u^\mu = p'_\mu u'^\mu = \left(\frac{E'}{c}, \bar{p}'\right)\begin{pmatrix} c \\ \bar{0} \end{pmatrix} = -\frac{E'}{\cancel{c}} \cdot \cancel{c} + \cancel{p'0_i} = -E' \quad ✓$$

scalarproduct is the same in all frames

– 4-force :

$$\mathfrak{f}^\mu \equiv \frac{dp^\mu}{d\tau} \qquad (\mathfrak{f}^\mu) = \gamma \frac{d}{dt}\left(\frac{E}{c}, \bar{p}\right) = \gamma\left(\frac{\bar{f}\bar{v}}{c}, \bar{f}\right)$$

① ②

① $dE = \bar{f}\, d\bar{x} = \bar{f}\, \bar{v}\, dt \Rightarrow \dot{E} = \bar{f}\bar{v}$ Work

② $\dot{\bar{p}} = \bar{f}$ 3-force

remember : a force is a change in momentum not $\bar{f} = m\bar{a}$

$$\bar{f} = \frac{d\bar{p}}{dt} = \frac{d}{dt}(m\bar{v}) = \underset{\uparrow \; m = \text{const.}}{m\dot{\bar{v}}}$$

Change in rest mass ?!

product rule (let m free)

$$u^\mu \mathfrak{f}_\mu = u^\mu \frac{dp_\mu}{d\tau} = u^\mu \frac{d}{d\tau}(m u_\mu) = u^\mu\left(\frac{dm}{d\tau} u_\mu + m \frac{du_\mu}{d\tau}\right) = \underbrace{u^\mu u_\mu}_{-c^2} \frac{dm}{d\tau} + m \underbrace{u^\mu a_\mu}_{=0} = -c^2 \frac{dm}{d\tau}$$

$\Rightarrow$ A force may change rest mass $\boxed{\frac{dm}{d\tau} = -\frac{u^\mu \mathfrak{f}_\mu}{c^2}}$ if $u^\mu \mathfrak{f}_\mu \neq 0$!

- If $u^\mu \mathfrak{f}_\mu = 0 \Rightarrow \frac{dm}{d\tau} = 0$ $\mathfrak{f}^\mu$ called pure force $\gamma\left(\frac{\bar{f}\bar{v}}{c}, \bar{f}\right)$

↓

acts on 3-acceleration
but not on rest energy

Variational approach

- Free particle : no external forces

- Action : $S \in \mathbb{R}$ must be Lorentz invariant (not dependent on motion of observer)

  $\Rightarrow$ S must depend on Lorentz scalar: τ is the only one characterizing the particle

$$S = \int L\,dt = \alpha \int d\overset{\swarrow}{\tau} = \alpha \int \gamma^{-1} dt$$

non-relat. limit $\beta << 1$ no external forces

$$\text{Find } \alpha: \quad L = \alpha\gamma^{-1} = \alpha(1-\bar\beta^2)^{1/2} = \alpha(1-\tfrac{1}{2}\bar\beta^2+\dots) \simeq \alpha - \frac{1}{2}\alpha\frac{\bar v^2}{c^2} \overset{!}{=} T + \overset{\downarrow}{\cancel{V}} = \frac{1}{2}m\bar v^2 \Rightarrow \underline{\alpha = -m_0c^2}$$

$$\boxed{L = -m_0c^2\gamma^{-1} = -m_0c^2(1-\bar\beta^2)^{1/2}}$$ ← Lagrangian of a free particle

· Note: S has a clear physical meaning in S.R. : interval of proper time along world line

$L \rightarrow L' = aL + b$ gives same eq. of motion ($\delta S = 0$) a = time rescaling, b = time shift

Non-relativistic limit to find α:

- The metric is hidden in there!

$$-c^2d\tau^2 = ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \Rightarrow d\tau = \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \qquad S = -m_0c\int\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$$

Very relevant in G.R.! $\eta \rightarrow g$

- Conjugate momentum :

$$\bar p = \frac{\delta \overset{\swarrow}{L}}{\delta \bar v} = \cancel{-}m_0\cancel{c^2}\frac{1}{\cancel{2}}(1-\bar\beta^2)^{-1/2}\cancel{(-)}\frac{1}{\cancel{c^2}}\cancel{2}\bar v = \gamma m_0\bar v \qquad \boxed{\bar p = \gamma m_0 \bar v}$$ as seen before

- Hamiltonian :

$$H = \bar v\overset{\swarrow}{\bar p} - \overset{\swarrow}{L} = m_0\gamma\bar v^2 + m_0c^2\gamma^{-1} = \gamma m_0c^2(\bar\beta^2 + \overset{(1-\bar\beta^2)}{\overset{\downarrow}{\gamma^{-2}}}) = \gamma m_0c^2 \qquad \boxed{E = \gamma m_0c^2}$$

This confirms our interpretation of the 4-momentum components $(p^\mu) = (\frac{E}{c}, \bar p)$

- Equation of motion : (Euler-Lagrange eq.)

$$\frac{d}{dt}\frac{\delta \overset{\swarrow}{L}}{\delta \bar v} - \frac{\delta \overset{\swarrow}{L}}{\delta \bar x} = \frac{d}{dt}(\overset{\gamma(v)}{\overset{\downarrow}{\gamma}} m_0\bar v) = m_0\left[\dot\gamma\bar v + \gamma\dot{\bar v}\right]$$

$$\dot\gamma = \cancel{-}\frac{1}{\cancel{2}}(1-\bar\beta^2)^{-3/2}(-\cancel{2})\bar\beta\dot{\bar\beta} = \gamma^3\frac{\bar v\dot{\bar v}}{c^2}$$

$$= m_0\left(\gamma^3\frac{\bar v\dot{\bar v}}{c^2}\cdot\bar v + \gamma\dot{\bar v}\right)$$

$$= m_0\dot{\bar v}\underbrace{\left(\gamma^3\frac{\bar v^2}{c^2} + \gamma\right)}_{>0} \overset{!}{=} 0 \Rightarrow \boxed{\dot{\bar v} = 0}$$ no acceleration as expected

↳ $\bar x = \bar a t + \bar b$ move along straight line

(instantaneous rest frame $\bar v = 0$) ↗

– Eq. of motion but directly from least action principle

$$S = -mc^2\int d\tau = -mc^{\cancel{2}}\int \frac{1}{\cancel{c}}\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} d\tau = -mc\int\left(-\eta_{\mu\nu}u^\mu u^\nu\right)^{1/2} d\tau$$

$$\delta S = \cancel{-}m\cancel{c}\int_A^B \frac{1}{\cancel{2}}\overbrace{\cancel{(-u_\nu u^\nu)^{-1/2}}}^{c^{-1}}(-)\cancel{2}u_\nu \delta u^\nu d\tau = m\int_A^B u_\nu \delta\overset{\leftrightarrow}{\frac{dx^\nu}{d\tau}} d\tau = m\int_A^B u_\nu \frac{d}{d\tau}\delta x^\nu d\tau \quad \leftarrow \text{by part}$$

$$= m\left[\underset{=0}{u_\nu\delta x^\nu}\Big|_A^B - \int_A^B \frac{du_\nu}{d\tau}\delta x^\nu d\tau\right] \overset{!}{=} 0 \quad \forall\, \delta x^\nu \Rightarrow \text{4-velocity} \quad \boxed{a^\nu = 0}$$

$$\Rightarrow \boxed{\frac{d^2x^\mu}{d\tau^2} = 0 \quad \rightarrow \quad x^\mu = \alpha^\mu\tau + \beta^\mu} \Rightarrow \begin{cases} \text{no acceleration, motion along a straight line} \\ \text{no change in particle' energy} \quad m\frac{du^0}{d\tau} = \frac{dp^0}{d\tau} = 0 \end{cases}$$

note: here $\eta_{\mu\nu}$ = const !
not the case in G.R ! You will see how important this is!

– Dispersion relation for massive particles

group and phase velocities can not be the same for massive particles

$$v_{ph} = \frac{H}{p} \neq \frac{dH}{dp} = v_{gr} \quad \text{but} \quad \boxed{v_{ph}\, v_{gr}} = \frac{H}{p}\frac{dH}{dp} = \frac{cp}{\sqrt{1+p^2}}\,\frac{c\sqrt{1+p^2}}{p} = \boxed{c^2}$$

i.e. geometric mean of v_{ph} and v_{gr} is c $\Rightarrow$ $v_{ph} > c$ if $v_{gr} < c$

$$\frac{H}{p}\frac{dH}{dp} = \frac{d(H^2)}{d(p^2)} = c^2 \qquad \text{integrate} \qquad d(H^2) = c^2 d(p^2) \Rightarrow H^2 = c^2p^2 + \underline{\underline{\text{const}}}$$

↑
rest mass as
integration constant.

Decay of particles

Example: body of mass M splits in two parts of mass m_1, m_2

$\Rightarrow M = m_1 + m_2$? NO!

- In rest frame of body : $E = Mc^2$
- Energy conservation : $Mc^2 = E_1 + E_2$ (1) split $\Rightarrow$ m_1, m_2 move apart $\Rightarrow |\bar{p}_i| > 0$

$$\Downarrow \qquad \leftarrow \quad \Rightarrow E_i^2 = m_i^2 c^4 + c^2 \bar{p}_i^2 > m_i^2 c^4$$

$M > m_1 + m_2$!

Spontaneous decay is possible

- if $M < m_1 + m_2$ $\Rightarrow$ decay is not possible (system is stable)

$\Rightarrow$ no spontaneous decay (or energy conservation would be violated)

$\Rightarrow$ to have decay, you need to give energy to the body of an amount $\Delta E = m_1 + m_2 - M$

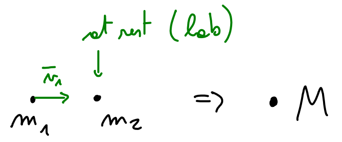
- Momentum conservation : $0 = \bar{p}_1 + \bar{p}_2$ $\Rightarrow$ $\bar{p}_1^2 = \bar{p}_2^2$ $\quad c^2 p_i^2 = E_i^2 - m_i^2 c^4$

$\Downarrow$ energies of the components E_1, E_2

$$E_1^2 - m_1^2 c^4 = E_2^2 - m_2^2 c^4 \quad \leftarrow \text{plug (1)}$$

$$\Rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 \qquad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2$$

Inverse process



- $E = E_1 + E_2 = E_1 + m_2 c^2$
- $\bar{p} = \bar{p}_1 + \bar{p}_2 = \bar{p}_1$

$\Rightarrow$ Find velocity: $\bar{p} = \frac{E \bar{v}}{c^2} \Rightarrow \bar{v} = \frac{c^2 \bar{p}}{E} = \frac{c^2 \bar{p}_1}{E_1 + m_2 c^2}$ velocity of the merged body in the lab frame

$\Rightarrow$ Find mass M : $E^2 = M^2 c^4 + c^2 \bar{p}^2$

$$c^4 M^2 = E^2 - c^2 \bar{p}^2$$

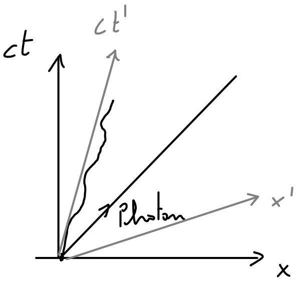
$$= (E_1 + m_2 c^2)^2 - (E_1^2 - m_1^2 c^4)$$

$$= \cancel{E_1^2} + m_2^2 c^4 + 2E_1 m_2 c^2 - \cancel{E_1^2} + m_1^2 c^4$$

$$= c^4 \left(m_1^2 + m_2^2 + \frac{2E_1 m_1}{c^2} \right) \quad \Rightarrow \quad \boxed{M^2 = m_1^2 + m_2^2 + \frac{2E_1 m_1}{c^2}}$$

Photons

Photons travel with the speed of light



$dx = c\,dt \quad 0 = -c^2 dt^2 + dx^2 \Rightarrow ds^2 = 0$

null 4-interval, null geodesic $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$

moves along null lines $\Rightarrow p_\mu p^\mu = 0 \quad \bar{p} = (p^0, p^1, 0, 0) \Rightarrow \frac{p^1}{p} = 1$ alternative way to state $v = c$

$E^2 = m^2 c^4 + c^2 p^2$ $\uparrow = 0$

$\Rightarrow$ must be massless! $\left(p_\mu p^\mu = -\frac{E^2}{c^2} + \bar{p}^2 \overset{!}{=} 0 \ : \ E = cp \right) \Rightarrow \boxed{m = 0\,!}$

For massive particles the proper time is not identically zero: $ds^2 < 0$ (i.e. $d\tau^2 \neq 0$!)

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau^2 = \eta_{\mu\nu} u^\mu u^\nu d\tau^2 < 0}$$ (interval along particle world-line $\bar{x}(\tau)$)

direction of motion is given by u^μ
shift can be parameterized by τ $\Big\}$ $dx^\mu = u^\mu d\tau$ (shift)

$\Rightarrow$ we can use $p^\mu = m u^\mu$

- **For photons** $ds^2 = 0 \quad -c^2 d\tau^2 = \overset{\downarrow}{ds^2} = 0$ proper time on the light-cone is identically zero!

  $\Rightarrow u^\mu = \frac{dx^\mu}{d\tau}$ 4-velocity is not well defined!

we must use another affine parameter (not τ)

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda^2 = \eta_{\mu\nu} k^\mu k^\nu d\lambda^2}$$

$k^\mu \equiv \frac{dx^\mu}{d\lambda}$ wave vector (sort of 4-velocity) $\quad dx^\mu = k^\mu d\lambda$ (shift)

$k_\mu k^\mu = 0$ null-type: along the light-cone

$$k_\mu k^\mu = -(k^0)^2 + \underbrace{(k^1)^2 + (k^2)^2 + (k^3)^2} = -(k^0)^2 + |\bar{k}|^2 \overset{!}{=} 0 \Rightarrow k^0 = \|\bar{k}\| \quad \boxed{(k^\mu) = k^0(1, \bar{e})}$$

- We can use k^μ to define the momentum of photons

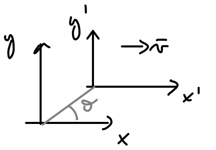
$(p^\mu) = \left(\frac{E}{c}, \bar{p}\right)$ as for any particle, photons have a E and a $\bar{p}$

1) null vector (massless) $E = cp$
2) energy is quantized $E = h\nu = \hbar\omega$ $\Big\}$ $\boxed{p = \frac{h\nu}{c} = \frac{\hbar\omega}{c} = \frac{h}{\lambda}}$ $\quad \omega = \frac{\nu}{2\pi} \quad \lambda = \frac{c}{\nu}$

$$(p^\mu) = \left(\frac{E}{c}, p\bar{e}\right) = \frac{E}{c}(1, \bar{e}) = \frac{\hbar\omega}{c}(1, \bar{e}) = \hbar(k^\mu) \Rightarrow \boxed{(k^\mu) = \frac{\omega}{c}(1, \bar{e}) = \frac{2\pi}{\lambda}(1, \bar{e})} = \left(\frac{2\pi}{\lambda}, \bar{k}\right)$$

"4-frequency"

- Lorentz transform of 4-frequency :



- $(k^\mu) = \left(\frac{2\pi}{\lambda}, \vec{k}\right) = \frac{2\pi}{\lambda}(1, \cos\vartheta, \sin\vartheta, 0)$
- $(k^{\mu'}) = \left(\frac{2\pi}{\lambda'}, \vec{k}'\right) = \frac{2\pi}{\lambda'}(1, \cos\vartheta', \sin\vartheta', 0)$

$k^{\mu'} = \Lambda^{\mu'}{}_{\alpha} k^\alpha$ $(\mu=0)$ time: $\frac{\lambda}{\lambda'} = \gamma(1-\beta\cos\vartheta)$ ← Doppler effect

$(\mu=i)$ space: $\tan\vartheta' = \frac{\tan\vartheta}{\gamma(1-\beta\sec\vartheta)}$ ← Aberration of light

Example:

$\mu=0$ $$K^0 = \Lambda^0{}_\alpha K^\alpha = \Lambda^0{}_0 K^0 + \Lambda^0{}_1 K^1 + \Lambda^0{}_2 K^2 + \Lambda^0{}_3 K^3$$

$$= \gamma K^0 - \beta\gamma K^1 + 0 + 0 = \frac{2\pi}{\lambda}(\gamma - \beta\gamma\cos\vartheta) \qquad \frac{2\pi}{\lambda'} = \frac{2\pi}{\lambda}\gamma(1-\beta\cos\vartheta)$$

Equation of motion of a particle in gravitational field

- Let us now investigate the motion of a particle in a weak and slowly evolving

1) Lagrangian of particle in gravitational field

- This is the only Lagrangian leading to Newtonian gravity
- Free particle with a perturbation

the two conditions are sort of connected: weak field = small acceleration. ⇒ low velocities

$$L = -mc\gamma^{-1} \cdot \left(1 + \frac{\varphi}{c^2}\right) = -mc^2\left(1-\bar{\beta}^2\right)^{1/2}\left(1+\frac{\varphi}{c^2}\right)$$

$$\simeq -mc^2\left(1-\frac{1}{2}\frac{\bar{v}^2}{c^2}\right)\left(1+\frac{\varphi}{c^2}\right)$$ free particle + correction $\phi \equiv \varphi/c^2$

is the only Lagrangian that gives the Newtonian grav. eq. of motion

$\varphi \ll c^2$ weak field

$\beta \ll c$ non relat. particle

$$= -mc^2 + \frac{1}{2}m\bar{v}^2 - m\cancel{c^2}\frac{\varphi}{\cancel{c^2}} + m\cancel{c^2}\frac{1}{2}\frac{\bar{v}^2}{c^2}\frac{\varphi}{\cancel{c^2}}$$

$$\simeq -mc^2 + \frac{1}{2}m\bar{v}^2 - m\varphi$$

ok, compatible with Newtonian gravity

- Note: flat space-time is implicit here! $\eta_{\mu\nu}$!

2) Euler-Lagrange equation for the eq. of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0 \qquad L = -mc^2\left(1-\bar{\beta}^2\right)^{1/2}(1+\phi) \qquad \phi \equiv \frac{\varphi}{c^2}$$ adimensional (no approx $v \ll c$, $\phi \ll 1$!)

$$\partial_\mu L = -mc^2\left(1-\bar{\beta}^2\right)^{1/2}\left(1+\partial_\mu\phi\right) \overset{\beta \ll 1}{\simeq} -mc^2\,\partial_\mu\phi \qquad \left(1-\bar{\beta}^2\right)^{-1/2}$$

$$\frac{\partial L}{\partial \dot{x}^\mu} \overset{\dot{x}^\mu = \beta^\mu}{=} \frac{1}{c}\frac{\partial L}{\partial \beta^\mu} = \cancel{+}\frac{mc^{\cancel{2}}}{\cancel{c}}\frac{1}{\cancel{2}}\left(1-\bar{\beta}^2\right)^{-1/2}\left(\cancel{+2}\beta^\mu\right)(1+\phi) = \frac{mc\beta^\mu}{\sqrt{1-\bar{\beta}^2}}(1+\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = mc(1+\phi)\left[\frac{\dot{\beta}^\mu}{\sqrt{1-\bar{\beta}^2}} - \frac{\beta^\mu}{\cancel{2}\left(1-\bar{\beta}^2\right)^{3/2}}\left(-\cancel{2}\beta^\mu\ \dot{\beta}\right)\right] + \frac{mc\beta^\mu}{\sqrt{1-\bar{\beta}^2}}\dot{\phi} \simeq mc(1+\phi)\dot{\beta}^\mu\left(1+\frac{1}{2}\beta^\mu\right)$$

$\bar{\beta} \ll 1$ $\quad \ll 1$

$$= m(1+\phi)\ddot{x}^\mu$$

set $\dot{\phi}=0$ → static potential (ok for Sun)

⇒ The E-L. equation become:

$$\cancel{m}(1+\phi)\ddot{x}^\mu + \cancel{m}c^2\partial_\mu\phi = 0$$

$$\ddot{x}^\mu = -c^2(1+\phi)^{-1}\partial_\mu\phi \simeq -c^2(1-\phi)\partial_\mu\phi = -c^2\left(\partial_\mu\phi - \phi\partial_\mu\phi\right) = c^2\partial_\mu\left(\phi - \frac{\phi^2}{2}\right)$$

$$\Rightarrow \boxed{\ddot{\bar{x}} = -c^2\bar{\nabla}\left(\phi - \frac{\phi^2}{2}\right)} = -c^2\bar{\nabla}V$$

$\neq \ddot{x}^\mu = -\partial_\mu\varphi \quad \left(\phi = \frac{\varphi}{c^2}\right)$

additional term $\frac{\phi^2}{2}$!

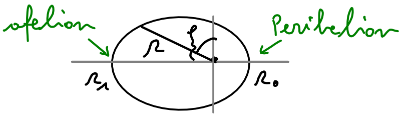
$V = V_0 + \delta V \qquad \delta V \ll V_0$

Newtonian $\quad V_0 = mc^2\phi = m\varphi = \frac{GmM}{r}$

Perturbation (quadratic) $\quad \delta V = mc^2\frac{\phi^2}{2} = \frac{m}{2}\left(\frac{GM}{r}\right)^2$ ⊛₁

3) Perihelion shift

- Kepler's orbit :



$L = GMm^2 a(1-e^2)$ orbital angular momentum $\circledast_2$

$\xi =$ polar angle

$V_L =$ effective potential energy

$$\frac{dr}{d\xi} = \frac{mr^2}{L}\sqrt{\frac{2}{m}(E-V_L)} \qquad V_L = V + \frac{L^2}{2mr^2}$$

$$= \frac{mr^2}{L}\left[\frac{2}{m}\left(E - V - \frac{L^2}{2mr^2}\right)\right]^{1/2}$$

$$= \frac{r^2}{L}\Big[\underbrace{2m(E-V) - \frac{L^2}{r^2}}_{\equiv A}\Big]^{1/2}$$

$$= \frac{r^2}{L}\sqrt{A}$$

- Derive perihelion shift by integrating $d\xi$

Newtonian ↓ Perturbation ↙

$$V = V_0 + \delta V \qquad \delta V \ll V_0$$

$$\Delta\xi = 2\int_{r_0}^{r_1}\frac{d\xi}{dr}\,dr \overset{\star}{=} -2\frac{\delta}{\delta L}\int_{r_0}^{r_1} dr\left[2m(E-\overset{\downarrow}{V}) - \frac{L^2}{r^2}\right]^{1/2}$$

$$= -2\frac{\delta}{\delta L}\int dr\left[2m(E-V_0) - 2m\delta V - \frac{L^2}{r^2}\right]^{1/2} \qquad \leftarrow A_0 \equiv 2m(E-V_0) - \frac{L^2}{r^2}$$

$$= -2\frac{\delta}{\delta L}\int dr\, A_0^{1/2}\left[1 - \frac{2m\delta V}{A_0}\right]^{1/2}$$

$$\simeq \underbrace{-2\frac{\delta}{\delta L}\int dr\, A_0^{1/2}\Big[1}_{=0 \text{ in Newton's gravity}} - \frac{\cancel{2}m\delta V}{\cancel{2}A_0}\Big] = +2\frac{\delta}{\delta L}\int_{r_0}^{r_1}\overset{\swarrow}{dr}\,\cancel{A_0^{1/2}}\frac{m\delta V}{A_0^{1/2}} \qquad \overset{\leftarrow}{\frac{dr}{d\xi} \simeq \frac{r^2}{L}\sqrt{A_0}} \quad \text{change of variable}$$

$$= 2\frac{\delta}{\delta L}\int_0^\pi d\xi\,\frac{r^2}{L}\cancel{\sqrt{A_0}}\,\frac{m\delta V}{\cancel{A_0^{1/2}}} = 2m\frac{\delta}{\delta L}\left[\frac{1}{L}\int_0^\pi d\xi\, r^2 \overset{\circledast_1}{\delta V}\right]$$

$$= \cancel{2}m\frac{\delta}{\delta L}\left[\frac{1}{L}\int_0^\pi d\xi\,\cancel{r^2}\frac{m}{\cancel{2}}\frac{M^2G^2}{\cancel{r^2}c^2}\right] = \frac{m^2M^2G^2}{c^2}\left[\frac{\delta}{\delta L}\frac{1}{L}\int_0^\pi d\xi\right] = -\frac{m^2M^2G\pi}{c^2L^2} \;\nwarrow \circledast_2$$

$$= -\frac{\cancel{m^2}M^{\cancel{2}}\cancel{G}^{\cancel{2}}\pi}{c^2\cancel{G^2}M\cancel{m^2}a(1-e^2)} = -\frac{\pi M}{c^2 a(1-e^2)}$$

$$= -0{,}017'' \text{ per orbit (88days)}$$

$M_\odot = 2\,10^{30}$ Kg

$a_{\text{mercury}} = 5{,}8\cdot 10^{9}$ m , $e_{\text{mercury}} = 0{,}2$

$L = GMm^2 a(1-e^2)$ $a =$ semi-major axis

$e =$ ellipticity

But: $(\Delta\xi)_{\text{true}} = +42''$!

$\Rightarrow$ Failure in flat space-time

$$\star\; \frac{\delta}{\delta L}\left[2m(E-\overset{\downarrow}{V}) - \frac{L^2}{r^2}\right]^{1/2} = \frac{1}{\cancel{2}}\left[\sim\right]^{-1/2}\cdot\left(\frac{-\cancel{2}L}{r^2}\right) = \frac{d\xi}{dr}$$

$$\Delta_{100}\xi = -7''$$

Classical, relativistic linear theory in Minkowski space

here we have one deg. of freedom ↓

1) Lagrangian density: Similar to electrodynamics, use a classical scalar field

$$\boxed{\mathcal{L}(\varphi,\partial_\mu\varphi)=\frac{1}{8\pi G}\partial_\mu\varphi\,\partial^\mu\varphi+\varphi\rho}$$

$\mu=0,1,2,3$ scalar classical field (adimensional)

$\varphi\rho$ → field-matter interaction

$\partial_\mu\varphi\,\partial^\mu\varphi$ → Self interaction, free field (kinetic term)

$\frac{1}{8\pi G}$ → Const. to get the Newtonian limit

$\eta_{\mu\nu}$ is there! $\eta_{\mu\nu}\partial^\nu\varphi\,\partial^\mu\varphi$ Flat space-time

↳ quadratic to have a linear theory

2) Euler-Lagrange eq., equation of motion of the field

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}-\frac{\partial\mathcal{L}}{\partial\varphi}=\frac{2}{8\pi G}\partial_\mu\partial^\mu\varphi-\rho=0 \quad \Rightarrow \quad \boxed{\square\varphi(x^\mu)=4\pi G\rho(x^\mu)}$$

"Poisson eq."

$\square\equiv\partial_\mu\partial^\mu=\eta_{\mu\nu}\partial^\nu\partial^\mu=-\frac{1}{c^2}\partial_t^2+\bar{\nabla}^2$ D'Alambert operator

- Solution = plane waves : $\varphi\propto e^{\pm i\eta_{\mu\nu}K^\mu x^\nu}=e^{\mp i(\bar{k}\bar{x}-\omega t)}$ $\quad K_\mu x^\nu=\left(\frac{\omega}{c},\bar{k}\right)\begin{pmatrix}ct\\ \bar{x}\end{pmatrix}=-\omega t+\bar{k}\bar{x}$
- Propagation in vacuum : $\rho=0$ $\quad\square\varphi=0\Rightarrow\eta_{\mu\nu}K^\mu K^\nu=K^\mu K_\mu=0$

  i.e. $K^\mu=$ null vector

  i.e wave propagates along light-cone with velocity c

- Yes!

↳ Newtonian gravity : $-c^{-2}\partial_t^2\phi\ll\bar{\nabla}^2\phi$ i.e. slow time evolution of ϕ, retardation can be ignored

↳ retardation : because of d'Alampert operator $\square\equiv-\frac{1}{c^2}\partial_t^2+\bar{\nabla}^2$

↳ gravitational waves : $\phi\neq0$ fluctuating in vacuum

- Energy-Momentum tensor of the field $\left(\mathcal{L}_f=\frac{1}{8\pi G}\partial_\alpha\varphi\,\partial^\alpha\varphi\right)$ field only!

$$T_f{}^\mu{}_\nu=\partial_\nu\varphi\frac{\partial\mathcal{L}_f}{\partial\varphi_{,\mu}}-\delta^\mu{}_\nu\mathcal{L}_f=\partial_\nu\varphi\frac{\overset{1}{\cancel{2}}}{\underset{4}{\cancel{8}}\pi G}\partial^\mu\varphi-\frac{1}{8\pi G}\delta^\mu{}_\nu\partial_\alpha\varphi\partial^\alpha\varphi=\frac{1}{4\pi G}\left(\partial_\nu\varphi\partial^\mu\varphi-\frac{1}{2}\delta^\mu{}_\nu\partial_\alpha\varphi\partial^\alpha\varphi\right)$$

$T_{grav}^{\mu\nu}$ is a conserved quantity : $\partial_\nu T_{grav}^{\mu\nu}=0$ $\quad\boxed{\partial_j T_f^{ij}=\rho\,\partial^i\varphi}$ gravitational force

OK!

- Join with $T^{\mu\nu}$ of fluids $\Rightarrow$ $T^{ij} \to T^{ij}_{fluid} + T^{ij}_{grav}$ $\boxed{\partial_j T^{ij}_{fluid} + \partial_j T^{ij}_{grav} = 0}$ Navier-Stokes eq. with gravity

  (Hydro + Poisson eq. to solve the system for: $\rho, \bar{v}, \varepsilon, P, \varphi$)

- But... still we have an inconsistency!

ρ is in the T^{00} component of matter energy-momentum tensor $\overset{T^{00}=\rho c^2}{\overset{\downarrow}{T^{\mu\nu}}}$ (as in special relativity)

- ρ transforms under Lorentz transf. (volumes transform)
- but ϕ = scalar $\in \mathbb{R}$ $\Rightarrow$ invariant under L.T.

$\Rightarrow$ Inconsistent!

$$\underset{\text{scalar}\nearrow}{\Box\phi} \propto \frac{T^{00}}{c^2}\nwarrow\text{component of tensor}$$

$\Rightarrow$ we need a vector potential component $\bar{A}$ such that $(A^\mu) = (\phi, \bar{A})$ can be sourced by $T^{\mu\nu}$

in this way ϕ would transform as well being one component of a 4-vector, but... we do not need those dog. of freedom

- You would use $\rho \longrightarrow -\frac{T}{c^2} \equiv -\frac{T^\mu{}_\mu}{c^2} = -\frac{\mathrm{tr}(T^{\mu\nu})}{c^2}$ where $\rho = -\frac{T}{c^2}$ when pressure terms T^{ii} can be ignored i.e. non relativistic matter

  ↳ still, just for non relat. case

  (–) because $T^\mu{}_\mu = \overset{\downarrow}{-}T^{00} + T^i{}_i$

- We will come back to the $T^{\mu\nu}$ business later

$$L = -mc^2\gamma^{-1} \quad \to \quad L = -mc^2\gamma^{-1}\left(1 + \frac{\varphi}{c^2}\right)$$

More about gravity

- Let's investigate T^{ij}_{grav} : potential energy

1) Re-express T^{ij} :

drop $\frac{1}{2c^2}\partial_t\varphi\partial^t\varphi$ $\qquad$ $\nabla^2\varphi = 4\pi G\rho$ Poisson eq.

$$T^{ij}_{grav} = \frac{1}{4\pi G}\left(\partial^i\varphi\partial^j\varphi - \frac{1}{2}\delta^{ij}\partial_\alpha\varphi\partial^\alpha\varphi\right) \simeq \frac{\partial^i\varphi\partial^j\varphi}{4\pi G} - \frac{\delta^{ij}}{8\pi G}\left(\partial_k(\varphi\partial^k\varphi) - \varphi\underbrace{\partial_k\partial^k\varphi}\right)$$

$$= \frac{\partial^i\varphi\partial^j\varphi}{4\pi G} - \frac{\delta^{ij}}{8\pi G}\partial_k(\varphi\partial^k\varphi) + \frac{\delta^{ij}}{\underset{2}{\cancel{8\pi G}}}\varphi\cancel{4\pi G}\rho = \frac{\partial^i\varphi\partial^j\varphi}{4\pi G} - A^{ij} + \frac{1}{2}\delta^{ij}\rho\varphi$$

2) Define $U^{ij} \equiv \int d^3x\, T^{ij}$ and study trace $U = Tr(U^{ij}) = \int d^3x\, Tr(T^{ij}) = \int d^3x\, T$

$$Tr(T^{ij}) = T = \frac{\partial^i\varphi\partial_i\varphi}{4\pi G} + \frac{1}{2}\overset{=3\rho\varphi}{\delta^i_i\rho\varphi} - A^i_i = \frac{\overbrace{\partial^i(\varphi\partial_i\varphi)}^{B} - \varphi\overset{\nabla^2\varphi=4\pi G\rho}{\underbrace{\partial^i\partial_i\varphi}}}{4\pi G} + \frac{3\rho\varphi}{2} - \overset{Tr(A^{ij})}{A} = B + \frac{1}{2}\rho\varphi - A$$

$$\Rightarrow \boxed{U = \frac{1}{2}\int d^3x\,\rho\varphi}$$ gravitational potential energy

here $\int d^3x\, A \overset{\text{Gauss theo.}}{=} 0 = \int d^3x\, B$ Because A, B = gradient of a function

in fact: Gauss theorem $\int_V d^3x\,\partial_k f = \int_{\delta V} dA\, f = 0$ δV surface enclosing V

because you want to integrate over the entire source body ρ
i.e. at ∞ where $f=0$
→ Physical meaning: φ is not unique ⇒ you can add a const.

- Chandrasekhar expression for grav. potential energy

- remember: $\partial_j T_f^{ij} = \rho\partial^i\varphi$ gravitational force

gradient of a function ⇒ drops in integral
("drop it" = rescale T^{ij})

- note: $$x^i\partial_k T^{jk}_{grav} = \overbrace{\partial_k(x^i T^{jk}_{grav})}^{C^{ij}} - T^{jk}_{grav}\underbrace{\partial_k x^i}_{\delta^i_k} \Rightarrow T^{ji}_{grav} = -x^i\rho\partial^j\varphi - \cancel{C^{ij}}$$

$$U = \int d^3x\, Tr(T^{ij}) = -\int d^3x\,\rho x^i\partial_i\varphi$$

- join to previous result:

$$\boxed{\frac{1}{2}\int d^3x\,\rho\varphi = -\int d^3x\,\rho x^i\partial_i\varphi}$$ Chandrasekhar expression for grav. potential energy

potential on shells, e.g. useful for stars ◎

Virial tensor theorem

- Generalization of the virial theorem : $T + \frac{1}{2} V = 0 \Rightarrow$ equilibrium

typically point mass in orbit $T = \frac{1}{2} m \bar{v}$

- Now express this concept in terms of momentum of inertia I^{ij} of orbiting body

- $I^{ij} \equiv \int_V d^3x\, \rho\, x^i x^j$ Inertial tensor of a body x^i, x^j not time dependent

V = fixed volume

- Go for $\frac{d^2 I^{ij}}{dt^2}$:

plug continuity eq.

$$\frac{dI^{ij}}{dt} = (\partial_t + \bar{v}\bar{\nabla})\int_V d^3x\, \rho\, x^i x^j = \int_V d^3x \left[x^i x^j \partial_t \rho + \cancel{\rho \partial_t (x^i x^j)} + \cancel{\bar{v}\bar{\nabla}(\rho x^i x^j)} \right]$$

$=0$ $\quad \bar{v} = \partial_t \bar{x} = 0$

$$= -\int_V d^3x\, x^i x^j \partial_k(\rho v^k) = -\int_V d^3x \left[\underbrace{\partial_k(\rho v^k x^i x^j)}_{=0 \text{ Gauss}} - \partial_k(x^i x^j)\rho v^k \right] = +\int_V d^3x \left(x^j \underbrace{\partial_k x^i}_{\delta^i_k} + x^i \underbrace{\partial_k x^j}_{\delta^j_k} \right) \rho v^k$$

$$= \int_V d^3x\, (v^i x^j + v^j x^i)\rho$$

$$\frac{d^2 I^{ij}}{dt^2} = \int_V d^3x \left[x^j \partial_t(\rho v^i) + x^i \partial_t(\rho v^j) \right]$$

- links $x^j \partial_t(\rho v^i)$ to T^{ij} :

a) $x^j \partial_0 T^{0i} = x^j \partial_0 (\rho c v^i) = c x^j \partial_0(\rho v^i) = x^j \partial_t(\rho v^i)$

b) mom. cons. : $\partial_0 T^{0i} + \partial_j T^{ji} = 0 \quad \partial_0 T^{0i} = -\partial_j T^{ji} \quad x^j \partial_0 T^{0i} = -x^j \partial_j T^{ji} = -\underbrace{\partial_j(x^j T^{ij})}_{\text{gradient will drop}} + T^{ij} \underbrace{\partial_j x^j}_{\delta^j_j = 1}$

$\Rightarrow T^{ji} = x^j \partial_t(\rho v^i) \underbrace{- c^{ij}}_{\text{irrelevant}}$

$$= \int_V d^3x\, (T^{ji} + T^{ij}) = 2\int_V d^3x\, T^{ij} = 2\left[\int_V \rho v^i v^j d^3x + \delta^{ij} \int_V P\, dx^3 + U^{ij} \right] = 0$$

(for perfect fluid : $T^{ij} = \rho v^i v^j + P\delta^{ij} + T^{ij}_{grav}$) (if $=0 \Rightarrow$ system is in equilibrium)

$\Rightarrow$ Virial tensor theorem :

$$\boxed{K^{ij} + \frac{1}{2}\left[\delta^{ij} \int d^3x\, P + U^{ij} \right] = 0}$$

(1) Kinetic (translational + rotational)

$K^{ij} \equiv \frac{1}{2} \int_V \rho v^i v^j d^3x$

(2) work by pressure

(3) grav. potential energy

$U^{ij} = \int_V d^3x\, T^{ij}_{grav}$

Grav. force :

$$T^{\mu}_{\text{grav}\,\nu} = \frac{1}{4\pi G}\left(\partial_\nu\varphi\,\partial^\mu\varphi - \frac{1}{2}\delta^\mu{}_\nu\,\partial_\alpha\varphi\,\partial^\alpha\varphi\right)$$

$$\partial_j T^{ij}_{\text{grav}} = \frac{1}{4\pi G}\left[(\partial_j\partial^i\varphi)\partial^j\varphi + \partial^i\varphi\,(\underbrace{\partial_j\partial^j\varphi}_{\bar\nabla^2\varphi = 4\pi G\rho \text{ Poisson eq.}}) - \frac{1}{2}\overbrace{\delta^{ij}\partial_j}^{\partial^i}\partial_\alpha\varphi\,\partial^\alpha\varphi - \frac{1}{2}\,\partial_\alpha\varphi\,\overbrace{\delta^{ij}\partial_j}^{\partial^i}\partial^\alpha\varphi\right]$$

$$= \frac{1}{\cancel{4\pi G}}\left(\cancel{\partial_j\partial^i\varphi\,\partial^j\varphi} + \partial^i\varphi\,\cancel{4\pi G}\rho - \cancel{\partial^i\partial_\alpha\varphi\,\partial^\alpha\varphi}\right) \simeq \rho\,\partial^i\varphi$$

approx: drop $\frac{1}{c^2}\partial_\epsilon\varphi\,\partial^\epsilon\varphi$

Toward G.R. : linking gravity to the metric of space-time

- All objects fall with the same acceleration (if same initial conditions) even if ≠ mass and ≠ substance

$\cancel{m_i}\vec{a} = -\cancel{m_G}\vec{\nabla}\varphi$ i.e. $m_i = m_G$ (e.g. $m\vec{a} = -q\vec{E}$ $q \neq m$!)

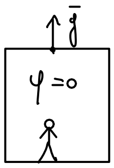
inertial mass

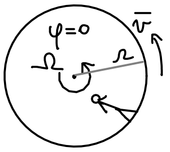
"gravitational charge"

$\Rightarrow$ $m_G = m_i$ is an inertial business!

- Idea: equivalence principle

a non inertial frame is equivalent to a gravitational field





locally equivalent to $\vec{g} = -\vec{\nabla}\varphi$
but physically very different

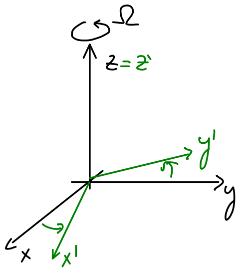
- Can we use this idea to construct a covariant theory of gravity?! Yes!

- e.g. rotating frame

- weak field limit $\varphi \Leftrightarrow g_{\mu\nu}$

Example: rotating system

- Equivalence principle : accelerated frame ↔ gravity
- Free particle : no field, i.e. no external force
- Inertial frame : $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$



- In a rotating system :

$$\begin{cases} x = x'\cos(\Omega t) - y'\sin(\Omega t) \\ y = x'\sin(\Omega t) + y'\cos(\Omega t) \\ z = z' \end{cases}$$

$$dx = dx'\cos(\Omega t) + x'\sin(\Omega t)\Omega dt - dy'\sin(\Omega t) + y'\cos(\Omega t)\Omega dt$$

$$dy = dy'\sin(\Omega t) - x'\cos(\Omega t)\Omega dt + dy'\cos(\Omega t) + y'\sin(\Omega t)\Omega dt$$

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$= \Big[c^2 - \Omega^2(x'^2 + y'^2)\Big] \underset{\substack{\text{time} \\ g_{00}}}{dt^2} \underbrace{- dx' - dy' - dz'^2}_{\substack{\text{space} \\ g_{ii}}} \underbrace{+ 2\Omega y'dx'dt - 2\Omega x'dy'dt}_{\substack{\text{off-diagonal (mixed)} \\ g_{10}\ g_{01}\ g_{20}\ g_{02}}}$$

$$= g_{\mu\nu}\, dx^\mu dx^\nu$$

→ centrifugal force

→ locally as a gravitational field

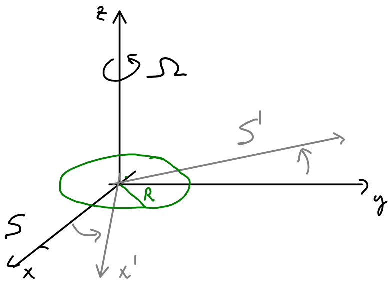
equivalence principle

⇒ Associate gravity to the metric of space-time ! $g_{\mu\nu} \neq \eta_{\mu\nu}$

- Careful! Here the space is still Minkowski! (flat)

- $g_{\mu\nu} \neq \eta_{\mu\nu}$ only because of the "weird" coord. system we used not because of an intrinsic property of space-time

- In fact non-locally, very different behavejor then with a grav. field
  - ↳ in a grav. field : for $r \to \infty$ frome source of field $\bar{g} = -\bar{\nabla}\varphi \to 0$
  - ↳ here $r \to \infty$ $\bar{g} \to \infty$!



Note:

measure the circonference of a circle in the x-y plane

S = inertial (Ω=0) C = 2πR

S = non-inertial (Ω≠0) C ≠ 2πR ! ≠ geometry! "curved space"

because of Lorentz contraction

Connecting the metric to a gravitational field

- Let's try to find the metric associated to a gravitational field
- Take a non relativistic particle in a given fixed gravitational field

inertial mass ↓

$$L = -mc^2 + \frac{1}{2} m \bar{v}^2 - m\varphi$$

note $m_i = m_G$!

freedom to add a const $\varphi \to \varphi + A$, e.g. $A = -c^2$

$$S = \int L\, dt = -mc^2 \int \left(1 - \frac{\bar{v}^2}{2c^2} + \frac{\varphi}{c^2}\right) dt \overset{!}{=} -mc^2 \int d\tau \; *$$

Equivalence principle: free particle $S = -mc^2 \int d\tau$ but with some specific metric $\eta_{\mu\nu} \to g_{\mu\nu}$

$$* \; d\tau = \frac{1}{c}\left(-g_{\mu\nu} dx^\mu dx^\nu\right)^{1/2} \overset{!}{=} \left(1 - \frac{\bar{v}^2}{2c^2} + \frac{\varphi}{c^2}\right) dt \simeq \left(1 - \frac{\bar{v}^2}{c^2} + \frac{2\varphi}{c^2}\right)^{1/2} dt$$

$-\frac{\bar{v}^2}{2c^2} + \frac{\varphi}{c^2} \ll 1$ i.e. non relativistic particle in weak field

$\Downarrow$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2\left(1 - \frac{\bar{v}^2}{c^2} + \frac{2\varphi}{c^2}\right) dt^2 = -\left(1 + \frac{2\varphi}{c^2}\right) c^2 dt^2 + \frac{d\bar{x}^2}{\cancel{dt^2}} \cancel{dt^2}$$

← identify $g_{\mu\nu}$

$\Downarrow$

$$\Rightarrow \boxed{g_{00} = -\left(1 + \frac{2\varphi}{c^2}\right), \quad g_{0i} = 0 = g_{i0}, \quad g_{ij} = \delta_{ij}}$$

← no coordinate transformation can make $g = \eta$ everywhere simultaneously, you can only locally

small correction of $\eta_{00} = -1$

In reality you have a correction of the same order of g_{00}

$g_{ii} = 1 - \frac{2\varphi}{c^2}$ (coming from Einstein eq. in weak field limit)

without it you get gravitational lensing wrong

- Limit of the approach:
  - we assumed a given and fixed φ (φ was not 'predicted / associated to a source)
  - not the correct value for g_{ii} terms

- But....
  - By assuming a free particle we can associate gravity to a curved space-time !! :-)

    welcome to General Relativity!

units: $\left[\frac{G}{c}\right] = \frac{m}{Kg}$ ← geometry ← mass

G enters through the weak field limit

c " " relativity

Vectors, 1-forms, tensors

(Körper)
$K =$ field, e.g. $K = \mathbb{R}$

- **Vector:** $\bar v \in V$ "it has no components" $V = \{\bar v \in K^n\}$ vector space

- **Basis set:** $\{\bar e_i\}$ $\bar e_{\overset{\uparrow}{i}} \in V$ linearly independent vectors defining the frame
  label

- **Components of a vector:** $\bar v = v^i \bar e_i$ as a linear combination of basis vectors

- **Transformation of vectors:** $x^{i'} = x^{i'}(x^j(\tau))$

$$\frac{dx^{i'}}{d\tau} = \frac{\partial x^{i'}}{\partial x^j}\frac{dx^j}{d\tau} \Rightarrow \boxed{dx^{i'} = \frac{\partial x^{i'}}{\partial x^j} dx^j} \qquad \frac{\partial x^{i'}}{\partial x^j} \equiv J^{i'}{}_j$$

Jacobian of the transformation

infinitesimal change of $x^{i'}$ with respect to x^j

Now: $\Lambda^{i'}{}_j \equiv \frac{\partial x^{i'}}{\partial x^j}$ are NOT just Lorentz transf!

$d\bar x$ = displacement
prototype of vector

$$\boxed{v^{i'} = \frac{\partial x^{i'}}{\partial x^j} v^j}$$

- **Transformation of basis:**

frames (S, S') basis sets $(\{\bar e_i\}, \{\bar e_{i'}\})$ vector components $(v^i, v^{i'})$ $\bar v = \begin{cases} v^i \bar e_i \\ v^{i'} \bar e_{i'} \end{cases}$ "$\bar v$ is $\bar v$!"

$$\bar v = v^i \underline{\bar e_i} = \overset{\searrow}{v^{i'}} \bar e_{i'} = \Lambda^{i'}{}_i v^i \bar e_{i'} \overset{}{\underset{\uparrow}{=}} v^i \underline{\Lambda^{i'}{}_i \bar e_{i'}} \Rightarrow \boxed{\bar e_i = \Lambda^{i'}{}_i \bar e_{i'}} \quad S' \to S$$

linearity of Λ

inverse:

$$(\Lambda^{-1})^{j'}{}_i \equiv \Lambda_{j'}{}^i \Rightarrow \Lambda_{j'}{}^{\overset{\downarrow}{i}} \bar e_i = \underbrace{\Lambda_{j'}{}^i \Lambda^{i'}{}_i}_{\delta^{i'}_{j'}} \bar e_{i'} = \bar e_{j'} \Rightarrow \boxed{\bar e_{j'} = \Lambda_{j'}{}^i \bar e_i} \quad S \to S'$$

Kronecker delta

- **Linear map** :

$$\boxed{T: V \to G \qquad T(\bar v) = T\bar v = T_{ij} v^i = \bar w \qquad v \in V \quad w \in G}$$

$T(\bar v + \bar u) = T(\bar v) + T(\bar u)$ $u \in V$ distributive

$T(a\bar v) = aT(\bar v)$ $a \in K$ linear

- **Bilinear map** : $\boxed{T: V \times V \to G \quad T(\bar v, \bar u) = T_{ij} v^i u^j}$

linear in both of its 2 arguments: $T(a\bar v + b\bar u) = aT(\bar v) + bT(\bar u)$ $a, b \in \mathbb{R}$

- <u>The metric</u>

– Now : a generic metric g! (not just η)

– It is a bilinear map identifying the scalar product

$$g : V \times V \to \mathbb{R} \quad (\bar{u}, \bar{v}) \to g(\bar{u}, \bar{v}) \equiv \langle \bar{u}, \bar{v} \rangle = a \qquad \bar{u}, \bar{v} \in V \quad a \in \mathbb{R}$$

– It is also the linear map "linking" the 2 spaces V and $\tilde{V}$

$$g : V \to \tilde{V} \quad (\bar{v}) \to g(\bar{v}, -) = \tilde{v} \qquad v_i = g_{ij} v^i \quad (v_i) \in \tilde{V} \quad (v^i)^T \in V$$

<u>Properties</u> :

- $g(\bar{u}, \bar{v}) = 0 \quad \forall \bar{u} \in V \Rightarrow \bar{v} = 0$ $\quad (\Longleftrightarrow \det(g) \neq 0)$ <u>non degenerate</u>
- $g(\bar{v}, \bar{v}) = \langle \bar{v}, \bar{v} \rangle = \|\bar{v}\|^2$ it defines the <u>norm</u> of a vector
- $g(\bar{u}, \bar{v}) = g(u^i \bar{e}_i, v^j \bar{e}_j) = u^i v^j g(\bar{e}_i, \bar{e}_j) = u^i v^j g_{ij}$ <u>components</u> of g $\quad g_{ij} \equiv g(\bar{e}_i, \bar{e}_j)$
- $d\bar{s}^2 = g_{ij} dx^i dx^j \overset{\text{relabeling}}{\overset{\downarrow}{=}} g_{ji} \overset{\rightleftarrows}{dx^j dx^i} = g_{ji} dx^i dx^j \Rightarrow g_{ij} = g_{ji}$ <u>g is symmetric</u> because of the quadratic form of $d\bar{s}^2$

  $ds^2 > 0 \quad \Rightarrow$ Riemanian space

  $ds^2 <> = 0 \quad \Rightarrow$ pseudo-Riemanian space
- $g(\bar{u}, \bar{v}) = a = g(u^i \bar{e}_i, v^j \bar{e}_j)$ frame S

  $\qquad\qquad\quad \overset{!}{=} g(u^{i'} \bar{e}'_i, v^{j'} \bar{e}'_j)$ frame S' $\qquad$ <u>$a \in \mathbb{R}$ is invariant</u>
- $(g^{-1})^{ab} \equiv g^{ab}$ defined as $g_{ab} g^{bc} = \delta^c{}_a$ $\qquad$ <u>inverse</u> of the metric, rank $\binom{2}{0}$

  g^{ij} exists because g is not degenerate, i.e. $\det(g) \neq 0$

  $\delta^c{}_a : V \to V$ identity, swaps indexes $\delta^c{}_b v^b = v^c$

- One-forms = dual vectors = covariant vectors = co-vectors

$$\boxed{\underbrace{\bar v^*: V \to K}_{\text{linear map}} \quad \bar u \to \bar v^*(\bar u) \equiv \underbrace{g(\bar v, \bar u)} = \langle \bar v, \bar u \rangle \quad \bar u, \bar v \in V \quad \bar v^* \in V^*}$$

$\bar v^* \equiv g(\bar v, -) \equiv \tilde v$

dual vector space of V
space of 1-forms

– Components, index lowering

$$\tilde v(\bar u) = \tilde v(u^i \bar e_i) = u^i \underline{\tilde v(\bar e_i)} = u^i v_i$$

$$= g(\bar v, \bar u) = g_{ij} u^i v^j = u^i v_i = u_j v^j$$

$\tilde v$ lower index, $\bar u$ upper index

g lowers the index

– Example:

$$g = \eta = \mathrm{diag}(-1, 1, 1, 1)$$

(!)

$$\bar v = (v^0, v^1, v^2, v^3)^T \in V$$

(!)

$$\tilde v = g(\bar v, -) = (-v_0, v_1, v_2, v_3) \in V^*$$

(!) take care!

– Both vectors and 1-forms are linear maps

$\tilde v : V \to \mathbb{R} \qquad g(\bar v, \bar w) \equiv \tilde v(\bar w) \qquad \tilde v = g(\bar v, -)$

$\bar v : \tilde V \to \mathbb{R} \qquad g(\bar v, \bar w) \equiv \bar w(\tilde v)$

– Basis of one-forms $\{\tilde\omega^i\}$

$\tilde v = v_i \tilde\omega^i$ in analogy to vectors: $\bar v = v^i \bar e_i$

$$\tilde v(\bar u) = v_i \tilde\omega^i(u^j \bar e_j) = v_i u^j \tilde\omega^i(\bar e_j) \overset{!}{=} v_i u^i \Rightarrow \boxed{\tilde\omega^i(\bar e_j) = \omega^i{}_j \overset{!}{=} \delta^i_j = \frac{\delta x^i}{\delta x^j}}$$

δ^i_j = Kronecker delta

j-th derivative of the i-th coordinate
is how x^i changes along j
$\tilde\omega^i$ are the prototype of gradients

i-th one-form basis in cartesian frame $\longrightarrow$ $\tilde\omega^i = (\omega^i{}_j) = (\omega^i{}_0, \omega^i{}_1, \dots) = (\delta_0 x^i, \delta_1 x^i, \dots \delta_m x^i) = \widetilde{\delta x^i}$

i = label
j = index of component

Careful:

$\tilde\omega^0 = (1, 0, \dots, 0) \qquad \bar e_0 = (1, 0, \dots, 0)^T$

$\tilde\omega^1 = (0, 1, \dots, 0) \qquad \bar e_1 = (0, 1, \dots, 0)^T$

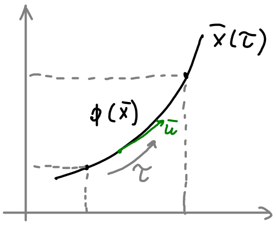
---- ----

$\tilde\omega^m = (0, 0, \dots, 1) \qquad \bar e_m = (0, 0, \dots, 1)^T$

$\tilde\omega^i \in \tilde V \qquad \bar e_i \in V$

but $\{\tilde\omega^i\} \neq \{\bar e_i\}$ they belong to a different spaces!

- One-forms are the prototypes of gradients



$\phi(\bar{x}): \mathbb{R}^m \to \mathbb{R}$ ← some scalar function

$\bar{x}(\tau)$ ← parametric curve

$\bar{u} \equiv \frac{d\bar{x}}{d\tau} = \left(\frac{dx^i}{d\tau}\right)^T$ ← vector tangent to $\bar{x}(\tau)$ $\quad \bar{dx} = \bar{u}\, d\tau$

notation: $\partial_i \to {}_{,i}$

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial x^i}\frac{dx^i}{d\tau} = \partial_i\phi\, u^i = \phi_{,i}\, u^i = a \in \mathbb{R}$$ ← Change of ϕ along $x(\tau)$: $\phi(\bar{x}(\tau))$

$$= \tilde{\partial\phi}(\bar{u})$$ ← Frameless representation with $\tilde{\partial\phi} = \phi_{,i}\,\tilde{\omega}^i \quad \bar{u} = u^i \bar{e}_i$

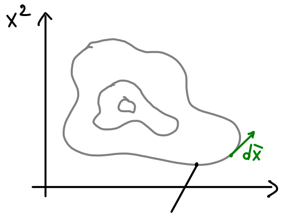
one-form ↗ ↖ vector

$\tilde{\partial\phi} \in V^*$ cotangent space "covectors" $\qquad \bar{u} \in V$ tangent space

Note: $u^i \partial_i \phi = \nabla_{\bar{u}}\phi \in \mathbb{R}$ directional derivative
change along direction $\bar{u}$

To visualize it:

$\tilde{\partial\phi}(d\bar{x}) = \partial_i\phi\, dx^i = d\phi$ change of ϕ along $\bar{dx}$



$\partial_i\phi \; dx^i$

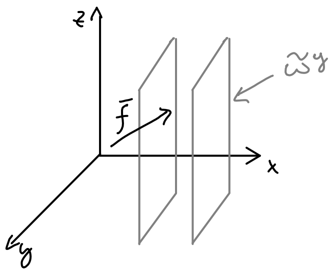
change ↗ ↖ displacement vector $\bar{dx}$

$\phi = \text{const} \Rightarrow \frac{d\phi}{d\tau} = \tilde{d\phi}\,\bar{dx} = 0$ iso-levels

$\bar{dx}$: prototype of vector (displacement)
$\tilde{\partial}^i$ " " one-form (gradient)

- Useful also to quantify fluxes across a surface

$$\int \bar{f}\cdot\bar{dA} = \int \tilde{dA}(\bar{f})$$



area as a one form

- Transformation of one-forms / basis

$$\boxed{v_{i'}} = \tilde{v}(\bar{e}_{i'}) = \tilde{v}(\Lambda^j{}_{i'}\,\bar{e}_j) = \Lambda^j{}_{i'}\,\tilde{v}(\bar{e}_j) = \boxed{\Lambda^j{}_{i'}\, v_j}$$

$$\tilde{v} = v_i\,\tilde{\omega}^i = v_{i'}\,\tilde{\omega}^{i'} = \Lambda^i{}_{i'}\, v_i\, \omega^{i'} \quad \Rightarrow \quad \boxed{\tilde{\omega}^i = \Lambda^i{}_{i'}\,\omega^{i'}} \quad \boxed{\tilde{\omega}^{i'} = \Lambda_i{}^{i'}\,\tilde{\omega}^i}$$

- This construction guarantees the invariance of 4-intervals

$$\bar{v}(\tilde{w}) = v^i \bar{e}_i(w_j\tilde{\omega}^j) \overset{!}{=} v^{i'}\bar{e}_{i'}(w_{j'}\tilde{\omega}^{j'}) = \Lambda^{i'}{}_i v^i \bar{e}_{i'}(\Lambda^j{}_{j'} w_j \tilde{\omega}^{j'}) = \Lambda^{i'}{}_i \Lambda^j{}_{j'} v^i w_j \underbrace{\bar{e}_{i'}(\tilde{\omega}^{j'})}_{\delta^{j'}_{i'}}$$

$$= \underbrace{\Lambda^{i'}{}_i \Lambda^j{}_{i'}}_{\delta^j_i} v^i w_j = v^i w_i \in \mathbb{R} \text{ invariant}$$

Tensors

Tensors are rulers in physics:

metric tensor (space-time)
elettromagnetic tensor
energy-momentum tensor
....

They are a "generalization":

Scalars : 0 indeces (a) , Vector : 1 index (a^i) , 1-forms : (a_i)
Matrix : 2 indeces (a_{ij}) , Tensors : N indeces up/down
↑ (tensors are matrixes but not all matrixes are tensors)

General definition:

a matematical object obaying certain transformation roles
i.e components have certain transformation properties
under a change of coordinates

Types of tensors:

a tensor of type $\begin{pmatrix} M \\ N \end{pmatrix}$ is a linear mapping of M 1-forms and N vectors to scalars (Lorentz invariant)

$$T : \underbrace{\tilde{V} \times \tilde{V} \times \cdots \times \tilde{V}}_{M} \times \underbrace{V \times V \times \cdots \times V}_{N} \rightarrow \mathbb{R} \quad \text{invariant}$$

$$\text{type} \begin{pmatrix} M \\ N \end{pmatrix} \begin{array}{l} \leftarrow \text{ \# of input 1-forms (index up)} \\ \leftarrow \text{ " " " vectors (index down)} \end{array}$$

Rank: total number of indeces

type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ rank $= 3$ $\quad i = 1, \ldots, m \quad j = 1, \ldots, m \quad k = 1, \ldots, m \qquad$ eg. $T_{ij}{}^{k} v^i w^j r_k = a \in \mathbb{R}$

m^3 components (m-dimensional vector space V eg. $V = \mathbb{R}^m$)

You already met tensors!

Type $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: scalar $a : \mathbb{R} \rightarrow \mathbb{R} \quad a(b) = a\,b = c \qquad a, b, c \in \mathbb{R}$

Type $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: "1-forms" = dual vector = covector = covariant vector

$\tilde{P} : V \rightarrow \mathbb{R} \quad \tilde{P}(\bar{v}) = P_i v^i = a \qquad a \in \mathbb{R} \quad \bar{v} \in V \quad \tilde{P} \in V^* \qquad$ linear map

Type $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: vector = contro-variant vector = tangent vector

$\bar{P} : V \rightarrow \mathbb{R} \quad \bar{P}(\tilde{v}) = P^i v_i = a \qquad a \in \mathbb{R} \quad \tilde{v} \in V^* \quad \bar{P} \in V \qquad$ linear map

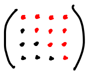
Type $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $P : V^* \times V \rightarrow \mathbb{R} \quad P(\tilde{v}, \bar{w}) = P^i{}_j v_i w^j = a \qquad a \in \mathbb{R} \quad \tilde{v} \in V^* \quad \bar{w} \in V \qquad$ bilinear map

Type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$: $P : V \times V \rightarrow \mathbb{R} \quad P(\bar{v}, \bar{w}) = P^i{}_j v_i w^j = a \qquad a \in \mathbb{R} \quad \bar{v}, \bar{w} \in V \qquad$ bilinear map

eg. the metric η, g ! $\quad \eta : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}$

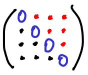
Symmetric tensors

$T(\bar v,\bar w)=T(\bar w,\bar v) \iff T_{ij}=T_{ji} \quad \forall \bar v,\bar w\in V \rightarrow$



Antisymmetric tensors

$T(\bar v,\bar w)=-T(\bar w,\bar v) \iff T_{ij}=-T_{ji} \quad \forall \bar v,\bar w\in V \rightarrow$



$T_{ij}\equiv \delta_i A_j-\delta_j A_i$ is guaranteed to be anti-symmetric

eg $S_{ab}T^{ac}=\quad \forall S\in\mathcal{T}^k_e$ symmetric, $T\in\mathcal{T}^k_e$ antisymmetric

Splitting in symmetric and antisymmetric parts:

eg. a type $\binom{0}{2}$ tensor T: $\quad T_{ij}=\underset{\substack{\uparrow\\ \text{you can always do it}}}{T_{(i,j)}+T_{[i,j]}}$ with $T_{(i,j)}\equiv\frac{1}{2}(T_{ij}+T_{ji})$ Symmetric part

$T_{[i,j]}\equiv\frac{1}{2}(T_{ij}-T_{ji})$ Antisymmetric part

Components of a tensor

components are the values of a function (linear mapping) when its arguments are the basis $\{\bar e_i\}$, $\{\tilde\omega^i\}$ of the frame. Eg.

$P: V\times V^*\to\mathbb{R} \quad P(\bar v,\tilde w)=P(v^i\bar e_i, w_j\tilde\omega^j)=v^i w_j\underbrace{P(\bar e_i,\tilde\omega^j)}=v^i w_j P_i{}^j$

$$\boxed{P(\bar e_i,\tilde w^j)=P_i{}^j}$$

Rising/Lowering indexes

$P(\overset{\downarrow}{\bar v},\overset{\downarrow}{\tilde\omega})=P(v^i\bar e_i, w_j\tilde\omega^j)=v^i \overset{\downarrow}{w_j} \overset{\downarrow}{P}(\bar e_i,\tilde\omega^j)=v^i w^\alpha\underbrace{g_{j\alpha}P_i{}^j}=v^i w^\alpha\underbrace{P_{i\alpha}}$

$\binom{M}{N}\to\binom{M-1}{N+1}$ lowering $\quad$ eg. $g_{\alpha\mu}T^\mu{}_{\beta\gamma}=T_{\alpha\beta\gamma}$

$\binom{M}{N}\to\binom{M+1}{N-1}$ rising $\quad$ eg. $g^{\alpha\gamma}T^\mu{}_{\beta\gamma}=T^\mu{}_\beta{}^\alpha \quad\Rightarrow\quad T_{ij}{}^k\neq T_i{}^{jk}$!

Transformation of the components

$P^{i'j'}=P(\tilde\omega^{i'},\tilde\omega^{j'})=P(\Lambda^{i'}{}_\alpha\tilde\omega^\alpha,\Lambda^{j'}{}_\beta\tilde\omega^\beta)=\Lambda^{i'}{}_\alpha\Lambda^{j'}{}_\beta P(\tilde\omega^\alpha,\tilde\omega^\beta)$

$P_{i'}{}^{j'}=P(\bar e_{i'},\tilde\omega^{j'})=P(\Lambda^\alpha{}_{i'}\bar e_\alpha,\Lambda^{j'}{}_\beta\tilde\omega^\beta)=\Lambda^\alpha{}_{i'}\Lambda^{j'}{}_\beta P(\bar e_\alpha,\tilde\omega^\beta)$

$P_{i'j'}=P(\bar e_{i'},\bar e_{j'})=P(\Lambda^\alpha{}_{i'}\bar e_\alpha,\Lambda^\beta{}_{j'}\bar e_\beta)=\Lambda^\alpha{}_{i'}\Lambda^\beta{}_{j'}P(\bar e_\alpha,\bar e_\beta)$

$P^{i'}{}_{j'}=P(\tilde\omega^{i'},\bar e_{j'})=P(\Lambda^{i'}{}_\alpha\tilde\omega^\alpha,\Lambda^\beta{}_{j'}\bar e_\beta)=\Lambda^{i'}{}_\alpha\Lambda^\beta{}_{j'}P(\tilde\omega^\alpha,\bar e_\beta)$

$$\boxed{\begin{aligned}
P^{i'j'}&=\frac{\delta x^{i'}}{\delta x^\alpha}\frac{\delta x^{j'}}{\delta x^\beta}P^{\alpha\beta}\\
P_{i'}{}^{j'}&=\frac{\delta x^\alpha}{\delta x^{i'}}\frac{\delta x^{j'}}{\delta x^\beta}P_\alpha{}^\beta\\
P_{i'j'}&=\frac{\delta x^\alpha}{\delta x^{i'}}\frac{\delta x^\beta}{\delta x^{j'}}P_{\alpha\beta}\\
P^{i'}{}_{j'}&=\frac{\delta x^{i'}}{\delta x^\alpha}\frac{\delta x^\beta}{\delta x^{j'}}P^\alpha{}_\beta
\end{aligned}}$$

Basis for tensors

- We want a basis in analogy to vectors and one-forms

we have: $\bar{v} = v^i \bar{e}_i \qquad \bar{v}(-) = v^i \bar{e}_i(-)$

$\tilde{w} = w_i \tilde{\omega}^i \qquad \tilde{w}(-) = w_i \tilde{\omega}^i(-)$

we want: e.g. $P : V \times V \to \mathbb{R} \qquad P(-,-) = P_{ij} \text{☺}^{ij}(-,-)$ P linear combination of a basis set

therefore: $P_{\alpha\beta} = P(\bar{e}_\alpha, \bar{e}_\beta) \overset{!}{=} P_{ij} \underbrace{\text{☺}^{ij}(\bar{e}_\alpha, \bar{e}_\beta)}_{\overset{!}{=} \delta^i_\alpha \delta^j_\beta} = P_{ij}\, \omega^i_\alpha \omega^j_\beta = P_{ij}\, \omega^i(\bar{e}_\alpha)\, \omega^j(\bar{e}_\beta) \equiv \underbrace{\tilde{\omega}^i \otimes \tilde{\omega}^j}_{\text{Basis}}(\bar{e}_\alpha, \bar{e}_\beta)$

$\delta^i_\alpha = \frac{\delta x^i}{\delta x^\alpha} = \omega^i_\alpha$

here we defined the outer product $\otimes$: $\tilde{\omega}^i(-)\, \tilde{\omega}^j(-) \equiv \tilde{\omega}^i \otimes \tilde{\omega}^j(-,-)$

(α,β) component of basis (i,j) ; $\{\tilde{\omega}^i \otimes \tilde{\omega}^j\}$ basis set of $\binom{0}{2}$ tensors

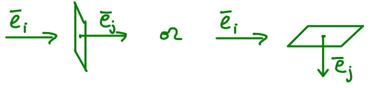
- You have various ways to construct a tensor

$$P = P_{ij}\, \tilde{\omega}^i \otimes \tilde{\omega}^j = P^i{}_j\, \bar{e}_i \otimes \tilde{\omega}^j = P^{ij}\, \bar{e}_i \otimes \bar{e}_j = P_i{}^j\, \tilde{\omega}^i \otimes \bar{e}_j$$

Examples:

type $\binom{0}{1}$	1-form: $\tilde{v} : V \to \mathbb{R}$	$\tilde{v} = v_i \tilde{\omega}^i$	$\{\tilde{\omega}^i\}$ basis of 1-forms	
type $\binom{1}{0}$	vector: $\bar{v} : V^* \to \mathbb{R}$	$\bar{v} = v^i \bar{e}_i$	$\{\bar{e}_i\}$ basis of vectors	
type $\binom{0}{2}$	$P : V \times V \to \mathbb{R}$	$P = P_{ij}\, \tilde{\omega}^i \otimes \tilde{\omega}^j$		eg. metric $g = g_{ij}\, \tilde{\omega}^i \otimes \tilde{\omega}^j$
type $\binom{3}{1}$	$P : V^* \times V^* \times V \times V^* \to \mathbb{R}$	$P = P^{ij}{}_k{}^\ell\, \bar{e}_i \otimes \bar{e}_j \otimes \tilde{\omega}^k \otimes \bar{e}_\ell$		

Intuitive understanding of tensors: is an object that "points" in multiple directions



Every tensor can be expressed as a combination of outer products

for example: given $\tilde{p}, \tilde{q} \in V^*$, you can always build a tensor $T = \tilde{p} \otimes \tilde{q} \equiv \tilde{p}(-)\tilde{q}(-) \qquad T : V \times V \to \mathbb{R}$

Warning $T(\bar{v}, \bar{w}) \equiv \tilde{p}(\bar{v})\tilde{q}(\bar{w}) \neq \tilde{p}(\bar{w})\tilde{q}(\bar{v}) \qquad \bar{v}, \bar{w} \in V$ in general $\otimes$ is not commutative

$a = p_i v^i q_j w^j \qquad p_i w^i q_j v^j = b$ in fact in general $a \neq b \qquad \tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$

Tensors operations

3 tensors of the same type, based on the same vector space

- Sum/Subtraction : $S = T + V \quad T_{ij} + V_{ij} = T(\bar{e}_i, \bar{e}_j) + V(\bar{e}_i, \bar{e}_j) = S(\bar{e}_i, \bar{e}_j) = S_{ij}$

- Multiplication by a scalar : $\alpha \in \mathbb{R} \quad S = \alpha T \quad S_{ij} = \alpha T_{ij}$

- Outer ptoduct : $(\tilde{q} \otimes \tilde{p})(\bar{v}, \bar{u}) \equiv \tilde{q}(\bar{v})\tilde{p}(\bar{u}) \quad \bar{v}, \bar{u} \in V \quad \tilde{q}, \tilde{p} \in V^* \quad (\tilde{q} \otimes \tilde{p})$ Rank 2

$$(\tilde{q} \otimes \tilde{p} \otimes \tilde{r})(\bar{v}, \bar{u}, \bar{s}) = \underbrace{\tilde{q}(\bar{v})\,\tilde{p}(\bar{u})}_{t(\bar{v},\bar{u})}\tilde{r}(\bar{s}) = t(\bar{v}, \bar{u})\tilde{r}(\bar{s})$$

$$\equiv (\underset{\uparrow \atop = \tilde{q} \otimes \tilde{p}}{t} \otimes \tilde{r})(\bar{v}, \bar{u}, \bar{s})$$

associative

- Inner product :

eg. $T(\bar{\omega}^i, \bar{e}_j) S(\bar{e}_i) = \boxed{T^i{}_j S_i = Q_j} = Q(\bar{e}_j)$ inner product between 2 tensors T and S gives a new tensor Q

$TS \neq ST : \quad T^{ij} S_i \neq T^{ij} S_j$ not commutative

- Contraction :

fixed, same index

eg. Rank-3 tensor $T(-,-,-) = (T^\alpha{}_{\beta\gamma}) \qquad T(\tilde{\omega}^\alpha, -, \bar{e}_\alpha) \equiv S(-)$ Rank-1 tensor

components : $\boxed{S_\beta} = T(\tilde{\omega}^\alpha, \bar{e}_\beta, \bar{e}_\alpha) = T^\alpha{}_{\beta\alpha} = \boxed{g^{\alpha\nu} T_{\nu\beta\alpha}}$ (repeated index up, down = contraction)

We obtain a lower rank tensor S

- Derivative :

eg. $\partial_\alpha T^{\mu\nu} = R_\alpha{}^{\mu\nu}$ (in flat S-T!) $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ tensor with an higher rank

only $\partial_\alpha \phi$ is a valid tensor in all S-T (1-form)

Mapping tensors on tensors

$T(-,-,\bar{v}) \quad \bar{v} \in V$ fixed

If we provide other two "inputs" we get a real number

$\Rightarrow T(-,-,\bar{v}) \equiv S(-,-)$ is a rank-2 tensor

T maps a vector $\bar{v}$ into a rank-2 tensor S

Components $\Rightarrow \boxed{S_{\alpha\beta}} \equiv S(\bar{e}_\alpha, \bar{e}_\beta) \equiv T(\bar{e}_\alpha, \bar{e}_\beta, \bar{v}) = \boxed{T_{\alpha\beta\gamma} v^\gamma}$

another eg. $W(-) \equiv T(\bar{u}, -, \bar{v}) \quad \bar{u}, \bar{v} \in V$ fixed $\Rightarrow W$ rank-1 tensor (vector)

$$W(\bar{e}_\alpha) = T(\bar{u}, \bar{e}_\alpha, \bar{v}) = T_{\gamma\alpha\delta} u^\alpha v^\delta$$

Special tensors:

- The Levi-CIvita symbol

$$\varepsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{even permutations of } 0,1,2,3 \\ -1 & \text{odd } \quad '' \quad\quad '' \\ 0 & \text{otherwise} \quad \text{(i.e. same index)} \end{cases} \qquad \text{starting with } \varepsilon_{0123} = 1$$

$$\text{e.g. } \varepsilon_{0123} = 1 \quad \varepsilon_{0\underbrace{21}3} = -1 \quad \varepsilon_{0\underbrace{321}} = -1 \quad \varepsilon_{\underbrace{10}23} = -1 \ \dots \ \varepsilon_{\underbrace{10}\underbrace{32}} = 1 \ \dots\dots$$

$$\varepsilon_{\nu 12 \nu} = -\varepsilon_{\nu 12 \nu} \Rightarrow \varepsilon_{\nu 12\nu} = 0 \quad \nu = 0,1,2,3$$

$$\text{e.g. of application: } \bar{\nabla} \times \bar{B} = \varepsilon^{ijk} \partial_j B_k = (\partial_2 B_3 - \partial_3 B_2)\, \bar{e}_1 + (\partial_3 B_1 - \partial_1 B_3)\, \bar{e}_2 + (\partial_1 B_2 - \partial_2 B_1)\, \bar{e}_3$$

is invariant under Lorentz transformations in flat space-time (not in general)

- Kroneker delta

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{The same in all coordinate systems and } \forall \text{ space-time}$$

Tensors are maps $\Rightarrow (\delta^i_j)$ is the identity map from vectors to vectors

$\delta : V \to V \quad \delta(\bar{v}) = \bar{v} \quad \delta^j_i v^i = v^j$ it "selects" a component without changing it

Pseudo tensors
e.g. e^{iklm} completly anti-symmetric unit tensor Landau pg. 34 (in flat)
pg 3-4 curvilinear

Space of all smooth scalar functions on M

$F = \{\phi \mid \phi : V \to K\}$

$\phi : V \to K \qquad \phi(\bar{v}) = a \quad \bar{v} \in V \; a \in K \qquad \phi$ linear map

Derivative : $\quad d : F \to \mathbb{R}$ linear map

(I) $\quad d(af+bg) = a\,d(f) + b\,d(g) \qquad f, g \in F \quad a, b \in K$

(II) $\quad d(fg) = d(f)\,g + f\,d(g)$

eg.
$f \in F \quad \underline{f(\bar{v}) \equiv c} \quad \forall \bar{v}$ (constant function)

(I) $\Rightarrow \quad d(c \cdot f) = c\,d(f)$

(II) $\Rightarrow \quad \underline{d(f^2)} = d(ff) = d(f) \cdot f + f\,d(f) = 2f\,d(f) = 2c\,d(f) \underline{= 2\,d(cf)}$

but $d(f^2) = d(cf) \quad \Rightarrow \quad 2c\,d(f) = c\,d(f)$ true only if $d(f) = 0$

$\Rightarrow$ derivatives of const. functions are zero

$\leadsto$ derivatives have the structure of a vector space Mathias 17

Part III

Curved space-time

Manifods and the tangent space: summary

We deal with a real n-dimensional, C^∞, manifold, M wich we equip with an atlas

1) Manifold : continuous set of points $\{P\}$; with a subset of open balls $\{O_\alpha\}$ covering M
- continuous each point P has infinitely close nei bours
- differentiable (C^∞) i.e smooth hypersurfaces
- covering: each $P \in M$ lies in at least one O_α

2) Chart / coordinate system:

$\forall P \in M \; \exists$ homomorphism $h: O \to U \subset \mathbb{R}^n$ $\quad h(P) = \bar{x}$ $\quad$ chart $= (O, h)$

3) C^∞Atlas: 1) $O_1 \cup O_2 \cup \dots O_n = M$

2) if $O_\alpha \cap O_\beta \neq \emptyset \Rightarrow \forall P \in O_\alpha \cap O_\beta \quad h_\alpha \circ h_\beta^{-1}(x^\mu) = P$

Maximal atlas: atlas containin all possible compatible charts

Differentiation on a manifold

map $F: M \to M'$ $\quad F(P) = P'$ $\quad$ 2 manifolds: M, M' $\quad P \in M \quad P' \in M'$

$\partial_\mu f(x^\nu) = \partial_\mu (h' \circ F \circ h^{-1})(x^\nu) \quad (x^\nu) \in U_\alpha \subset \mathbb{R}^m \quad (x'^\nu) \in U'_\beta \subset \mathbb{R}^{m'}$

Directional derivatives and vectors

$\mathcal{F} = \{\phi : M \to \mathbb{R}, C^\infty\}$ $\qquad$ space of all smooth scalar functions on M

$\varphi(\lambda) \equiv (\phi \circ h^{-1}) \circ (h \circ \gamma)(\lambda) = \varphi(\bar{x}(\lambda))$ $\qquad$ parametric curve $\gamma: \mathbb{R} \to M$

$$\frac{d\varphi(\lambda)}{d\lambda} = \frac{d}{d\lambda}\Big[(\phi \circ h^{-1}) \circ \underbrace{(h \circ \gamma)}_{\bar{x}(\lambda)}(\lambda)\Big] = \frac{\partial(\phi \circ h^{-1})}{\partial x^\nu} \frac{dx^\nu(\lambda)}{d\lambda} \Rightarrow \frac{d}{d\lambda} = \frac{dx^\nu}{d\lambda} \partial_\nu$$

linear map $\frac{d}{d\lambda}: \mathcal{F} \to \mathbb{R}$ obaying Leibnitz product rule

Tangent space $\quad M_pT \equiv \{\frac{d}{d\lambda}\}$ has structure of a vector space

Coord. transformation $\quad \bar{x}(\bar{x}') \Rightarrow \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu = \frac{\widetilde{\partial x^\mu}}{\partial x^{\nu'}} \frac{\widetilde{dx^{\nu'}}}{d\lambda} \partial_\mu \qquad \partial_{\nu'} = \frac{\partial x^\mu}{\partial x^{\nu'}} \partial_\mu$ transformation of basis

Dual vectors

$v^*: M_pT \to \mathbb{R}$ linear map $\qquad v^* \in M_pT^*$ dual space of M_pT / cotangent space

- defined as: $\left.\begin{array}{l} \{e_\mu\} \text{ basis set of } M_pT \\ \{\omega^\nu\} \;\; '' \;\; '' \;\; '' \;\; M_pT^* \end{array}\right\}$ $\quad \omega^\nu(e_\mu) \overset{!}{=} \delta^\nu_\mu$

or

- defined as: $v^* \equiv g(v, -)$ with metric $g: M_pT \to M_pT^*$ (!) no need to specify $\{e_\mu\}$

4) Induce a metric

g: as a linear map $\quad g(\cdot, -): T_p \to T_p^* \quad g(\bar{v}, -) = \tilde{v} \quad \bar{v} \in T_p \quad \tilde{v} \in T_p^* \qquad v_i = g_{ij} v^j$

g: as a bilinear map $\quad g(\cdot, \cdot): T_p \times T_p \to \mathbb{R} \quad g(\bar{v}, \bar{u}) \equiv \langle \bar{v}, \bar{u} \rangle$

5) Introduce the affine connectin, covariant derivatives...

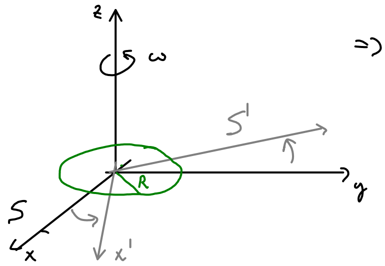
A theory of gravity based on differential geometry: General Relativity

- **Equivalence principle** → gravitational field <=> non inertial frame

- **Frame**
  - → Inertial ~ Minkowski metric η (Euclidean space)
  - → Non-inertial ~ "generic" metric g (curved space)

- Recall $\phi \propto r^{-1}$ and Gauss theorem

e.g. rotating frame : around fixed z-axis
ω = angular velocity



=> measure the circonference of a circle in the x-y plane
S = inertial ($\omega = 0$) $c = 2\pi R$
S = non-inertial ($\omega \neq 0$) $c \neq 2\pi R$! "curved space"
↑
because of Lorentz contraction

Can you recall the relationship between $\phi \propto r^{-1}$ and the Gauss theorem?

↳ in flat space, because $\phi \propto r^{-1}$ spheres have surface πr^2

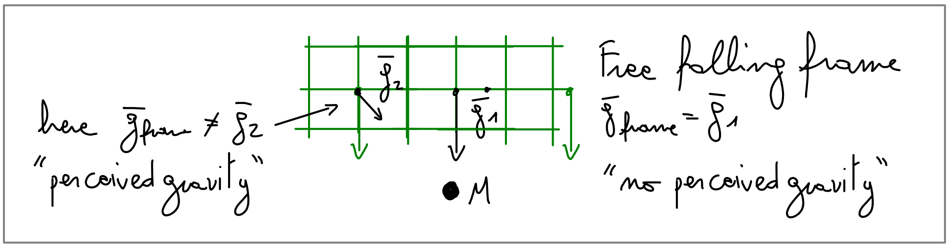
↳ But in curved space, surface of sphere $\neq \pi r^2$
=> deviation from $\phi \propto r^{-1}$

=> We need to describe curved spaces!!

Particle in flat s. experiencing a force → free falling particle in curved s.
Gravitational field ~ curved space-time characterized by g

- **Space-Time** :

flat → with a transformation you can get a Minkowski metric everywhere
curved → Such transf. not possible everywhere, only locally : non-Euclidean metric
↳ frame comoving with a free falling observer



- As gravitational fields, g depends on the position x^μ : $g(x^\mu)$!

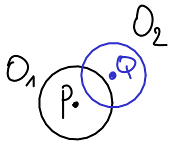
Curved spaces

We deal with a real n-dimensional, C^∞, manifold, M wich we equip with an atlas

1) **Manifold** : continuous set of points $\{P\}$ with a subset of open balls $\{O_\alpha\}$ covering M

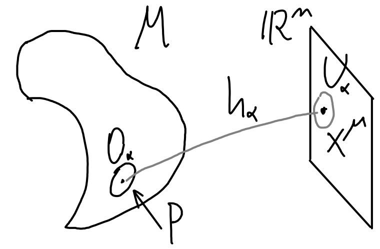
$O_\alpha \subset M$

- continuous : each point P has infinitely close neighbours
  scalar functions $\phi(P)$ can be defined as continuous
- differentiable (C^∞) : exist scalar functions $\phi(P)$ ∞ differentiable everywhere :
  $\exists \; \delta^r \phi(P) \in K \quad r=0,\dots\infty \quad \forall P \in M$
  i.e smooth hypersurfaces, no discontinuities
- covering : each $P \in M$ lies in at least one O_α



2) **Chart / coordinate system:**

you can define coordinate systems on manifolds because manifolds are locally homomorphic to $\mathbb{R}^n$ (because manifolds are homogeneous)



(1 to 1 invertible function)

i.e.: $\forall P \in M \; \exists \; h: O \to U$ homomorphism

O_α = open ball, $P \in O_\alpha$ (open set in M)

$U_\alpha \subset \mathbb{R}^n$ (open set in $\mathbb{R}^n$)

$h_\alpha(O_\alpha) = \{x^\mu\}$ image of h

n = dimension (M) # of indep parameters

$\text{chart}_\alpha = (O_\alpha, h_\alpha)$

- in general there might not be one non-degenerate coordinate system to cover the entire manifold, e.g. spherical coordinates on a sphere -> divergent at the pole but... you can use coordinate patches

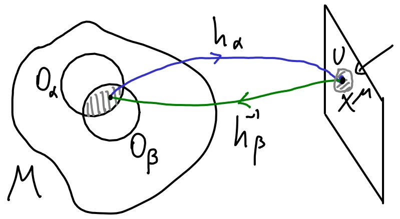
- one chart might not be sufficient to cover a the manifold => need a "collection" of charts

3) **C^∞ Atlas:**

1) $O_1 \cup O_2 \cup \dots O_r = M$ (i.e. $\{O_\alpha\}$ open balls covering M)

2) if $O_\alpha \cap O_\beta \neq \emptyset$ (i.e. they overlap) $\Rightarrow$ $\forall \; P \in O_\alpha \cap O_\beta \quad h_\alpha \circ h_\beta^{-1}(x^\mu) = P$

i.e. you have redundancies but no "inconsistencies"



same image $h_\alpha(U_\alpha \cap U_\beta) = h_\beta(U_\alpha \cap U_\beta)$

- **Maximal atlas** : atlas containin all possible compatible charts, i.e. all possible coord. systems

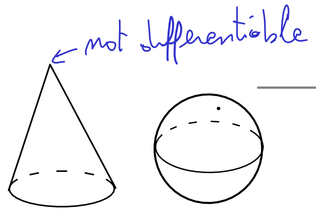
- You can have of course complex manifolds, homomorphic to $\mathbb{C}$
- We can induce a topology on the manifolds by requiring the maps h_α to be homeomorphysms

↳ here we would consider Hausdorf and paracompact topological spaces see eg. General Relativiti (Wald, appendix A)

Example of manifolds

- $\mathbb{R}^n$ you never bothered to use the machinery above because it is not necessary

Example: NOT a manifold: surface of a cone
Example: Euclidean space, surface of a sphere



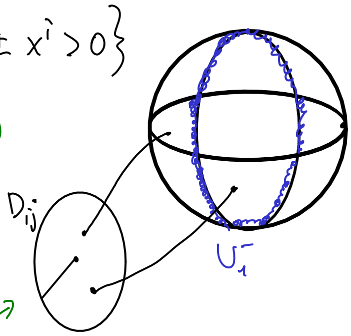
Sphere : $S^2 \equiv \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$

$= U_1^+ \cup U_1^- \cup U_2^+ \cup U_2^- \cup U_3^+ \cup U_3^- \quad U_i^\pm \equiv \{(x^1, x^2, x^3) \in S^2 \mid \pm x^i > 0\}$

Charts : $D_{ij} = \{(x^i, x^j) \in \mathbb{R}^2 \mid (x^i)^2 + (x^j)^2 < 1, i \neq j\}$ (6 half spheres)

e.g. $h_1^+ : U_1^+ \to D_{23} \quad h_1^+(x^1, x^2, x^3) = (x^2, x^3)$

$h_i^{+/-}$ projections of $U_i^{+/-}$ on $D_{ij} \in \mathbb{R}^2$



the sphere surface is a 2D object

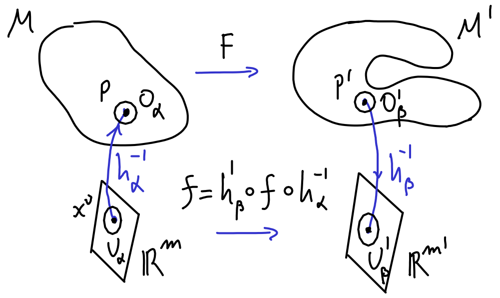
In the 1° definition of S^2:

(x^1, x^2, x^3) = 3 degrees of freedom
$(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$: 1 constrain
$\Big\}$ 2 d.o.f. i.e. 2 parameters, the chart makes it obvious

- For visual purposes here we are embedding S^2 in a 3D space but this is not necessary, S^2 is an object in itself !!!!!!!!

• Differentiation on a manifold

- Not obvious how... manifolds are 'just' a set of points => how can you define derivatives?
- Manifolds are locally like $\mathbb{R}^m$ (isomorphic) => you can differenciate and integrate on a manifold



2 manifolds: M, M' (prime "'" labels ≠ manifolds)

map $F : M \to M'$ $\quad F(P) = P'$

$g \circ F \circ h^{-1} : \mathbb{R}^m \to \mathbb{R}^{m'}$ $\quad f(x^\nu) = (h' \circ F \circ h^{-1})(x^\nu) \equiv h'[F(h^{-1}(x^\nu))]$

$\partial_\mu f(x^\nu) = \partial_\mu (h' \circ F \circ h^{-1})(x^\nu)$ $\quad \bar{x} \in U_\alpha \subset \mathbb{R}^m$ $\quad \bar{x}' \in U'_\beta \subset \mathbb{R}^{m'}$

- the chart allows to compute derivatives with respect to a frame
- F is C^∞ if $\forall\, h_\alpha, h'_\beta$ the map $h'_\beta \circ f \circ h_\alpha^{-1} : U_\alpha \to U'_\beta$ if is C^∞ in the sense of advanced calculus

• Directional derivative

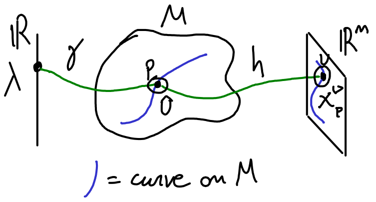
Derivative of a scalar function along a path on M crossing $P \in M$

$\mathcal{F} = \{ \phi : M \to \mathbb{R}, C^\infty \}$ space of all smooth scalar functions on M

$\phi(P) = a \in \mathbb{R}$ $\quad \phi : M \to \mathbb{R}$ smooth scalar function (here $M' = \mathbb{R}$)

$\gamma(\lambda) = P_\lambda$ $\quad \gamma : \mathbb{R} \to M$ parametric curve $C = \{\gamma(\lambda) \in M\}$ $C \subset M$

$h(P) = x_P$ $\quad h : O \to U \subset \mathbb{R}^m$ omomorphysm of the chart (O, h)



$$\boxed{\varphi(\lambda) = \overset{(2)}{(\phi \circ h^{-1})} \circ \overset{(1)}{(h \circ \gamma)}(\lambda) = \varphi(x^\nu(\lambda))}$$

function expressed as a function of λ along the curve

(1) $\bar{x}(\lambda) = (h \circ \gamma)(\lambda)$ Curve γ mapped in $\mathbb{R}^m$

(2) $\hat{\phi}(x^\nu) = (\phi \circ h^{-1})(x^\nu)$ function ϕ "expressed as a function of x^ν"

$$\boxed{\frac{d\varphi(\lambda)}{d\lambda} = \frac{d}{d\lambda}\Big[(\phi \circ h^{-1}) \circ \underbrace{(h \circ \gamma)}_{\bar{x}(\lambda)}(\lambda)\Big] \overset{*}{=} \underbrace{\frac{\partial(\phi \circ h^{-1})}{\partial x^\nu}}_{\partial_\nu} \frac{dx^\nu}{d\lambda}(\lambda) \equiv \frac{dx^\nu}{d\lambda} \partial_\nu(\phi)}$$

Derivative of ϕ along the path and evaluated in P

* chain rule

ϕ is arbitrary => $\boxed{\frac{d}{d\lambda} = \frac{dx^\nu}{d\lambda} \partial_\nu}$ $\quad \frac{d}{d\lambda} : \mathcal{F}_P \to \mathbb{R}$ directional derivative at a point P

They tell you how ϕ changes along the the curve γ, i.e. direction tangent to the curve

In Euclidean 3D space you might have seen $\nabla_{\vec{v}} \equiv \vec{v} \cdot \vec{\nabla} = v^\nu \partial_\nu$

Vectors as directional derivatives: the tangent space

- How do we define vectors on a manifold? Not obvious
- Convenient construction : vector = directional derivative

space of all smooth scalar functions ϕ on M $\quad \mathcal{F}=\{\phi: M\to\mathbb{R}, C^\infty\}$ $\quad \phi(P)=a\in\mathbb{R}$ $\quad P\in M$

$M_pT \equiv \{\frac{d}{d\lambda}\}$ $\qquad$ Tangent vector space defined in $P\in M$

$\frac{d}{d\lambda}\equiv v:\mathcal{F}\to\mathbb{R}$ $\qquad \underset{\uparrow \frac{d}{d\lambda}}{v} = \underset{\uparrow \frac{d}{dx^\nu}}{v^\nu \delta_\nu}$

Vector: (a) linear map obeying (b) Leibnitz product rule

$\{\delta_\nu\}$ $\nu=1,\dots,n$ $\quad n=\dim(M)$ a good basis set for M_pT

- Properties of vectors (derivatives)

1) Linear map : $\qquad v(\alpha f+\beta g)=\alpha v(f)+\beta v(g) \qquad f,g\in\mathcal{F} \qquad \alpha,\beta\in\mathbb{R}$

2) Leibnitz product rule: $\qquad v(fg)=f v(g)+g v(f)$

because they are derivatives... eg. $v(fg)=\frac{d}{d\lambda}(fg)=f\frac{dg}{d\lambda}+g\frac{df}{d\lambda}=fv(g)+gv(f)$

- They behave as vectors

Given $\frac{d}{d\lambda},\frac{d}{d\gamma}$ dir. deriv. , is $\frac{d}{d\eta}\equiv\alpha\frac{d}{d\lambda}+\beta\frac{d}{d\gamma}$ a directional derivative? Check...

$$\frac{d}{d\eta}(fg)=\left(\alpha\frac{d}{d\lambda}+\beta\frac{d}{d\gamma}\right)(fg)=\alpha\frac{d(fg)}{d\lambda}+\beta\frac{d(fg)}{d\gamma}\overset{(2)}{=}\alpha f\frac{dg}{d\lambda}+\alpha g\frac{df}{d\lambda}+\beta f\frac{dg}{d\gamma}+\beta g\frac{df}{d\gamma}$$

$$=f\underbrace{\left(\alpha\frac{dg}{d\lambda}+\beta\frac{dg}{d\gamma}\right)}+g\underbrace{\left(\alpha\frac{df}{d\lambda}+\beta\frac{df}{d\gamma}\right)}=f\frac{dg}{d\eta}+g\frac{df}{d\eta} \Rightarrow \frac{d}{d\eta} \text{ obey Leibnitz rule } \checkmark \text{ yes}$$

- Looking inside the ingredients:

$$\frac{d\varphi(\lambda)}{d\lambda}=\frac{d}{d\lambda}\Big[(\phi\circ h^{-1})\circ\underbrace{(h\circ\gamma)}_{\bar{x}(\lambda)}(\lambda)\Big]\overset{*}{=}\underbrace{\underbrace{\frac{\delta(\phi\circ h^{-1})}{\delta x^\nu}}_{\delta_\nu}}\underbrace{\frac{dx^\nu(\lambda)}{d\lambda}}\equiv\frac{dx^\nu}{d\lambda}\delta_\nu(\phi) \qquad \forall\phi \Rightarrow \boxed{v=v^\nu\delta_\nu}$$

- ν-th component : $\boxed{v^\nu=\frac{dx^\nu}{d\lambda}\equiv\frac{d}{d\lambda}\underbrace{(h\circ\gamma)}_{(x^\nu)}\Big|_{\lambda_P}}$ real number depending on the curve γ and the frame h

- ν-th basis : $\boxed{\delta_\nu(-)\equiv\frac{\delta(-\circ h^{-1})}{\delta x^\nu}\Big|_{h(P)}}$ vector depending on the frame h $\qquad \delta_\nu:\mathcal{F}\to\mathbb{R}$
  (derivative along the x^ν "axis")

- v^ν "how fast" x^ν is changing with respect to $d\lambda$: $dx^\mu=v^\mu d\lambda$ (shift)
- $\{\delta_\nu\}$ linearly independent tangent vectors spanning M_pT $\Rightarrow$ basis set (not proven here)
- here you see that the vector itself $(\frac{d}{d\lambda})$ does not depend on h : it contains $h\circ h^{-1}$ = identity
  only the basis and the components do ("compensating each other", $h\circ h^{-1}$)

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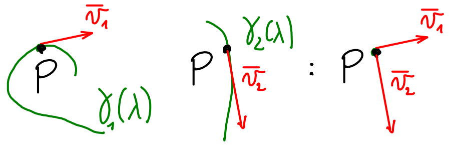
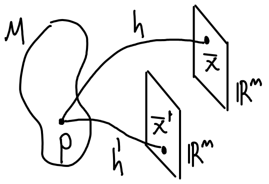


might look more familiar to you: $\bar{v}=v^\nu\bar{\delta}_\nu$ $\quad$;)

Generic coordinate transformations come for free from Leibnitz product rule



- ∂_μ and $\partial_{\mu'}$ coordinate basis based on the chart h and h'

- Coordinate Transformations:

$$\bar{x}' = (h' \circ h^{-1})(\bar{x}) \equiv \bar{x}'(\bar{x})$$

$$\bar{x} = (h \circ h'^{-1})(\bar{x}') \equiv \bar{x}(\bar{x}')$$

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu \underset{\uparrow \text{ Chain rule}}{=} \widetilde{\frac{\partial x^\mu}{\partial x^{\nu'}}} \underset{\uparrow \text{ transformed component}}{\widetilde{\frac{dx^{\nu'}}{d\lambda}}} \partial_\mu \qquad \partial_{\nu'} = \frac{\partial x^\mu}{\partial x^{\nu'}}\partial_\mu \quad \underline{\text{transformation of basis}}$$

- For vector components and tensors all what we have learned holds

e.g. $$\underbrace{\boxed{v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\alpha} v^\alpha}}_{\text{evaluated in P}} \qquad \underbrace{\boxed{T_{\mu'\nu'}(P) = \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}} T_{\alpha\beta}(P)}}_{\text{evaluated in P}}$$

1-forms as Differentials

$\phi : \mathcal{M} \to \mathbb{R}$ scalar function $\quad \phi \in \mathcal{F}(\mathcal{M})$

$\tilde{w} : T_{P\mathcal{M}} \to \mathbb{R} \qquad \tilde{w} : \bar{v} \to \tilde{w}(\bar{v}) \equiv \bar{v}[\phi] \qquad \bar{v} \in M_pT \qquad \tilde{w} \in M_pT^*$ 1-form

defined by its action on a vector

in analogy to vectors we want: $\quad \tilde{w} = w_\mu \tilde{\omega}^\mu$ with $\tilde{w}(\bar{v}) = w_\mu v^\mu$

we had: $\quad \frac{d\varphi(\lambda)}{d\lambda} = \left.\frac{\delta(\phi \circ h^{-1})}{\delta x^\nu}\right|_p \left.\frac{dx^\nu(\lambda)}{d\lambda}\right|_p = a \in \mathbb{R}$

for consistency $\quad w^\nu \equiv \left.\frac{\delta(\phi \circ h^{-1})}{\delta x^\nu}\right|_{h(P)}$ component of the diferential $d(\overbrace{\phi \circ h^{-1}}^{f(x)})$

$$d(\phi \circ h^{-1})|_p = \left.\frac{\delta(\phi \circ h^{-1})}{\delta x^\mu}\right|_{h(P)} (dx^\mu)|_p \qquad (dx^\mu)|_p \equiv dx^\mu \text{ basis of 1-forms}$$

$$\tilde{w} = w_\mu \, dx^\mu$$

=> 1-forms as differentials

Cotangent vectors = dual vectors = 1-forms

All things you already know... just under another angle

$V = M_pT$ for simplicity

- Introduce $\qquad v^* : V \to \mathbb{R}$ linear map
- $V^* = \{v^*\}$ has structure of a vector space $\qquad$ dual space of V
- Assume a basis for $V = \{v\}$ $\qquad \{e_\mu\}$ basis set of M_pT
- Define basis for v as $\qquad \{\omega^\mu\}$ " " " $M_pT^* \mid \omega^\nu(e_\mu) \overset{!}{=} \delta^\nu_\mu = \begin{cases} 1 & \nu = \mu \\ 0 & \nu \neq \mu \end{cases}$
- One can show $\qquad \dim(V) = \dim(V^*)$
- correspondance $\omega^\nu \leftrightarrow e_\nu$ => $\qquad$ isomorphism between V and V^*

↓

but it depends on the choice of basis set $\{e_\nu\}$

⇓

"need" to specify a new structure, 2 possibilities:

just set a preferred basis or induce a metric

↑

- Do you see? The metric is not part of the manifold!!
- The metric specifies the isomorphysm between V and V^* and allow us to measure norms, distances, volumes, ...

4) Inducing a metric on M:

- Equip M with a metric:

g : as a linear map $g(\cdot,-) : T_p \to T_p^*$ $g(\bar{v},-) = \tilde{v}$ $\bar{v} \in T_p$ $\tilde{v} \in T_p^*$ $v_i = g_{ij} v^j$

g : as a bilinear map $g(\cdot,\cdot) : T_p \times T_p \to \mathbb{R}$ $g(\bar{v},\bar{u}) \equiv \langle \bar{v},\bar{u} \rangle$

$ds^2 \lessgtr 0$ => here pseudo-Riemanian space

orthonormal basis $\{e_\mu\}$ of M_pT if $g(e_\mu, e_\nu) = \begin{cases} 0 \\ \pm 1 \end{cases}$ in $P \in M$

- To define distances, volumes, norms of vectors
- The metric is choosen to fit the physics we want to reproduce
- M_pT has a "flat geometry" ---> Minkowski metric (locally!)
  Minkowski because Special Relativity must hold in M_pT once a suitable frame is choosen (free fall)
- Careful! g is induced in P, but all P are in general different vector spaces => g depends on P! $g_{\mu\nu}(x^\alpha)$
- In a real gravitational field, it is IMPOSSIBLE to define one transformation such that $g(P) = \eta \ \forall \ P \in M$
- You can do it for each P separately, each P with its own coordinate transformation

* * *

- All of this was referring to each single points $P \in M$ separately...
  given another point $q \in M$ no way to associate M_pT with M_qT i.e. $\bar{v}_p + \bar{v}_q$??
  => Need an additional structure : the connection => parallel transport from P to q

* * *

5) The affine connection, covariant derivatives

- Every point on a manifold has its own T_p
- I.e. to each point is associatate its OWN vector space
- How do we relate vectors belonging to different vector spaces??
- We make invinitesimal 'moves' -> differential geometry
- Parallel transpor to "move vectors around"
- This is given by the affine connection, see covariant derivatives later on...

Abstract index notation

- The concept:

no need to specify a basis

True tensorial equations holds in any frame

they are not just equations of components of tensors specified by a frame

latin letters are used to "say" : arbitrary basis => "real" tensorial eq. valid in all frames

greek letters " " " " : specific frame => eq of components of tensors valid in that frame

some frames are build to simplify eq.s by exploiting the symmetries of a specific system

eg. spherical symmetry: from $S(\vec{x})$ general to $S(|\vec{x}|)=S(r)$ only for spherical systems

- Example:

$T^{ab}{}_c \in \mathcal{T}(2,1)$ tensor of type $\binom{2}{1}$ it is like writing $\tilde{w} \in \mathcal{T}(0,1) \Leftrightarrow w_a$

labels a, b, c do not refer to a basis, they serve to remind the type of tensor

$T^{ab}{}_c$ is the tensor itself

$T^{\alpha\beta}{}_\gamma$ = components of tensor $T^{ab}{}_c$ in a specific frame given by chart (h, O)

- Roles:

$T^{ab}{}_a$ contraction

$G^{ab}{}_{cdf} = T^{ab}{}_c S_{df}$ outher product (the symbol $\otimes$ is omitted)

$g_{ab} v^b = v_a$ metric g_{ab} isomorphysm between v^a and v_b

$(g^{-1})^{ab} \equiv g^{ab}$ inverse of the metric, rank $\binom{2}{0}$ $\Rightarrow$ $g_{ab} g^{bc} = \delta^c{}_a$ $\delta^c{}_a$ map from V_p into V_p (swap of indexes)

$g^{ab} w_b = w^b$

$T_{ab} = T_{ba}$ symmetric

$T_{ab} = -T_{ba}$ anti-symmetric

$T_{(ab)}$ = symmetric part of T_{ab}

$T_{[ab]}$ = anti-symmetric part of T_{ab}

more generally

$$T_{(a_1 \dots a_m)} = \frac{1}{m} \sum_{\pi} T_{a_{\pi(1)} \cdots a_{\pi(\ell)}}$$

$$T_{[a_1, \dots a_\ell]} = \frac{1}{m} \sum_{\pi} \delta_\pi T_{a_{\pi(1)} \cdots a_{\pi(\ell)}}$$

$$\delta_\pi = \begin{cases} 1 & \text{even } \pi \\ -1 & \text{odd } \pi \end{cases}$$

$T_{a_1 \dots a_\ell} = T_{[a_1, \dots a_\ell]}$ totally antisymmetric tensor (differential ℓ-form)

$T^{(ab)c}{}_{[de]}$ same roles also for "grouped" indeces

"APPENDIX"

Recall: a vector space is a set, V, combined with a field, K, and equipped with an addition and a multiplication with scalars (if the sum and multiplication are distributive and associative, then the vector field is called Abelian).

$V = \{\bar{v}\}$

$+ : V \times V \to V \quad +(\bar{v}, \bar{w}) \to \bar{v} + \bar{w} = \bar{u} \quad \bar{v}, \bar{w}, \bar{u} \in V$ sum of vectors is a vector

$\cdot : K \times V \to V \quad \cdot(a, \bar{v}) \to a\bar{v} = \bar{u} \quad a \in K$ scalar · vector " " "

Zero vector $\in V$ (identity element for +)

A vector is not something that goes from A to B, is an abstract object in a certain point P !!

Tangent field $\bar{v}|_p$

Idea: assign one tangent vector $v|_p \in V_p$ at each point $P \in M$ in a "smooth way"

$M_pT \neq M_qT \quad P, q \in M$ but v vary smoothly from point to point, i.e.

f C^∞ function at each $P \in M$, $v|_p(f) \in M$ i.e. $v(f)$ is a function on M $\frac{df}{d\lambda}$

v is smooth if $\forall$ smooth f, $v(f)$ is also smooth

Construction of a Vector field Set up:

$\{\phi_t\}$ 1 parameter group of diffeomorphisms $\phi_t : M \to M$

$\{\phi_t\} : \mathbb{R} \times M \to M$ is a C^∞ map if for $t \in \mathbb{R}$ fixed, $\phi_t : M \to M$ diffeomorphism

for $t, s \in \mathbb{R}$ $\phi_t \circ \phi_s = \phi_{t+s}$ ($\Rightarrow$ $\phi_{t=0}$ = identity map: $\phi_0(P) = P$)

Define the vector of the field in P, $v|_p$:

for a $P \in M$ fixed, $\phi_t(P) : \mathbb{R} \to M$ is a curve passing through P at $t=0$, $\phi_0(P) = P$ (called orbit of ϕ_t)

$v|_p \equiv$ tangent to this curve at $t=0$

$\Rightarrow$ vector field $\longleftrightarrow$ $\{\phi_t\}$

Just a fancy way to generate curves on M though a diffeomorphysm

Vector field $v \to \{\phi_t\}$ The way around

find $\{\phi_t\}$ resulting in curves such to generate the $v|_p$ $\forall P \in M$

find curve solving the system $\frac{dx^\mu}{dt} = v^\mu(x^1, \dots, x^n)$ v^μ vector component in basis $\{\partial_\mu\}$

ordinary differential equations in $\mathbb{R}^n$ given a P, the solution is unique

Physical insights in the tangent space

- A tangent space can be associate to every $P \in M$
- Physical meaning: locally, manifolds have a Minkowski metric
- You need it: in any point you can have a free falling observer, i.e. you can set a reference frame such that no gravity is felt <-> flat space-time

  i.e. ∃a coordinate transformation that transforms the metric g in the vicinity of a point P into a diagonal form with only +1 and -1 as elementrs
- In curved space-time (i.e. with real grav. fields.) yu can not define a transformation that makes $g = \eta \ \forall P \in M$

— In P the S.-T. is Minkowski, in the surrounding is locally Minkowski

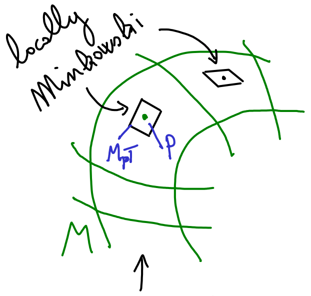
Given a point $P \in M$

1) g is symmetric ⇒ can be diagonalized: $g' = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ λ_i = eigenvalues

2) rescale the coordinates $x^{\mu'} \rightarrow \frac{x^{\mu'}}{\sqrt{|\lambda_\mu|}}$ ⇒ $g' = \text{diag}(\text{sign}(\lambda_0), \dots, \text{sign}(\lambda_3))$

$$\Rightarrow \boxed{g'_{\mu\nu}(\bar{x}') = \eta_{\mu\nu} + O[(\bar{x}' - \bar{x}'_p)^2]}$$

valid in the proximity of P $\bar{x} \in U_p$
(locally)

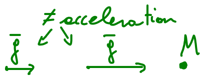


recall the elevator!

$$\underbrace{g_{\alpha\beta}(P) = \eta_{\alpha\beta} \quad g_{\alpha\beta,\gamma}(P) = 0}_{\text{Locally Minkowski at } 1^\circ \text{order}} \qquad \underbrace{g_{\alpha\beta,\gamma\mu}(P) \neq 0}_{\text{Tydal effects}}$$

2° order derivatives of $g \neq 0$

- The higher orders are responsible for the tydal effects



- The higher orders "hide" an intrinsic property of S-T that can not be removed by coord. transf.: the curveture!

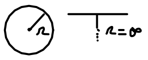
- The global geometry of the manifold is determined by the topology
  G.R. is a local theory ⇒ can not constrain the topology

Validity of the approximation to flat geometry

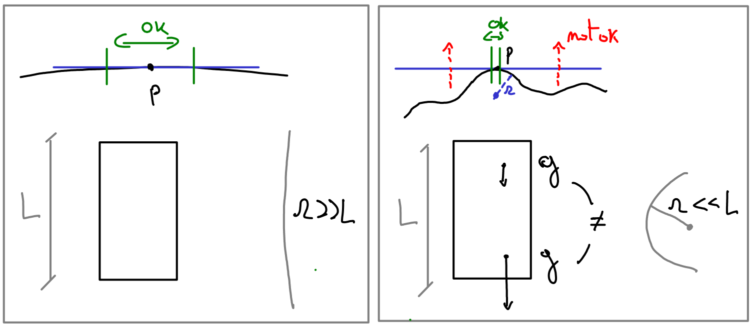
How far can you go from P, such that the approximation to flat space is valid?
It depends on the local curvature of the manifold

Note: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad [ds] = m \quad \Rightarrow \quad [g] = 1 \quad \Rightarrow \quad \left[\frac{\delta^2 g_{\mu\nu}}{\delta x^2}\right] = m^{-2}$

Define the curvature radius: $r(P) \equiv \left|\frac{\delta^2 g(P)}{\delta x^2}\right|^{-1/2}$ "$r \to \infty \Rightarrow$ flat space-time"



based on 2° derivatives of g in P because you can set $g_{\alpha\beta,\gamma}(P) = 0$
but 2° derivatives do not vanish if space has an intrinsic curvature



- large curvature (i.e. small curv. radius)
  $\Rightarrow$ visible tydal effects
- In different positions
  $\to$ different "grav. force"
  $\Rightarrow$ " accelerations
  $\Rightarrow$ not inertial motions

More on the impossibility to have a Minkowski metric simultaneously everywhere

- Impossible to bring the entire space to Lorentzian form and set cartesian coordinates everywhere
- Prove: $x^\mu(x^{\nu'})$ arbitrary transformation such that $g = \eta$ in P

  Taylor expand g and count the degrees of freedom (i.e. # of coefficints in expansion)

$$x^\nu(\bar{x}') = x^\nu(P) + \frac{\delta x^\nu}{\delta x^{\alpha'}}\bigg|_P (x^{\alpha'} - x^{\alpha'}_P) + \frac{1}{2}\frac{\delta^2 x^\nu}{\delta x^{\alpha'}\delta x^{\beta'}}\bigg|_P (x^{\alpha'} - x^{\alpha'}_P)(x^{\beta'} - x^{\beta'}_P) + \frac{1}{3!}\frac{\delta^3 x^\nu}{\delta x^{\alpha'}\delta x^{\beta'}\delta x^{\delta'}}\bigg|_P (-)(-)(-) + \ldots$$

$$\begin{array}{llll} & \uparrow & \uparrow & \uparrow \\ \text{d.f.f. of coeff.:} & n \cdot n = n^2 & n\,\frac{n(n+1)}{2} & n\,\frac{n(n+1)(n+2)}{3!} \\ & \wedge \quad \text{ok} \checkmark & = \quad \text{ok} \checkmark & \vee \quad \text{Not enough!!!} \\ \text{d.f.f. } \delta^2 g: & \frac{n(n+1)}{2} & \frac{n^2(n+1)}{2} & \frac{n(n+1)^2}{4} \\ & (g') & (\delta g') & (\delta^2 g') \end{array}$$

already at 3° order not enough constraints from $\delta^2 g'$ to fix the coefficients of $x^\nu(\bar{x}')$ to impose $g = \eta$ everywhere

$\Rightarrow$ FAIL! or the space is Lorentian from the start or you can not make it so

$\Rightarrow$ The curvature belongs to the manifold M, not to the coordinates!

B 23

Affine connectin and covariant derivative: Summary

- Problem: $\partial_a \phi \in \mathcal{T}(0,1)$, ϕ scalar but for $T \in \mathcal{T}(l,k)$ $\partial_a T \notin \mathcal{T}(l,k+1)$
  Need a new derivative to keep things covariant: $\nabla_a T \in \mathcal{T}(l,k+1)$
- Covariant derivative: we want it
  1) linear 2) Leibnitz rule 3) commute with contraction 4) preserves notation $v = v^a \nabla_a$
  5) torsion free (used in G.R. but optional)
- Not unique, Affine connection C: $\nabla_a v^c = \hat{\nabla}_a v^c + C^{b}{}_{ac} v^c$ $\nabla_c, \hat{\nabla}_c$ obeying rules above
- In flat space we want $\nabla_a = \partial_a$: $\nabla_a v^c = \partial v^c + \Gamma^c{}_{ab} v^b$ Γ: Christoffel symbols
- Examples: $\nabla_a \phi = \partial_a$; $\nabla_a w_b = \partial_a w_b - \Gamma^\gamma_{ab} w_\gamma$; $A_a{}^b{}_{;c} = A_a{}^b{}_{,c} - \Gamma^\gamma_{ca} A_\gamma{}^b + \Gamma^b_{c\gamma} A_a{}^\gamma$ ...
- Γ not a tensor: matrix of coefficients. Very important! This is why it works
- ∇_a and vector commutator: $[v,u] = (v^a \nabla_a u^b - u^a \nabla_a v^b)\nabla_b + \underbrace{(v^a u^b \nabla_a - u^a v^b \nabla_a)\nabla_b}_{=0 \text{ Torsion free}}$
- Torsion $T(\bar{v},\bar{u}) : M_pT \times M_pT \to M_pT$ $T(\bar{v},\bar{u}) \equiv \nabla_{\bar{v}} \bar{u} - \nabla_{\bar{u}} \bar{v} - [\bar{v},\bar{u}]$ $T(\bar{v},\bar{u})(-)$
- Torsion tensor $T : M_pT^* \times M_pT \times M_pT \longrightarrow \mathbb{R}$ $T(\bar{v},\bar{u},\tilde{w}) \to T(\bar{v},\bar{u})(\tilde{w})$ $T^\gamma{}_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}$
- Torsion free $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ $\Rightarrow$ $[v,u] = (v^a \nabla_a u^b - u^a \nabla_a v^b)\nabla_b$
  $T^c{}_{ab} = 0 \Leftrightarrow \Gamma^c_{ab} = \Gamma^c_{ba}$
- Metric compatibility $\nabla_a g_{bc} = 0$ $\Rightarrow$ $g(u,v) = \text{const}$ if $\nabla_{\bar{w}} u = 0\ \nabla_{\bar{w}} v$
  $v_{\mu;\beta} = g_{\mu\alpha} v^\alpha{}_{;\beta}$ (g: lowering on $\bar{v}$)
  $g_{\mu\nu,\beta} = g_{\gamma\nu} \Gamma^\gamma_{\beta\mu} + g_{\mu\gamma} \Gamma^\gamma_{\beta\nu}$
  $\partial_\beta g = g(\Gamma^\mu_{\beta\mu} + \Gamma^\nu_{\beta\nu})$ $g \equiv \det(g_{\mu\nu})$
- G.R.: $T=0$ + $\nabla_a g_{bc} = 0$ $\Rightarrow$ Γ is unique: $\Gamma^a_{bc} = \frac{1}{2} g^{a\gamma}(\partial_b g_{\gamma c} + \partial_c g_{b\gamma} - \partial_\gamma g_{cb})$

useful expressions
$$\left\{\begin{array}{l}
\Gamma_{abc} \equiv g_{a\gamma}\Gamma^\gamma_{bc} = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{cb}) \\
\Gamma^a_{ac} = \frac{1}{2g}\partial_c g = \frac{1}{2}\partial_c \ln g = \partial_c \ln(\sqrt{|g|}) \\
\nabla_a v^a = \frac{1}{\sqrt{|g|}} \partial_a(\sqrt{|g|} v^a)
\end{array}\right.$$

- There are many theories of gravity ---

Issue with "standard" derivatives

Partial derivatives are NOT good tensorial operators on a manifold

- We have seen in flat space $\partial_\mu T^{\nu\sigma} = S_\mu{}^{\nu\sigma}$ S = rank-3 tensor $\begin{pmatrix} M \\ N \end{pmatrix} \to \begin{pmatrix} M \\ N+1 \end{pmatrix}$
- But this is not the case on manifolds ... ⍨

  i.e. derivatives do not transform a tensor into another tensor

For example:

- $\phi \in \mathbb{R}$ $\partial_\mu \phi$ is a type $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor also on curved manifolds, OK!
- $\tilde{w} \in \hat{V}$ i.e. type $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor, is $\partial_i w_j$ a type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor? Check if it transforms as a tensor

coord transf.

$$\partial_{i'} w_{j'} = \left(\frac{\partial x^i}{\partial x^{i'}} \partial_i\right)\left(\frac{\partial x^j}{\partial x^{j'}} w_j\right) = \underbrace{\frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} w_j} + \underline{\frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial w_j}{\partial x^i}}$$

like a tensor of type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, "ok"

- with $\frac{\partial}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}} \neq 0$ (non linear transformation) $\partial_i w_j$ is NOT a tensor!
- it remains because the transformation now is not constant and its derivative is not vanishing (as it was the case for the Lorentz transf.)

- Same problem with higher rank tensors

- Do you remember the field eq. of the linear theory of gravity? $\Box \phi = 4\pi G \rho$

  we will get something like that but look... $\Box = \partial_\mu \partial^\mu$ standard derivatives!

- This is a problem if we want to build a covariant theory!

$\Rightarrow$ We need to define a new operator: the covariant derivative ∇_μ

- derivative operators are coordinate dependent

  i.e. not associated to the intrinsic structure manifold alone (a chart is involved)

Covariant derivative and affine connection

Linear map: takes each smooth tensor field $\binom{k}{l}$ to another smooth tensor field $\binom{k}{l+1}$ satisfying 4 (+1 optional) conditions

$\nabla_\alpha : \mathcal{T}(k,l) \to \mathcal{T}(k+1,l)$ $\mathcal{T}(k,l) = \{$tensors of type $\binom{k}{l}\}$ derivative along coordinate α

1) linearity : $\nabla_i(\alpha A + \beta B) = \alpha \nabla_i A + \beta \nabla_i B$ $A,B \in \mathcal{T}(k,l)$, $\alpha,\beta \in \mathbb{R}$
2) Leibnitz rule : $\nabla_i(AB) = A\nabla_i B + B \nabla_i A$ $A \in \mathcal{T}(k,l)$, $B \in \mathcal{T}(k',l')$
3) Commutativity with contraction : $\nabla_c\left(g_{er} A^{a_1 \dots e \dots a_m}{}_{b_1 \dots r \dots b_l}\right) = \nabla_i A^{a_1 \dots r \dots a_m}{}_{b_1 \dots r \dots b_l}$ $A \in \mathcal{T}(k,l)$
4) Notation consistency: tangent vectors as directional derivatives on scalar fields $v(f) = v^i \nabla_i f \in M_pT$
5) Torsion free (Optional!) : $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ $\forall f \in \mathcal{F}$ i.e. ∇_a commutes

Warning: abuse of notation, ∇_α not a 1-form! α just a label

Warning: for now we do NOT use (5)

This makes $\nabla_a v^b$ a tensor

- ∇ is not unique

- we just specified some rules. Anything that satisfies them is a covariant derivative.
- take 2 different covar. derivatives: ∇_a , $\hat{\nabla}_a$
- one can show that (using conditions 2, 4) $\boxed{\hat{\nabla}_a v^b = \nabla_a v^b + C^b{}_{ac} v^c}$ type $\binom{1}{1}$ tensor

$C^b{}_{ac}$ = connection coefficients / affine connection : possible disagreement between ∇ and $\hat{\nabla}$ depends on the properties of space captured by C

=> Lot of freedom to define the operator ∇, you can choose any $C^c{}_{ab}$!

64 coefficients Γ^b_{ac} → $n^3 = 4^3 = 64$ (in general)

later on, we will impose further constraints

- In flat space we want $\nabla_a = \partial_a$ => $\boxed{\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c}$ Γ = Christoffel symbol

$\nabla_a v^b \equiv v^b_{;a}$ in analogy to $\partial_a v^b = v^b{}_{,a}$

- Another way to look at it:

$\nabla_i = (\partial_i) + ($some linear transformation$)$ such that $\nabla_a T \in \mathcal{T}$

- Parallel transport: We can use it to set a condition to "move around" vectors

$u^a \nabla_a v^b \overset{!}{=} 0$ v transported along u u, v vectors

Transformation of Γ:

$$\nabla_{\beta'} v^{\mu'} = X^{\beta}_{,\beta'} X^{\mu'}_{,\mu} \nabla_\beta v^\mu = \underline{X^{\beta}_{,\beta'} X^{\mu'}_{,\mu} \partial_\beta v^\mu} + X^{\beta}_{,\beta'} X^{\mu'}_{,\mu} \Gamma^{\mu}_{\beta\gamma} v^\gamma$$

product rule

$$\nabla_{\beta'} v^{\mu'} = \partial_{\beta'} v^{\mu'} + \Gamma^{\mu'}_{\beta'\gamma'} v^{\gamma'} = (X^\beta_{,\beta'}\partial_\beta)(X^{\mu'}_{,\mu} v^\mu) + \Gamma^{\mu'}_{\beta'\gamma'} v^{\gamma'} = X^\beta_{,\beta'} v^\mu \partial_\beta X^{\mu'}_{,\mu} + \underline{X^\beta_{,\beta'} X^{\mu'}_{,\mu} \partial_\beta v^\mu} + \Gamma^{\mu'}_{\beta'\gamma'} X^{\gamma'}_{,\gamma} v^\gamma$$

$\mu \to \gamma$

$$\Rightarrow \quad X^\beta_{,\beta'} X^{\mu'}_{,\mu} \Gamma^\mu_{\beta\gamma} v^\gamma = X^\beta_{,\beta'} \partial_\beta X^{\mu'}_{,\mu} v^\mu + \underline{\Gamma^{\mu'}_{\beta'\gamma'}} X^{\gamma'}_{,\gamma} v^\gamma$$

$\forall v^\gamma$

$\cdot X^\gamma_{,\gamma'}$ apply inverse

$$\Rightarrow \quad \boxed{\Gamma^{\mu'}_{\beta'\gamma'} = X^\gamma_{,\gamma'} X^\beta_{,\beta'} X^{\mu'}_{,\mu} \Gamma^\mu_{\beta\gamma} - \underbrace{X^\gamma_{,\gamma'} X^\beta_{,\beta'} \partial_\beta X^{\mu'}_{,\gamma}}}$$

$=0$ only if $X^{\mu'}_{,\gamma\beta} = 0$ ⇒ Tensor only for linear transf.!

- This shows that Γ is NOT a $\binom{1}{2}$ tensor (it does not transform as such)

This is very relevant!

1) $\Gamma_a : (\Gamma_a)^b_c$ connection coefficient matrix to make things covariant "along a"
it "connects" vectors from one point to another point $Q: M_PT \Leftrightarrow M_QT$

2) Γ not a tensor is why it works in making $\nabla_a v^b$ a tensor
↳ the first term above "cancel" the non tensorial part that you would have in $\partial_\mu v^\nu$"

3) You can choose a transformation in which space is flat in a given $P \in M$
⇒ $\nabla_\mu = \partial_\mu$ i.e. $\Gamma^\alpha_{\mu\beta} = 0$ in that $P \in M$, not everywhere at the same time
but if Γ would be a tensor ⇒ Γ null $\forall P \in M$ but this can not be in general!
(a null Tensor is null everywhere)

Physics: frame of a free falling observer $g = \eta$ but if gravity ⇒ $g \neq \eta$ in other locations
Careful! $\Gamma \neq 0$ does not mean flat space : e.g. in flat space + spherical coord $\Gamma \neq 0$

- But... $\Gamma^\gamma_{\beta\alpha}$ with α = fixed are components of a $\binom{1}{1}$ tensor (β, γ free indeces)

in short: $\nabla_\beta \bar{e}_\mu \equiv \Gamma^\gamma_{\beta\mu} \bar{e}_\gamma$ $\quad \nabla_\beta \bar{e}_\mu \in \mathcal{T}(0,2)$ $\quad \bar{e}_\gamma \in \mathcal{T}(1,0)$ ⇒ $(\Gamma^\gamma_\beta)_\mu \in \mathcal{T}(1,1)$ μ = fixed

in long:

$\bar{v} = v^\mu \bar{e}_\mu$ $\qquad \nabla \bar{v} = \nabla_\beta v^\mu (\tilde{\omega}^\beta \otimes \bar{e}_\mu)$ $\qquad \nabla_\beta v^\mu = \partial_\beta v^\mu + \Gamma^\mu_{\beta\gamma} v^\gamma$

$\bar{e}_\mu = \delta^\alpha_\mu \bar{e}_\alpha$ $\qquad \nabla \bar{e}_\alpha = \nabla_\beta \delta^\mu_\alpha (\tilde{\omega}^\beta \otimes \bar{e}_\mu)$ ← $\nabla_\beta \delta^\mu_\alpha = \cancel{\partial_\beta \delta^\mu_\alpha} + \Gamma^\mu_{\beta\gamma} \delta^\gamma_\alpha = \Gamma^\mu_{\beta\alpha}$ ⟶ $\nabla_\beta \delta^\mu_\alpha = \Gamma^\mu_{\beta\alpha}$

$= \Gamma^\mu_{\beta\alpha} (\tilde{\omega}^\beta \otimes \bar{e}_\mu)$ ✓

$=0$

$\delta^\mu_\alpha = 0, 1 \Rightarrow \partial_\beta 0 = 0 = \partial_\beta 1$

it tells us how the basis change from one P to the next one" on the manifold M

⇒ it transforms as a $\binom{1}{1}$ tensor $\Gamma^{\mu'}_{\beta'\alpha} = X^{\mu'}_{,\mu} X^\beta_{,\beta'} \Gamma^\mu_{\beta\alpha}$

– This allows us to write: $\Gamma_{\mu\beta\alpha} \equiv g_{\mu\gamma} \Gamma^\gamma_{\beta\alpha}$ this is a bit stretching the notation

What about covariant derivatives of other objects?

- For a scalar (gradient):

$\boxed{\nabla_a \phi = \partial_a \phi}$ $\phi \in \mathbb{R}$ scalar function "ϕ do not have a basis ⇒ does not change"

Prove: $\phi = v_a w^a \in \mathbb{R}$: $\phi_{;b} \underset{\uparrow}{=} v_{a;b} \underset{\uparrow}{w^a} + v_a \underset{\uparrow}{w^a{}_{;b}} = v_{a,b} w^a - \cancel{\Gamma^{\gamma}_{ba} v_\gamma w^a} + v_a w^a{}_{,b} + \cancel{\Gamma^{a}_{b\gamma} v_a w^\gamma} = \phi_{,b}$

Leibnitz rule

$a \leftrightarrow \gamma$

- For vectors:

$\boxed{\nabla_a v^b \equiv v^b{}_{;a} = \partial_a v^b + \Gamma^b_{a\gamma} v^\gamma}$ covariant derivative of a vector

$\nabla \bar{v}$ is a $\binom{1}{1}$ tensor $\nabla \bar{v} = \nabla_\beta \tilde{\omega}^\beta \cdot v^\alpha \bar{e}_\alpha = \nabla_\beta v^\alpha (\tilde{\omega}^\beta \otimes \bar{e}_\alpha)$

- For 1-forms:

Take scalar $\phi = v_\alpha w^\alpha$

$$\begin{aligned} \phi_{;\beta} = \phi_{,\beta} &= v_{\alpha,\beta} w^\alpha + v_\alpha w^\alpha{}_{,\beta} && \leftarrow w^\alpha{}_{;\beta} = \underline{w^\alpha{}_{,\beta}} + \Gamma^\alpha_{\beta\gamma} w^\gamma \\ &= v_{\alpha,\beta} w^\alpha + v_\alpha w^\alpha{}_{;\beta} - v_\alpha w^\gamma \Gamma^\alpha_{\beta\gamma} && \leftarrow \alpha \leftrightarrow \gamma \text{ relabeling} \\ &= \underbrace{\left(v_{\alpha,\beta} - v_\gamma \Gamma^\gamma_{\beta\alpha}\right)}_{\Downarrow} w^\alpha + v_\alpha w^\alpha{}_{;\beta} && \leftarrow \phi_{;\beta} \overset{!}{=} v_{\alpha;\beta} w^\alpha + v_\alpha w^\alpha{}_{;\beta} \end{aligned}$$

(must respect Leibnitz product rule)

$\boxed{v_{\alpha;\beta} = v_{\alpha,\beta} - v_\gamma \Gamma^\gamma_{\beta\alpha}}$ covariant derivative of a 1-form

↑ (!)

$\nabla \tilde{v}$ is a $\binom{1}{1}$ tensor $\nabla \tilde{v} = \nabla^\beta \bar{e}_\beta v_\alpha \tilde{\omega}^\alpha = \nabla^\beta v_\alpha (\bar{e}_\beta \otimes \tilde{\omega}^\alpha)$

- For tensors:

$$\boxed{\begin{aligned} A^{ab}{}_{;c} &= A^{ab}{}_{,c} + \Gamma^a_{c\gamma} A^{\gamma b} + \Gamma^b_{c\gamma} A^{a\gamma} \\ A_a{}^b{}_{;c} &= A_a{}^b{}_{,c} - \Gamma^\gamma_{ca} A_\gamma{}^b + \Gamma^b_{c\gamma} A_a{}^\gamma \\ A^a{}_{b;c} &= A^a{}_{b,c} + \Gamma^a_{c\gamma} A^\gamma{}_b - \Gamma^\gamma_{cb} A_{a\gamma} \\ A_{ab;\beta} &= A_{ab,c} - \Gamma^\gamma_{ca} A_\gamma{}^b - \Gamma^\gamma_{cb} A_{a\gamma} \end{aligned}}$$

same "rule" as for 1-forms/vectors for the lower (upper) indeces and (+)/(−)

← $\nabla A = \nabla_\beta A^\mu{}_\nu (\tilde{\omega}^\beta \otimes \bar{e}_\mu \otimes \tilde{\omega}^\nu)$ type $\binom{1}{2}$ tensor

← Type $\binom{0}{3}$ tensor

- Tensors of higher rank: same "rules", you get a Γ for each index

Commutator of vectors

- $[v,u] \equiv \overset{\downarrow}{v}\,\overset{\downarrow}{u} - \overset{\downarrow}{u}\,\overset{\downarrow}{v} = v^a\partial_a(u^b\partial_b) - u^a\partial_a(v^b\partial_b) = (v^a\partial_a u^b - u^a\partial_a v^b)\,\partial_b$ $\qquad u,v\in M_pT$

$\boxed{[v,u]^b = v^a\partial_a u^b - u^a\partial_a v^b}$ b-th component of commutator

- From 2,4: express commutator of vectors in terms of ∇_a

$$v^a\nabla_a u^b - u^a\nabla_a v^b = v^a\partial_a u^b - u^a\partial_a v^b + v^a\Gamma^b_{ac}u^c - u^a\overset{a\leftrightarrow c}{\Gamma^b_{ac}v^c} = [v,u]^b + \underbrace{(\Gamma^b_{ac}-\Gamma^b_{ca})}_{\equiv T^b{}_{ac}\ \text{torsion}}v^a u^c$$

$\boxed{[v,u]^b = v^a\nabla_a u^b - u^a\nabla_a v^b - T^b{}_{ac}v^a u^c}$

- You can prove that : $\nabla_a\nabla_b f - \nabla_b\nabla_a f = -T^c{}_{ab}\nabla_c f$ => torsion free: $T=0 \Longleftrightarrow \nabla_a\nabla_b f = \nabla_b\nabla_a f$

Torsion map

$\boxed{\hat{T}(\bar v,\bar u) : M_pT\times M_pT \to M_pT \qquad \hat{T}(\bar v,\bar u)\equiv\nabla_{\bar v}\bar u - \nabla_{\bar u}\bar v - [\bar v,\bar u]}$ antisymmetric

Meaning:

2 vectors $u,v\in M_pT$, $\delta\lambda\in\mathbb{R}$ small

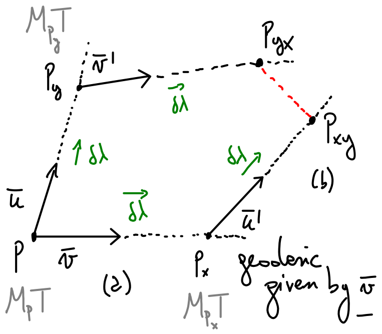
1) Parallel transport u along geodesic given by v ($\delta\lambda$ "shift): get $u'\in M_pT$

2) Run along geodesic given by u' ($\delta\lambda$ "shift): get point P_{xy}

Same for v to get P_{yx}

=> Torsion quantifies how far P_{xy} is from P_{yx}, i.e. how parallelograms close

$x^c(P_{xy}) - x^c(P_{yx}) = (T^c{}_{ab}v^a u^b)_p\,\delta\lambda$



Torsion Tensor

since $\hat{T}(a\bar v,b\bar u) = ab\,\hat{T}(\bar v,\bar u)$ we can define

$\boxed{T : M_pT^*\times M_pT\times M_pT \longrightarrow \mathbb{R} \qquad T(\tilde\omega,\bar v,\bar u)\equiv\tilde\omega\,\hat{T}(\bar v,\bar u)}$

- Components on basis $\{\tilde{dx}^\alpha\}$, $\{\bar\partial_\alpha\}$, $\delta^\nu_\alpha\nabla_\nu = \nabla_{\bar\partial_\alpha}\equiv\nabla_\alpha$ $\qquad$ recall: $\bar\partial_\alpha = (\delta^\beta_\alpha)$

$$T^\gamma{}_{\alpha\beta} = dx^\gamma_\nu\,\hat{T}^\nu(\partial_\alpha,\partial_\beta) = dx^\gamma_\nu\,(\overset{\swarrow}{\nabla_\alpha}\delta^\nu_\beta - \overset{\swarrow}{\nabla_\beta}\delta^\nu_\alpha - \partial_\alpha\delta^\nu_\beta + \partial_\beta\delta^\nu_\alpha)$$

$$= dx^\gamma_{\underset{\smile}{\nu}}\,(\cancel{\partial_\alpha\delta^\nu_\beta} + \Gamma^{\overset{\frown}{\nu}}_{\alpha\sigma}\delta^\sigma_\beta - \cancel{\partial_\beta\delta^\nu_\alpha} - \Gamma^{\overset{\frown}{\nu}}_{\beta\sigma}\delta^\sigma_\alpha - \cancel{\partial_\alpha\delta^\nu_\beta} + \cancel{\partial_\beta\delta^\nu_\alpha})$$

$dx^\gamma_\nu = \delta^\gamma_\nu$

$$= \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}$$

expresses anti-symmetry of $T^\gamma_{\alpha\beta}$

- $T^{k}{}_{ij}$ is indeed a tensor, check transformation

$$\Gamma^{\mu'}_{\beta'\gamma'} = x^{\gamma}_{,\gamma'} x^{\beta}_{,\beta'} x^{\mu'}_{,\mu} \Gamma^{\mu}_{\beta\gamma} - x^{\gamma}_{,\gamma'} x^{\beta}_{,\beta'} \partial_\beta x^{\mu'}_{,\gamma} \qquad (\Gamma \text{ not a tensor})$$

$$\Gamma^{\mu'}_{\beta'\gamma'} - \Gamma^{\mu'}_{\gamma'\beta'} = x^{\gamma}_{,\gamma'} x^{\beta}_{,\beta'} x^{\mu'}_{,\mu} \Gamma^{\mu}_{\beta\gamma} - \underline{x^{\gamma}_{,\gamma'} x^{\beta}_{,\beta'} \partial_\beta x^{\mu'}_{,\gamma}} - x^{\beta}_{,\beta'} x^{\gamma}_{,\gamma'} x^{\mu'}_{,\mu} \Gamma^{\mu}_{\gamma\beta} + \underline{x^{\beta}_{,\beta'} x^{\gamma}_{,\gamma'} \partial_\gamma x^{\mu'}_{,\beta}} \qquad \text{relabel } \gamma \longleftrightarrow \beta$$

cancel each other

$\partial_\gamma \partial_\beta$ commute

$$= x^{\gamma}_{,\gamma'} x^{\beta}_{,\beta'} x^{\mu'}_{,\mu} \left(\Gamma^{\mu}_{\beta\gamma} - \Gamma^{\mu}_{\gamma\beta}\right) \quad \checkmark$$

- Being a tensor, if $T \neq 0$ in one frame $\Rightarrow$ $T \neq 0$ $\forall$ frame
  T describes an intrinsic property of space!

- Decompose the connection in a symmetric connectin + torsion

$$\Gamma^{c}_{ab} = \Gamma^{c}_{(ab)} + \Gamma^{c}_{[ab]} \qquad \Gamma^{c}_{(ab)} \equiv \frac{1}{2}\left(\Gamma^{c}_{ab} + \Gamma^{c}_{bc}\right) \qquad \text{transforms as a connection}$$

$$\Gamma^{c}_{[ab]} \equiv \frac{1}{2}\left(\Gamma^{c}_{ab} - \Gamma^{c}_{bc}\right) = \frac{1}{2} T^{c}{}_{ab} \qquad \text{transforms as a tensor}$$

$$T = T^{c}{}_{ab}\left(\bar{e}_c \otimes \tilde{\omega}^a \otimes \tilde{\omega}^b\right) \qquad T \in \mathcal{T}(1,2) \text{ Torsion tensor}$$

- $\Gamma^{c}_{(ab)}$ transforms as a connection because $\Gamma^{c}{}_{ab}$ is a connection and $T^{c}{}_{ab}$ is a tensor

$$\Gamma^{c}{}_{ab} = \Gamma^{c}_{(ab)} + T^{c}{}_{ab} \qquad \Rightarrow \qquad \Gamma^{c}_{(ab)} = \Gamma^{c}{}_{ab} - \frac{1}{2} T^{c}{}_{ab}$$

$$\Gamma^{c'}{}_{(a'b')} = (*)\Gamma^{c}{}_{ab} + (**)\Gamma^{c}{}_{ab} - \frac{1}{2}(*)T^{c}{}_{ab} = (*)\left(\Gamma^{c}{}_{ab} - \frac{1}{2}T^{c}{}_{ab}\right) + (**)\Gamma^{c}{}_{ab}$$

$*$ = tensorial transf.
$**$ = non-tensorial transf

$\Rightarrow$ Can think of $\Gamma^{c}{}_{ab}$ as composed by a symmetric connection + Torsion (!)
"Torsion is the anti-symmetric part of a connection"

- If you whish, Torsion pops up from this decomposition

— ◦ —

- Torsion do not affect the geodesic eq. becaus of its anti-symmetry, see later

$$u^a \partial_a u^c + \Gamma^{c}_{ab} u^a u^b = 0 \qquad \Gamma^{c}{}_{ab} = \Gamma^{c}{}_{(ab)} + \frac{1}{2} T^{c}{}_{ab}$$

$$u^a \partial_a u^c + \Gamma^{c}_{(ab)} u^a u^b + \cancel{\frac{1}{2} T^{ab} u^a u^b} = 0 \qquad \underbrace{T^{ab}}_{\text{anty-symmetric}} \underbrace{u^a u^b}_{\text{symmetric}} = 0$$

— ◦ —

Torsion free: the 5th condition

$T^c_{ab} = 0 \longleftrightarrow \Gamma^c_{ab} = \Gamma^c_{ba} \longleftrightarrow \nabla_a \nabla_b f = \nabla_b \nabla_a f$ characteristic of the covar. derivative!

$\Rightarrow [v, u]^b = (v^a \nabla_a u^b - u^a \nabla_a v^b)$

$\Rightarrow$ Number of independent coefficients: $\Gamma^{\alpha}_{\beta\gamma}$ $\leftarrow N$ possibilities $\Rightarrow \frac{N^2(N+1)}{2}$

symmetric $\Rightarrow \frac{N(N+1)}{2}$ combinations

$N=2$: 6 symbols

$N=3$: 18 "

$N=4$: 40 "

Still many d.o.f.! The connection not uniquely defined, need another assumption!

- G.R. assumes $T=0$ but this is not necessary in general

Another approach investigate the torsion free condition

look for the condition for which we have $\phi_{;\alpha\beta} = \phi_{;\beta\alpha}$ $\phi \in \mathcal{F}$

take a 1-form $\phi_{,\alpha} = \nabla_\alpha \phi = \partial_\alpha \phi$, compute $\nabla_\beta \phi_{;\alpha} = \phi_{;\alpha\beta}$

$$\phi_{;\alpha\beta} = \cancel{\phi_{,\alpha\beta}} - \Gamma^\gamma_{\beta\alpha}\phi_{,\gamma} \overset{!}{=} \cancel{\phi_{,\beta\alpha}} - \Gamma^\gamma_{\alpha\beta}\phi_{,\gamma} = \phi_{;\beta\alpha} \iff \underbrace{\left(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}\right)} \phi_{,\gamma} = 0 \checkmark$$

<u>Torsion Tensor</u> $T^\gamma{}_{\alpha\beta} \equiv \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\gamma}$

Metric compatibility, a 6th condition

- So far we never used the metric!
- One could use Γ as a field... but "to much" freedom is left

=> Impose a new condition involving the metric g

Metric compatibility: $\boxed{Q_{\beta\mu\nu} \equiv \nabla_\beta g_{\mu\nu} \overset{!}{=} 0 \quad \forall P \in M}$

$\uparrow$ non-metricity tensor

like in flat space $\partial_\beta \underset{}{\overset{\text{const}}{\eta_{\mu\nu}}} = 0$!

Γ = metric compatible connection

Riemannian geometry (Cartan was the first)

Meaning:

$\bar{u}, \bar{v}$ parallely transported along w i.e. $\nabla_{\bar{w}} \bar{u} = 0 \quad \nabla_{\bar{w}} \bar{v} = 0$ any $\bar{w}, \bar{u}, \bar{v} \in M_pT$

it would be natural that their scalar product do not change i.e. $\nabla_{\bar{w}}(\bar{u}\,\bar{v}) \overset{!}{=} 0$

i.e. $\nabla_{\bar{w}}(\bar{v}\,\bar{v}) \overset{!}{=} 0$ no change in norm

$$w^a \nabla_a (g_{bc} u^b v^c) = u^b v^c w^a \nabla_a g_{bc} + g_{bc} v^c \underbrace{w^a \nabla_a u^b}_{=0} + g_{bc} u^b \underbrace{w^a \nabla_a v^c}_{=0} \overset{!}{=} 0 \Rightarrow u^b v^c w^a \nabla_a g_{bc} \overset{!}{=} 0 \ \checkmark$$

Consequences:

a) $\boxed{v_{\mu;\beta}} = (g_{\mu\alpha} v^\alpha)_{;\beta} = \underbrace{\cancel{g_{\mu\nu;\beta}}}_{=0} v^\alpha + g_{\mu\alpha} v^\alpha{}_{;\beta} = \boxed{g_{\mu\alpha} v^\alpha{}_{;\beta}}$ (g: lowering on $\bar{v}$)

b) $g_{\mu\nu;\beta} = g_{\mu\nu,\beta} - g_{\gamma\nu}\Gamma^\gamma_{\beta\mu} - g_{\mu\gamma}\Gamma^\gamma_{\beta\nu} \overset{!}{=} 0$ $\boxed{g_{\mu\nu,\beta} = g_{\gamma\nu}\Gamma^\gamma_{\beta\mu} + g_{\mu\gamma}\Gamma^\gamma_{\beta\nu}}$

metric compatibility condition

partial derivative of g in terms of Γ

c) $\det(g_{\mu\nu}) \equiv g$ $\boxed{\partial_\beta g} = \underbrace{g\, g^{\mu\nu}}_{\text{minors of } \det(g_{\mu\nu})} (\partial_\beta g_{\mu\nu}) = g\, g^{\mu\nu} (\underbrace{g_{\gamma\nu}}_{\delta^\mu_\gamma} \Gamma^\gamma_{\beta\mu} + \underbrace{g_{\mu\gamma}}_{\delta^\nu_\gamma} \Gamma^\gamma_{\beta\nu}) = \boxed{g(\Gamma^\mu_{\beta\mu} + \Gamma^\nu_{\beta\nu})}$

With (b) we can easily prove....

We can have a transformation such that in a point P we have $\Gamma^\alpha_{bc}(P) = 0$

$g_{\mu\nu} = \eta_{\mu\nu}$, $\partial_\beta g_{\mu\nu}|_P = 0$ at 1° order (free falling observer)

$\Rightarrow \partial_\beta g_{\mu\nu} = g_{\nu\gamma}\Gamma^\gamma_{\beta\mu} + g_{\gamma\mu}\Gamma^\gamma_{\beta\nu} = 0$ $)\cdot g^{-1} = (g^{\nu\gamma})$

$\Gamma^\gamma_{\beta\mu} + \delta^\nu_\mu \Gamma^\gamma_{\beta\nu} = 2\Gamma^\gamma_{\beta\mu} = 0 \Rightarrow \boxed{\Gamma^\gamma_{\beta\mu}(P) = 0}$ in cartesian coordinates!

$\Rightarrow \nabla_\mu = \partial_\mu$ as desired free falling = Minkowski => standard derivative

Note: This holds also without metric compatibility (!), you can prove it by explicitely transforming Γ

We now make the connection unique

1) $T^{\mu}_{\ \beta\nu} \equiv \Gamma^{\mu}_{\beta\nu} - \Gamma^{\mu}_{\nu\beta} \overset{!}{=} 0$ <u>torsion free</u>

2) $\nabla_\beta g_{\mu\nu} \overset{!}{=} 0$ <u>Metric compatibility</u>

$\Big\}$ we link Γ to the metric! $\Gamma \Leftrightarrow g$
G.R.

"3)" $g_{\mu\nu} = g_{\nu\mu}$ The metric is symmetric (intrinsic property, not an assumption)

$$\left[\begin{array}{l} \partial_b g_{ca} = \Gamma^{\gamma}_{bc} g_{\gamma a} + \Gamma^{\gamma}_{ba} g_{\gamma c} \\ \partial_c g_{ab} = \Gamma^{\gamma}_{ca} g_{\gamma b} + \Gamma^{\gamma}_{cb} g_{\gamma a} \\ \partial_a g_{bc} = \Gamma^{\gamma}_{ab} g_{\gamma c} + \Gamma^{\gamma}_{ac} g_{\gamma b} \end{array}\right.$$

from metric compatibility

by cycling the indeces

$$\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{cb} = \underbrace{\Gamma^{\gamma}_{bc} g_{\gamma a}}_{+} + \cancel{\Gamma^{\gamma}_{\underset{\leftrightarrow}{ba}} g_{\gamma c}} + \cancel{\Gamma^{\gamma}_{\underset{\leftrightarrow}{ca}} g_{\gamma b}} + \underbrace{\Gamma^{\gamma}_{cb} g_{\gamma a}}_{+} - \cancel{\Gamma^{\gamma}_{ab} g_{\gamma c}} - \cancel{\Gamma^{\gamma}_{ac} g_{\gamma b}} = 2\Gamma^{\gamma}_{bc} g_{\gamma a}$$

$$\Rightarrow \boxed{\Gamma^{a}_{bc} = \frac{1}{2} g^{a\gamma}\left(\partial_b g_{\gamma c} + \partial_c g_{b\gamma} - \partial_\gamma g_{cb}\right)}$$

$\gamma \leftrightarrow a$
$\cdot\, g^{a\gamma}$
$\left(g_{\gamma a} g^{\gamma b} = \delta^b_a\right)$

<u>Christoffel symbol (of the 2° kind)</u>

Christoffel / Levi-civita / Riemannian connection

This is the connection we use in G.R.!

<u>Other useful expressions:</u>

$$g_{a\gamma}\Gamma^{\gamma}_{bc} = g_{a\gamma}\frac{1}{2}g^{\gamma\sigma}\left(\partial_b g_{\sigma c} + ..\right) = \frac{1}{2}\delta^{\sigma}_{a}\left(\dots\right)$$

- $\boxed{\Gamma_{abc} \equiv g_{a\gamma}\Gamma^{\gamma}_{bc} = \frac{1}{2}\left(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{cb}\right)}$ <u>Christoffel symbol of the 1° kind</u>

- $$\begin{aligned} \boxed{\Gamma^{a}_{ac}} &= \frac{1}{2} g^{a\gamma}\left(\partial_a g_{\gamma c} + \partial_c g_{a\gamma} - \partial_\gamma g_{ca}\right) \\ &= \frac{1}{2}\left(\cancel{g^{a\gamma}\partial_a g_{\gamma c}} + g^{a\gamma}\partial_c g_{a\gamma} - \cancel{g^{a\gamma}\partial_\gamma g_{ca}}\right) \\ &= \frac{1}{2} g^{a\gamma}\partial_c g_{a\gamma} \\ &= \frac{1}{2g}\partial_c g = \frac{1}{2}\partial_c \ln g = \partial_c \ln\left(\sqrt{|g|}\right) \end{aligned}$$

"Contraction Γ^{a}_{ac}"

$a \leftrightarrow \gamma$ in 3° term

$g = \det(g)$ $\quad dg = g\, g^{ab}\, dg_{ab} = -g\, g_{ab}\, dg^{ab}$
minors of determinant $g\, g^{ab}$

- $$\boxed{\nabla_a v^a} = \partial_a v^a + \Gamma^{a}_{a\gamma} v^\gamma = \partial_a v^a + v^a \partial_a \ln\left(\sqrt{|g|}\right) = \boxed{\frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}\,v^a\right)}$$

$\gamma \to \beta$

- Similar expressions for scalars, tensors, ... there is a full zoo out there for you.

Other operators

(*) Torsion $=0$ and $\nabla g=0$

Gradient $\nabla_\alpha \phi = \partial_\alpha \phi$ is a 1-form (ϕ scalar function)

Covariant divergence

$$\nabla_\alpha v^\alpha \equiv v^\alpha{}_{;\alpha} \equiv \delta^\alpha_\beta v^\beta{}_{;\alpha}$$

$$= \partial_\alpha v^\alpha + v^\gamma T^\alpha_{\gamma\alpha} \quad \Big\downarrow (*)$$

$$= \frac{1}{\sqrt{|g|}} \partial_\alpha\left(\sqrt{|g|}\, v^\alpha\right) \qquad g \equiv \det(g_{\mu\nu})$$

$$\nabla_\nu A^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu\left(\sqrt{-g}\, A^{\mu\nu}\right) + \underbrace{T^\mu_{\nu\gamma} A^{\gamma\nu}}_{=0}$$

A antisymmetric and torsion free

Laplacian

$$\nabla_\alpha \nabla^\alpha \phi \equiv \Box\phi \equiv \phi^{;\alpha}{}_{;\alpha} \in \mathbb{R}$$

$$= \frac{1}{\sqrt{|g|}} \partial_\alpha\left(\sqrt{|g|}\, g^{\alpha\beta} \partial_\beta \phi\right) \quad \Big\downarrow (*)$$

(is given by a scalar product
⇒ is frame independent!)

Curl

$$(\mathrm{curl}\, \tilde{v})_{\alpha\beta} = \nabla_\alpha v_\beta - \nabla_\beta v_\alpha$$

$$= \partial_\alpha v_\beta - v_\gamma T^\gamma_{\alpha\beta} - \partial_\beta v_\alpha + v_\gamma T^\gamma_{\beta\alpha}$$

$$= \partial_\alpha v_\beta - \partial_\beta v_\alpha - v_\gamma \underbrace{\left(T^\gamma_{\alpha\beta} - T^\gamma_{\beta\alpha}\right)}_{\text{Torsion}=0}$$

Type $\binom{0}{2}$ antisymmetric tensor

same story for vectors

- intrinsic derivative of a vector along a curve → Book 71

Few other comments

G.R.
- is a local theory based on differential geometry
- being local it does not constrain the topology of space
- Assumptions: Torsion free
  Metric compatibility

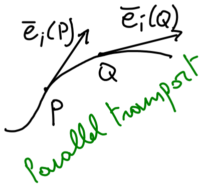
Other Theories of gravity

- One can also build up theories with $T^\alpha_{\beta\gamma} \neq 0$
- One can even build up a theory by setting curvature $=0$ and base everything on torsion!
- eg. Einstein-Cartan gravity theories (no Levi-Civita connection, g and $T \neq 0$ are independent)
  (not energy-momentum tensor as source but spin tensor)
- One could also directly T as a "field"

Some more insights in the affine connection

- All what we said about vectors is valid in the tangent space
- But what happens when we move on the manifold from a point P to a point Q ?
- How can we relate two vectors belonging to the two tangent spaces T_P and T_Q ?

- To visualize it, we embed the manifold M in a space with a larger dimension (as we did in the example of a 2D spherical surface embedded in $V=\mathbb{R}^3$)



Tangent spaces T_P and T_Q have different basis

Take P and Q infinitesimally close one to each other

$\bar{e}_\alpha(Q) \in M_Q T$ not $M_P T$

(this is why we speak about differential geometry)

$\bar{e}_\alpha(P) \in T_P$ basis in $P=(x^\mu)$

$\bar{e}_\alpha(Q) = \bar{e}_\alpha(P) + \underset{\uparrow \text{ small difference}}{\delta\bar{e}_\alpha} \in T_Q$ basis in $Q=(x^\mu+\delta x^\mu)$ $\delta x^\mu \to 0$

$$\delta\bar{e}_\alpha = \partial_\beta \bar{e}_\alpha \delta x^\beta$$

$$\downarrow$$

$$\partial_\beta \bar{e}_\alpha \equiv \underset{(1)}{\lim_{\delta x_c \to 0} \partial_\beta \bar{e}_\alpha \Big|_{T_P}} \overset{!}{=} \underset{(2)}{\Gamma^\gamma_{\beta\alpha}} \bar{e}_\gamma(P)$$

(1) Define derivatives of $\bar{e}_\alpha(Q)$ by projecting them on T_P

(2) We assume $\partial_\beta \bar{e}_\alpha$ to be a vector, i.e. linear combination of the basis

coefficients Γ tels you about the basis change

- Covariant derivative:

is now position dependent

$$\partial_\mu \bar{v} = \partial_\mu(v^\alpha \bar{e}_\alpha) = \partial_\mu v^\alpha \bar{e}_\alpha + v^\alpha \partial_\mu \bar{e}_\alpha$$

$\leftarrow$ metric compatibility $\Rightarrow$ $\partial_\mu \bar{e}_\alpha = \Gamma^\gamma_{\mu\alpha} \bar{e}_\gamma$

$$= \partial_\mu v^\alpha \bar{e}_\alpha + v^\alpha \Gamma^\gamma_{\mu\alpha} \bar{e}_\gamma$$

relabeling $\alpha \leftrightarrow \gamma$

$$= \left(\partial_\mu v^\alpha + v^\gamma \Gamma^\alpha_{\mu\gamma}\right) \bar{e}_\alpha$$

$$\equiv \underbrace{D_\mu v^\alpha \bar{e}_\alpha}_{\bar{v}}$$

- Link between $\partial_\beta \bar{e}_\mu \equiv \Gamma^\gamma_{\beta\mu} \bar{e}_\gamma \iff \nabla_\beta g_{\mu\nu} = 0$

The metric is a key tensor in G.R. lets give a close look at its cov. derivative:

$$\boxed{g_{\mu\nu;\beta}} = g(\bar{e}_\mu, \bar{e}_\nu)_{;\beta} = \langle \bar{e}_\mu, \bar{e}_\nu \rangle_{;\beta}$$

$$= (\bar{e}_\mu \bar{e}_\nu)_{,\beta} - (\bar{e}_\gamma \bar{e}_\nu)\,\Gamma^\gamma_{\beta\mu} - (\bar{e}_\mu \bar{e}_\gamma)\,\Gamma^\gamma_{\beta\nu}$$

$$= \cancel{\bar{e}_{\mu,\beta} \bar{e}_\nu} + \cancel{\bar{e}_\mu \bar{e}_{\nu,\beta}} - \cancel{\bar{e}_{\mu,\beta} \bar{e}_\nu} - \cancel{\bar{e}_\mu \bar{e}_{\nu,\beta}} = \boxed{0}$$

$\partial_\beta \bar{e}_\mu \equiv \Gamma^\gamma_{\beta\mu} \bar{e}_\gamma$ metric compatibility

i.e. $\partial_\beta \bar{e}_\mu$ assumed to be a vector

$\Rightarrow$ in flat space $\eta = \text{const} \Rightarrow \eta_{\mu\nu,\beta} = 0$

using ∇_μ we have the same in a curved space $g_{\mu\nu;\beta} = 0$

$\Rightarrow$ metric compatibility condition: $\boxed{\partial_\beta \bar{e}_\mu \equiv \Gamma^\gamma_{\beta\mu} \bar{e}_\gamma \iff \nabla_\beta g_{\mu\nu} = 0}$ $\quad \nabla_\beta g^{\mu\nu} = 0$

- As we did before you can easily get $\partial_c g_{ab} = \Gamma^\gamma_{ca} g_{\gamma b} + \Gamma^\gamma_{cb} g_{\gamma a}$ but using $\partial_\beta \bar{e}_\mu \equiv \Gamma^\gamma_{\beta\mu} \bar{e}_\gamma$

$$\partial_c g_{ab} = \partial_c(\bar{e}_a \bar{e}_\beta) \overset{\text{Leibnitz product rule}}{=} \underbrace{(\partial_c \bar{e}_a)}_{\Gamma^\alpha_{ca} \bar{e}_\alpha} \bar{e}_b + \bar{e}_a \underbrace{(\partial_c \bar{e}_b)}_{\Gamma^\alpha_{cb} \bar{e}_\alpha} = \Gamma^\gamma_{ca} \underbrace{\bar{e}_\gamma \bar{e}_b}_{g_{\alpha b}} + \Gamma^\gamma_{cb} \underbrace{\bar{e}_\gamma \bar{e}_a}_{g_{\alpha a}} = \Gamma^\gamma_{ca} g_{\gamma b} + \Gamma^\gamma_{cb} g_{\gamma a}$$

metric compatibility

A slightly different look at the affine connection and the covariant derivative

$\bar{v}$ vector : parallely transported $\bar{x} \to \bar{x} + \delta\bar{x}$ $\quad \delta\bar{x} \to 0$

$$v^\mu_{\parallel}(\bar{x} + \delta\bar{x}) = v^\mu(\bar{x}) - \Gamma^\mu_{\beta\gamma} v^\gamma(\bar{x})\,\delta x^\beta + \dots$$

Taylor expansion

$$\nabla_\beta v^\mu \equiv \lim_{\delta x^\beta \to 0} \frac{v^\mu(\bar{x} + \delta\bar{x}) - v^\mu_{\parallel}(\bar{x} + \delta\bar{x})}{\delta x^\beta}$$

$$= \lim_{\delta x^\beta \to 0} \frac{v^\mu(\bar{x} + \delta\bar{x}) - v^\mu(\bar{x}) + \Gamma^\mu_{\beta\gamma} v^\gamma(\bar{x})\,\delta x^\beta}{\delta x^\beta} = \partial_\beta v^\mu + \Gamma^\mu_{\beta\gamma} v^\gamma(\bar{x})$$

$\Rightarrow$ Γ_β coefficients of the first order in a Taylor expansion showing the change in a vector when parallely transported to an infinitesimally close point

Appendix

Nonmetricity tensor

$Q(\bar{v},\bar{u}) : M_pT \times M_pT \to M_pT^*$ $\quad Q : (\bar{v},\bar{u}) \to \nabla_a g(\bar{v},\bar{u}) dx^a$

$Q_{abc} \equiv \nabla_a g_{bc}$ $\quad Q \in T(0,3)$ change of scalar product / norm

$Q_{abc} v^b u^c = \nabla_a g_{bc} v^b u^c$ $\quad [Q(\bar{v},\bar{u})]_a = \nabla_a g(\bar{v},\bar{u})$

$\hat{Q}(\tilde{w},\bar{v},\bar{u}) : M_pT \times M_pT \times M_pT \to \mathbb{R}$ $\quad \hat{Q} : (\bar{w},\bar{v},\bar{u}) \to \bar{w}\, Q(\bar{v},\bar{u})$

$w^a Q_{abc} v^b u^c = w_a \nabla_a g_{bc} v^a u^b \in \mathbb{R}$ change of the scalar product along direction $\bar{w}$

- Quantifies the conge of norm of vectors when parallely trasported along $\bar{w}$

Contorsion, distorsion tensors

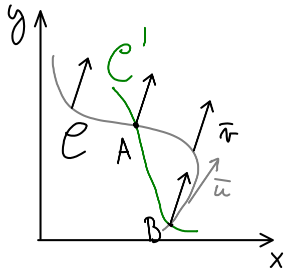
exercise 5 ...

Parallel transport

Parallel transport = move a vector (tensor...) along a path by keeping it "unchanged"

In flat space-time:

- if we go from A to B along different paths the "final" vector will always be the same



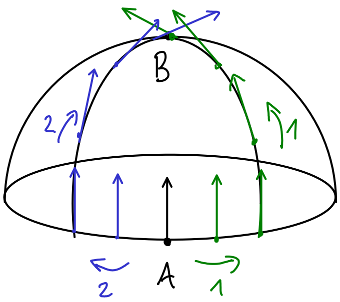
Curve: $\mathcal{C}=\{x^\mu \mid x^\mu = x^\mu(\lambda)\}$ $\gamma: \mathbb{R} \to \mathbb{R}^n$ $\gamma: \lambda \to x^\mu(\lambda)$

$\lambda =$ parameter along the curve, e.g. τ

$\bar{u}=\frac{d\bar{x}}{d\lambda}$ Vector tangent to curve

Condition: $$\boxed{\frac{d\bar{v}}{d\lambda}=u^\mu\partial_\mu\bar{v}\overset{!}{=}0}\ (*)$$ $\forall\, \bar{v}\in V$ $\forall\, \bar{x}\in\mathcal{C}$ directional derivative

In curved space-time:



- Going from A to B along different paths ("1" and "2") leads to different results
- Parallel transport possible whenever we have a connection
- Generalization of (*)

$\bar{v}$ is parallely transported along a curve C with tangent $\bar{u}$ if:

$$\boxed{\frac{D}{d\lambda}\bar{v}=u^\mu\nabla_\mu\bar{v}=0}$$

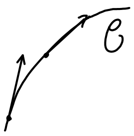
$(v^\nu{}_{,\mu}+\Gamma^\nu_{\mu\gamma}v^\gamma)$

$$\underbrace{u^\mu\partial_\mu v^\nu}_{\frac{dv^\nu}{d\lambda}}+\Gamma^\nu_{\mu\gamma}u^\mu v^\gamma=0$$ Pararallel transport equation

$u^a\nabla_a T^{b_1\dots b_l}{}_{c_1\dots c_k}=0$ $T\in\mathcal{T}(l,k)$ valid for all tensors

- Facts:
  - ordinary linear differential equation ⇒ one solution for a given initial condition on v^ν
  - the connection determines parallel transport

Geodesic equation



Set $\bar{v} \overset{!}{=} \bar{u} \equiv \frac{d\bar{x}}{d\lambda}$ i.e. vector tangent to a world line (auto parallel transport)

$\boxed{u^a \nabla_a u^c = 0}$ $\quad \frac{du^c}{d\lambda} + \Gamma^c_{ab}\, u^a u^b = 0 \quad \underbrace{\frac{dx^\mu}{d\lambda}\partial_\mu}_{\frac{d}{d\lambda}} \frac{dx^\nu}{d\lambda} + \Gamma^\nu_{\mu\gamma} \frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda} = \boxed{\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0}$

(4 equations)

coupled system of 4 second order ordinary differential eq. for $X^\mu(\lambda)$
=> 1 unique solution give initial $X^\nu, dX^\nu/d\lambda$

- geodesic: curve whose tangent vectors are parallely transported along the curve itself
  generalization of a straight line on a manifold
  In flat space you can choose a frame such that $\Gamma = 0 \Rightarrow \frac{dx^\nu}{d\lambda^2} = 0$ straight line

- Geodesics extremize curve between 2 points as measured by the metric (more later)
  - length $\quad \ell = \int (g_{ab}u^a u^b)^{1/2}\, d\lambda \quad u^a$ = tangent to curve at $\lambda \quad$ curve $C = \{\gamma(\lambda)\}$
  - prop. time: $\tau = \int (-g_{ab}u^a u^b)^{1/2}\, d\lambda$

  along geodesic lines, ℓ = shortest, τ = longest between 2 Points

- Geodesics preserve their time/null/space nature $\forall$ frame because parallel transport preserves inner product of parallely transported vectors => $v_\mu w^\mu >/=/</0/$ always

  (in Lorenzian geometry)

  time-like curve if $g_{ab}u^a u^b < 0 \ \forall P \in C$
  null " " $g_{ab}u^a u^b = 0 \ \forall P \in C \quad \ell = \tau = 0$
  space-like " " $g_{ab}u^a u^b > 0 \ \forall P \in C$

- ℓ, τ do not depend on parameterization of curve :
  => new tangent $u^\nu = \frac{dX^\nu}{ds}\frac{ds}{d\lambda} = s^\nu \frac{ds}{d\lambda} \quad \bar{s} = \frac{d\lambda}{ds}\bar{u}$: $\ell' = \int (g_{ab}s^a s^b)^{1/2} ds = \int (g_{ab}u^a u^b)^{1/2} \frac{d\lambda}{\cancel{ds}} \cancel{ds} = \ell$

- Torsion does not affect the geodesic eq. becaus of its anti-symmetry

  $u^a \partial_a u^c + \Gamma^c_{ab}\, u^a u^b = 0 \qquad T^c{}_{ab} = T^c{}_{(ab)} + \frac{1}{2}T^c{}_{ab}$

  $u^a \partial_a u^c + T^c{}_{(ab)}\, u^a u^b + \frac{1}{2}\cancel{T^{ab}\, u^a u^b} = 0 \qquad \underbrace{T^{ab}}_{\text{anty- symmetric}}\ \underbrace{u^a u^b}_{\text{Symmetric}} = 0$

Geodesic equation from variational principle

- Length and time intervals: $\eta \to g(x^\nu)$ (g is given)

$$\tau = \int (-g^{\mu\nu}u^\mu u^\nu)^{1/2} d\lambda$$
$$l = \int (g^{\mu\nu}u^\mu u^\nu)^{1/2} d\lambda$$

- look for extrema, variation of path $\{x^\mu\}$ $\partial l = 0$ $\partial\tau = 0$

$$\partial l = \int_A^B \frac{1}{2}\underbrace{(g_{\mu\nu}u^\mu u^\nu)}_{=1 \text{ reparametrize } \lambda}^{-1/2}(\partial g_{\mu\nu}u^\mu u^\nu + 2g_{\mu\nu}u^\mu \partial u^\nu)d\lambda$$

$\downarrow$ $\partial g_{\mu\nu} = \delta_\alpha g_{\mu\nu}\partial x^\alpha$ $\quad \partial u^\nu = \partial\frac{dx^\nu}{d\lambda} = \frac{d}{d\lambda}(\partial x^\nu)$

$$= \int_A^B \frac{1}{2}\delta_\alpha g_{\mu\nu}u^\mu u^\nu \partial x^\alpha + g_{\mu\nu}u^\mu\frac{d(\partial x^\nu)}{d\lambda}\,d\lambda$$

$\downarrow$ 2° term by part

$$= \int_A^B \overset{\alpha \leftrightarrow \nu}{''} \, d\lambda + \Big[\overset{=0}{\cancel{g_{\mu\nu}u^\mu \partial x^\nu}}\Big]_A^B - \int_A^B \frac{d}{d\lambda}(g_{\mu\nu}u^\mu)\,\partial x^\nu d\lambda$$

$$= \int_A^B \Big[\frac{1}{2}\delta_\nu g_{\mu\alpha}u^\mu u^\alpha - u^\mu\frac{dg_{\mu\nu}}{d\lambda} - g_{\mu\nu}\frac{du^\mu}{d\lambda}\Big]\partial x^\nu d\lambda \overset{!}{=} 0 \quad \forall\, \partial x^\nu$$

$$\Rightarrow \frac{1}{2}\delta_\nu g_{\mu\alpha}u^\mu u^\alpha - u^\mu\frac{dg_{\mu\nu}}{d\lambda} - g_{\mu\nu}\frac{du^\mu}{d\lambda} = 0$$

$\downarrow$ $\frac{d}{d\lambda}g_{\mu\nu} = \delta_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\lambda} = \delta_\alpha g_{\mu\nu}u^\alpha$

$$\frac{1}{2}\delta_\nu g_{\mu\alpha}\underline{u^\mu u^\alpha} - \delta_\alpha g_{\mu\nu}\underline{u^\mu u^\alpha} - g_{\mu\nu}\frac{du^\mu}{d\lambda} = 0$$

$$\Big(\frac{1}{2}\delta_\nu g_{\mu\alpha} - \delta_\alpha g_{\mu\nu}\Big)u^\mu u^\alpha = g_{\mu\nu}\frac{d^2x^\mu}{d\lambda^2} \xrightarrow{\cdot g^{\mu\nu}} \frac{d^2x^\mu}{d\lambda^2} = g^{\mu\nu}\frac{1}{2}\big(\delta_\nu g_{\mu\alpha} - 2\,\underline{\delta_\alpha}\,\underline{g_{\mu\nu}}\big)\overline{u^\mu u^\alpha}^{\text{symmetric}}$$

$$\frac{d^2x^\mu}{d\lambda^2} + \underbrace{g^{\mu\nu}\frac{1}{2}\big(\delta_\alpha g_{\mu\nu} + \delta_\mu g_{\alpha\nu} - \delta_\nu g_{\mu\alpha}\big)}_{}\frac{dx^\mu}{d\lambda}\frac{dx^\alpha}{d\lambda} = 0 \qquad \text{Geodesic eq. !}$$

$\Gamma^\mu_{\alpha\mu}$ affine connection with metric compatibility + Torsion free pops up!

- $\partial\tau \overset{!}{=} 0$ Same computation (but with a minus), same resulting eq.

$\Rightarrow$ Geodesics :
- lines minimizing path / maximizing time between 2 points x^μ_A, x^μ_B
- generalization of straight line, this is the trajectory of free particles
- same for null geodesics : trajectory of photons

Is there any physics?

Up to now, just characterized space-time: look for some physical meanings

1) "Inertial" motion in curved space

(no acceleration)

In special relativity: free particle => $\frac{du^\mu}{d\lambda}=0$ inertial motion e.g. $\lambda=\tau$ $u^\mu=\frac{dx^\mu}{d\lambda}$

Generalizing: $\frac{Du^\mu}{d\lambda}=0$ ~~inertial~~ → free falling!

"1° axiom of Newtonian mechanics!"

2) Free particle in Special Relativity:

$$S=-mc\int ds=-mc\int(-\eta_{\mu\nu}dx^\mu dx^\nu)^{1/2}=-mc\int(-\eta_{\mu\nu}u^\mu u^\nu)^{1/2}d\tau \Rightarrow \text{straight line}$$

we had in mind a particle, now forget about it, $S=\int ds$ is an interval (time, length, null)

3) Free particle in general Relativity:

$S=-mc\int(-g_{\mu\nu}u^\mu u^\nu)^{\frac{1}{2}}d\tau$ eg. action of massive particle => eq. of motion = geodesic equation

$$L(\bar{x},\bar{u})=-mc(-g_{\mu\nu}u^\mu u^\nu)^{\frac{1}{2}}$$

4) Generalized momentum

$$P_\mu=\frac{\delta L}{\delta u^\mu}=-m\cancel{c}\,\frac{1}{\cancel{2}}\cancel{(-g_{\mu\nu}u^\mu u^\nu)^{-1/2}}(-g_{\mu\nu}\,u^{\hat{\nu}}\cancel{2})=+m\,u_\mu$$

tangent to the geodesic as in S.R.

5a) 4-momentum conservation:

$$\frac{d}{dt}\Bigg(\underbrace{\frac{\delta L}{\delta u^\mu}}_{P_\mu}\Bigg)-\frac{\delta L}{\delta x^\mu}=0 \quad \text{if} \quad \frac{\delta L}{\delta x^\mu}=m\frac{1}{2}(-\partial_\mu g_{\alpha\beta}u^\alpha u^\beta)=0 \Rightarrow \frac{dP_\mu}{dt}=0 \quad \text{i.e. } P_\mu \text{ is conserved!}$$

$$\Updownarrow$$

$$\partial_\mu g=0$$

Solution when $g(\bar{x})$ is constant along axis X^μ

clearly this depends on the frame => need better ways to study conservation laws

↳ Killing vectors / Lie derivatives .... later

5b) 4-momentum conservation:

← use Torsion free + Metric compatibility

$$\frac{Du_\nu}{d\tau} = \frac{du_\nu}{d\tau} - \Gamma^\beta_{\alpha\nu} u^\alpha u_\beta$$

$$= \frac{du_\nu}{d\tau} - \frac{1}{2} g^{\beta\gamma}\left(\partial_\alpha g_{\gamma\nu} + \partial_\nu g_{\alpha\gamma} - \partial_\gamma g_{\nu\alpha}\right) u^\alpha u_\beta$$

$$= \frac{du_\nu}{d\tau} - \frac{1}{2}\left(\cancel{\partial_\alpha g_{\gamma\nu}} + \partial_\nu g_{\alpha\gamma} - \cancel{\partial_\gamma g_{\nu\alpha}}\right) u^\alpha u^\gamma$$

$$= \boxed{\frac{du_\nu}{d\tau} - \frac{1}{2}\partial_\nu g_{\alpha\gamma} u^\alpha u^\gamma = 0}$$

→ useful to look for conserved quantities

$$m\frac{du_\nu}{d\tau} = \frac{dP_\nu}{d\tau} = \frac{1}{2}\partial_\nu g_{\alpha\gamma} u^\alpha u^\gamma \Rightarrow \text{if } \partial_\nu g_{\alpha\beta} = 0 \quad P_\nu = \text{const} \quad P_\nu \text{ is conserved}$$

6) A force !? No, but we 'perceived' it like one

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0 \Rightarrow \boxed{m\frac{du^\mu}{d\lambda} = -m\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta}$$

"2° axiom of Newtonian mechanics"

$$\underset{\text{inertia}}{a^i} = \underset{\text{force}}{-\partial^i\phi} \rightarrow \underset{\text{free}}{a^i - \partial^i\phi = 0}$$

$\updownarrow$

charged particle, Lorentz force: $m\frac{du_\nu}{d\tau} = \frac{q}{c}F_{\nu\mu}u^\mu$ do you see the similarity?!

⇒ Derivatives of g give what looks like a force: what we call gravity

Again.... There is no force here! Just things related to the manifold

p.s. on the right-end side you could include a 4-force acting as a source term

⇒ Free (falling) particles follow geodesic curves of space-time!

- Careful: not yet at G.R. just geodesic eq.

7) The way around... Geodeisc equation in Newton's gravity

$\ddot{x}^i + \partial^i\varphi = 0 \quad i = 1,2,3$ Netonian mechanics $\quad \varphi =$ grav. potential $\quad \dot{x} = \frac{dx}{dt} \quad \lambda = t$

$$\ddot{x}^i + \partial^i\frac{\varphi}{c^2}c\cdot c \overset{⊛}{=} \ddot{x}^i + \partial^i\frac{\varphi}{c^2}\dot{x}^0\dot{x}^0 = 0$$

$$= \ddot{x}^i + \Gamma^i_{00}\dot{x}^0\dot{x}^0 = 0 \qquad \boxed{\Gamma^i_{00} \overset{!}{=} \partial^i\frac{\varphi}{c^2}}$$

$$⊛ \quad (\dot{x}^\mu) = (u^\mu) = \gamma\begin{pmatrix} c \\ \dot{x}^i \end{pmatrix} \underset{v \ll c}{\simeq} \begin{pmatrix} c \\ \dot{x}^i \end{pmatrix} \rightarrow \dot{x}^0 = c$$

- meaning of the affine parametrization $t' = a + bt$ $\quad$ a = time shift
b = rescaling of time

$\ddot{x} = \frac{d^2x}{dt^2} \rightarrow \ddot{x} = \frac{d^2x}{dt'^2}$ change by a factor of $\frac{1}{b^2}$

φ has units of velocity (i.e. time^{-2})

⇒ The rescalings cancel
Same eq. of motion

Weak field limit

- Assumptions:

1) Stationary metric: $\partial_0 g_{\mu\nu} = 0$ (not evolving with time) (not strictly necessary)

2) Weak gravitational field: small perturbation of the Minkowski metric

$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad |h_{\alpha\beta}| \ll 1$ (perturbative approach)

$\Downarrow$

small accelerations $\Rightarrow$ small velocities $\Leftarrow$

2b) Non relativistic objects: $|\dot{x}^i| \ll c \qquad u^\mu = \gamma \begin{pmatrix} c \\ \dot{x} \end{pmatrix} \simeq \begin{pmatrix} c \\ \dot{x} \end{pmatrix} \qquad u^i \ll c = u^0$ is conserved

- Geodesic eq.:

$$\frac{du^\nu}{d\tau} + \Gamma^\nu_{\underset{\smile}{\alpha\beta}} u^{\bar\alpha} u^{\hat\beta} = \frac{du^\nu}{d\tau} + \Gamma^\nu_{00} \overset{c}{u^0} \overset{c}{u^0} + 2\Gamma^\nu_{0i} u^0 u^i + \Gamma^\nu_{ij} u^i u^j \overset{\overset{|u^i| \ll |u^0|}{\downarrow}}{\simeq} \frac{du^\nu}{d\tau} + \Gamma^\nu_{00} c^2 = 0 \qquad \boxed{\frac{du^\nu}{d\tau} = -\Gamma^\nu_{00} c^2}$$

$$\Gamma^\nu_{00} = \frac{1}{2} g^{\nu\gamma} \Big(\underset{=0}{\cancel{g_{\gamma 0,0}}} \overset{\text{static}}{+} \underset{=0}{\cancel{g_{0\gamma,0}}} - g_{00,\gamma} \Big) = -\frac{1}{2} g^{\nu\hat\gamma} g_{00,\underline{\gamma}} = -\frac{1}{2} \Big(g^{\nu 0} \underset{\substack{=0 \\ \text{not evolving}}}{\cancel{g_{00,0}}} + g^{\nu i} g_{00,i} \Big) = -\frac{1}{2} \overset{\downarrow}{g}^{\nu i} \overset{\downarrow}{g}_{00,i} \simeq \frac{c^2}{2} \eta^{\nu i} h_{00,i}$$

$\Gamma^a_{bc} = \frac{1}{2} g^{a\gamma} (\partial_b g_{\gamma c} + \partial_c g_{b\gamma} - \partial_\gamma g_{cb})$

$$\boxed{\frac{du^\nu}{d\tau} = \frac{c^2}{2} \eta^{\nu i} h_{00,i} \overset{!}{=} -\eta^{\nu i} \partial_i \varphi} \quad \Rightarrow \quad \boxed{\partial_i h_{00} = -2 \partial_i \frac{\varphi}{c^2}} \qquad h_{00} = -\frac{2\varphi}{c^2} + A \qquad \overset{r \to \infty}{A = 0}$$

acceleration must be given by gradient of potential

$$\boxed{g_{00} = -\left(1 + \frac{2\varphi}{c^2}\right)}$$

by including higher order terms we get: $\boxed{g_{ii} = \left(1 - \frac{2\varphi}{c^2}\right)}$ (more later...)

correction of the same order as in g_{00}

$\Rightarrow$ Poisson eq.: $\bar\nabla^2 \varphi = \boxed{\bar\nabla^2 h_{00} = 4\pi G \rho}$

- With parallel transport

$$\frac{du_\gamma}{d\tau} = \frac{1}{2} u^\alpha u^\beta \partial_\gamma g_{\alpha\beta} \overset{\eta \text{ constant}}{=} \frac{1}{2} u^\alpha u^\beta \partial_\gamma h_{\alpha\beta}$$

linear relation $t \leftrightarrow \tau$

- $\underline{\gamma = 0}$ (time) $\quad \dfrac{du_0}{d\tau} = \dfrac{1}{2} u^\alpha u^\beta \overset{=0}{\partial_0 h_{\alpha\beta}} = 0 \Rightarrow \overset{\overset{\text{const}}{\downarrow}}{u_0} = g_{0\nu} u^\nu = g_{00} u^0 + g_{0i} u^i \simeq \eta_{00} \overset{\overset{c \frac{dt}{d\tau}}{\downarrow}}{u^0} \qquad \boxed{+\dfrac{d^2 t}{d\tau^2} = 0}$

- $\underline{\gamma = i}$ (space) $\quad \dfrac{du_i}{d\tau} \underset{\gamma \simeq 1}{\simeq} \dfrac{du_i}{dt} \Big| = \dfrac{1}{2} \overset{c}{u^0} \overset{c}{u^0} \partial_i h_{00} + u^0 \underset{\smile}{u^j} \partial_i h_{0j} + \dfrac{1}{2} \underset{\smile}{u^j} \underset{\smile}{u^k} \partial_i h_{jk} \simeq \dfrac{c^2}{2} \partial_i h_{00} \overset{!}{=} -\partial_i \varphi$

$u^i \ll u^0$

Affine parameters

- We derived the geodesic equation finding world lines (trajectories) parametrized by λ
- From variational approach we got $\lambda = \tau$
- Can we use any arbitrary function of $\lambda = f(\tau)$?

$$\frac{d^2x^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\lambda = f(\tau)$$

$$d\lambda = \frac{\delta f}{\delta\tau}d\tau = f' d\tau \quad \Rightarrow \quad \frac{d}{d\lambda} = \frac{1}{f'}\frac{d}{d\tau}$$

$$\frac{d^2}{d\lambda} = \frac{d}{d\lambda}\left(\frac{1}{f'}\frac{d}{d\tau}\right) = \frac{d}{f'd\tau}\left(\frac{1}{f'}\frac{d}{d\tau}\right)$$

$$= \frac{1}{f'^2}\frac{d^2}{d\tau^2} - \frac{f''}{f'^2}\frac{d}{d\tau}$$

$$\Downarrow$$

$$\frac{1}{\cancel{f'^2}}\frac{d^2x^\nu}{d\tau^2} - \frac{f''}{\cancel{f'^2}}\frac{dx^\nu}{d\tau} + \Gamma^\nu_\beta\frac{1}{\cancel{f'^2}}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_\beta\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = f''\frac{dx^\nu}{d\tau}$$

as geodesic eq. if $f''(\tau) \overset{!}{=} 0$

i.e. $\boxed{f(\tau) = a + b\tau}$ $a, b \in \mathbb{R}$ const

a = shift
b = rescale

λ = affine parameter

- Geodesic eq. are invariant under affine re-parameterization
  $\Rightarrow$ parallel transport do not only constrain the path, it also constrain the way we parameterize it
- Clearly $\tau \to \lambda = a + b\tau$ does not change the result of variational approach either
- You could also use another parametrization but then the geodesic eq. would look like above. with the additional $f''\frac{dx^\nu}{d\tau}$

* * *

A weaker but equivalent condition:

$$u^a\nabla_a u^c = \alpha u^c \quad \alpha \in \mathbb{R}$$

$\uparrow$
still a "straight line"

maintain direction but allowing change in norm
"weaker condition" but with same solution as the other one
equivalent, reparametrize with an affine parameter
$\Rightarrow$ no loss in generality when using $u^a\nabla_a u^c = 0$

Final remarks:

- we can just move infinitesimally from one point to another
- it does not really make sense to compare vectors in ≠ locations ....
  if we shift them along different paths we get different results

  eg. 2 particles, what is their relative velocity $\bar{v}_i$? Just not defined if they are in ≠ points

  eg. cosmic expansion : redshift of galaxies is not a doppler effect associated to velocities,
  it is due to the time evolution of the metric

- Photons => grav. lensing comes out naturally from the geodesic eq.
  they follow geodesics

- You can have geodesics that can run into singularities (there you can not construct a T_p)
  => geodesically incomplete manifold ↳ eg. in Schwarzschild solution

C.G8 Dyson formula

The Curvature of space

- Geodesic eq.: line of free falling particles
  rate of change of $\dot{x}$ (i.e. $\frac{dx}{d\tau}$) across the particle
  This is completly independent from the presence or not of curvature in Space-time
  also $g_{\mu\nu}$, $\Gamma^{\mu}_{\alpha\gamma}$, ∇_α contains no information about gravity
  build only to deal with arbitrary coordinate choices

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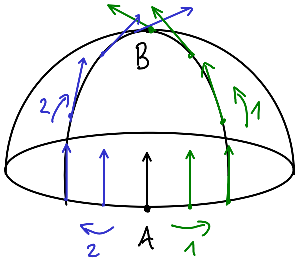
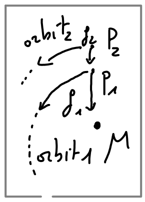
- Gravitational fields: "do not exist" in one single point because you can always obtain $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\mu}_{\alpha\beta} = 0$ locally

⇒ To quantify gravit you need a non local "measure" to be sensitive to $\partial^2 g$

eg. "going around" with parallel transport:
going from A to B along different paths gives ≠ results



eg. 2 points, they are booth free falling
but their relative acceleration is different
⇒ detect presence of gravity
→ tydal forces, geodesic deviation equation



$$\frac{d^2 \ddot{x}_2}{d\tau^2} \neq \frac{d^2 \ddot{x}_1}{d\tau^2}$$

- Mathematically, this is expressed by the Riemann (curvature) tensor $R^{\alpha}{}_{\mu\nu\beta}$

- Intrinsic properties of space-time are identified by tensors
  ↳ If a tensor A is $A \neq 0$ in one frame ⇒ $A \neq 0$ $\forall$ frame
  i.e. it can not be made vanish with a suited coord transformation
  ⇒ We need tensors to express intrinsic properties of the space-time
  e.g. Γ is not good because not a tensor

Curvature map

commutation of the directions

$$\boxed{\hat{R} : TM \times TM \times TM \to TM \qquad \hat{R}(\bar{v},\bar{u})\bar{s} \equiv \underbrace{\nabla_{\bar{v}}\nabla_{\bar{u}}\bar{s} - \nabla_{\bar{u}}\nabla_{\bar{v}}\bar{s}}_{[\nabla_{\bar{v}},\nabla_{\bar{u}}]\bar{s}} - \nabla_{[\bar{v},\bar{u}]}\bar{s}}$$

commutator of double covariant derivatives

Meaning 1:

- $\hat{R}$ tells you by how much covariant derivatives do not commute $[\nabla_{\bar{v}},\nabla_{\bar{u}}]$
- "Double derivatives" like in functional analysis <=> info about curvature

Meaning 2:

- P with coords $\{x^\mu\}$

vectors with infinitesimal components

- 2 curves: $\left\{ \mathcal{C}_1 = \mathcal{C}_x \cup \mathcal{C}_{xy} \right.$ $\left\{ \mathcal{C}_2 = \mathcal{C}_y \cup \mathcal{C}_{yx} \right.$

$\mathcal{C}_x = \{ P \in M \mid x = h(P),\ x_P^\mu + \lambda\delta^\mu \}$

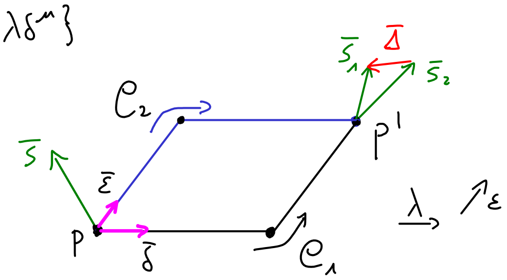
$\mathcal{C}_{xy} = \{ \text{"} \mid \text{"},\ x_P^\mu + \delta^\mu + \lambda\varepsilon^\mu \}$

$\mathcal{C}_y = \{ \text{"} \mid \text{"},\ x_P^\mu + \lambda\varepsilon^\mu \}$

$\mathcal{C}_{yx} = \{ \text{"} \mid \text{"},\ x_P + \varepsilon^\mu + \lambda\delta^\mu \}$

$\lambda, \varepsilon \in [0,1] \quad \varepsilon^\mu, \delta^\mu \ll 1$

- $\bar{v} \in M_PT$
- parallel transport it to P' along $\mathcal{C}_1 \Rightarrow \bar{s}_1 \in M_{P'}T$
- " " " " $\mathcal{C}_2 \Rightarrow \bar{s}_2 \in M_{P'}T$



$$\Rightarrow \boxed{\Delta^\mu = s_1^\mu - s_2^\mu = R^\mu{}_{\gamma\alpha\beta}\,\delta^\alpha\delta^\beta s^{\bar{\gamma}}}$$

- quantifies the difference in $\bar{s}$ when parallely transported along 2 different infinitesimal paths
- "input vectors $\bar{v}, \bar{u}$ give the direction of the path,
  e.g. $\nabla_{\bar{v}}\nabla_{\bar{u}}\bar{s}$ -> parallel transport of $\bar{s}$ along $\bar{u}$ and then along $\bar{v}$

  (α,β): directions along which to run; "2 vectors $\bar{v},\bar{u}$"; antisymmetric pair
- indeces (μ): component d of the output vector

  (γ): dummy

Basic properties: $\hat{R}(\bar{v},\bar{u})\bar{s} = -\hat{R}(\bar{u},\bar{v})\bar{s}$ antisymmetric

$\hat{R}(f\bar{v}, g\bar{u})h\bar{s} = fgh\hat{R}(\bar{v},\bar{u})\bar{s}$ $f,g,h \in \mathcal{F}$ C^∞ functions on M => trilinear map

=> implies a type $\binom{1}{3}$ tensor $R^d{}_{cab} \equiv \langle \tilde{\omega}^d, \hat{R}(\bar{e}_a, \bar{e}_b, \bar{e}_c) \rangle \in \mathcal{T}(1,3)$

curvature tensor

Curvature / Riemann tensor

$\hat{R} : TM \times TM \times TM \to TM$ $\quad \hat{R}(\bar{v},\bar{u},\bar{s}) \equiv \nabla_{\bar{v}}\nabla_{\bar{u}}\bar{s} - \nabla_{\bar{u}}\nabla_{\bar{v}}\bar{s} - \nabla_{[\bar{v},\bar{u}]}\bar{s}$ curvature map

$$\boxed{T^{*}M \times TM \times TM \times TM \to \mathbb{R} \qquad R(\tilde{w},\bar{v},\bar{u},\bar{s}) \equiv \tilde{w}\,\hat{R}(\bar{v},\bar{u},\bar{s}) \quad \tilde{w}\in T^{*}M \quad \bar{v},\bar{u},\bar{s}\in TM \quad R\in\mathcal{T}(1,3)}$$

- Components on $\{\tilde{dx}_i\}, \{\partial_j\}$ $\quad \nabla_{\partial_k} \equiv \nabla_k$ (with abstract notation you should use greek letters)

$R^{d}{}_{cab} = \langle \tilde{\omega}^{d}, \hat{R}(\bar{e}_a,\bar{e}_b,\bar{e}_c)\rangle$

$$\boxed{R^{i}{}_{jkl}} = \tilde{dx}^{i}\left[R(\bar{\partial}_k,\bar{\partial}_l)\bar{\partial}_j\right]$$

(2) $[\bar{\partial}_a,\bar{\partial}_b] = \partial_a\partial_b - \partial_a\partial_b = 0$ the directions commute and $\overset{=\bar{0}}{[\bar{\partial}_k,\bar{\partial}_l]}{}^{\mu}\nabla_{\mu} = 0$

$$= dx^{i}{}_{\alpha}\left[\underbrace{\nabla_k\nabla_l\partial^{\alpha}_{j}}_{(1)} - \nabla_l\nabla_k\partial^{\alpha}_{j} - \overset{=0}{\nabla_{[\bar{\partial}_k,\bar{\partial}_l]}}\partial^{\alpha}_{j}\right]$$

(1) $\nabla_k\nabla_l\partial^{\alpha}_{j} = \nabla_k\left(\underbrace{\partial_l\partial^{\alpha}_{j}}_{=0} + \Gamma^{\alpha}_{l\beta}\partial^{\beta}_{j}\right) = \nabla_k\left(\Gamma^{\alpha}_{lj}\right) = \partial_k\Gamma^{\alpha}_{lj} + \Gamma^{\alpha}_{k\gamma}\Gamma^{\gamma}_{lj}$ $\qquad \partial^{\alpha}_{j} = \begin{cases} 0 & \alpha\neq j \\ 1 & \alpha=j \end{cases}$

$$= dx^{i}{}_{\alpha}\left[\partial_k\Gamma^{\alpha}_{lj} + \Gamma^{\alpha}_{k\gamma}\Gamma^{\gamma}_{lj} - \partial_l\Gamma^{\alpha}_{kj} - \Gamma^{\alpha}_{l\gamma}\Gamma^{\gamma}_{kj}\right]$$

$$= \boxed{\partial_k\Gamma^{i}_{lj} - \partial_l\Gamma^{i}_{kj} + \Gamma^{i}_{k\gamma}\Gamma^{\gamma}_{lj} - \Gamma^{i}_{l\gamma}\Gamma^{\gamma}_{kj}}$$

- Comments:

- All symmetries of R (such as $R^{i}{}_{cab} = -R^{i}{}_{cba}$) will be discussed later

- Warning: $R^{i}{}_{jkl}$ the order of the indexes is conventional and there is no agreement!!

- Being a tensor, if $R\neq0$ in one frame $\Rightarrow$ $R\neq0$ $\forall$ frame
  it is an intrinsic property of space-time

- Flatness of space and $R^{d}{}_{cab}$

$R^{a}{}_{bcd} = 0$ covariant eq. $\Rightarrow$ characterizes an intrinsic property of manifolds
sufficient and necessary condition for a flat manifold
if $\neq0$ $\Rightarrow$ curved space = gravity

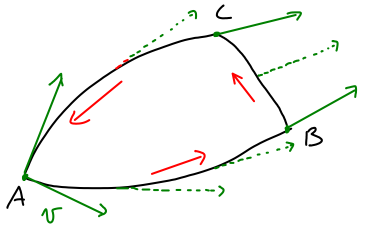
↑

$R^{a}{}_{bcd}$ contains $\partial\Gamma$, i.e. double derivatives of g : $\partial\partial g$, the famous 2° order that can not be made vanish with a simple choice of frame

"Alternative" derivation of R

④

- Parallel transport of a vector along a closed loop
- Difference between initial and final vector
- Exploits Stokes theorem



1) Parallel transport along ∂_α direction:

$$\nabla_{\partial_\alpha} u^\nu = 0 \qquad \partial_\alpha u^\nu + \Gamma^\nu_{\alpha\gamma} u^\gamma = 0 \qquad \partial_\alpha u^\nu = -\Gamma^\nu_{\alpha\gamma} u^\gamma \qquad (1)$$

2) Change after infinitesimal shift dx^α:

$$du^\nu = \partial_\alpha u^\nu \, dx^\alpha = -\Gamma^\nu_{\alpha\gamma} u^\gamma dx^\alpha$$

3) Total change:

$$\Delta u^\nu \equiv \oint du^\nu \qquad \text{integral over a closed loop}$$

$$= -\oint \Gamma^\nu_{\alpha\gamma} u^\gamma dx^\alpha \qquad B^\nu_{\ \alpha} \equiv \Gamma^\nu_{\gamma\alpha} u^\gamma \quad (2)$$

$$= -\oint B^\nu_{\ \alpha} dx^\alpha.$$

$$= -\int \partial_\beta B^\nu_{\ \alpha} \, ds^{\alpha\beta}, \qquad \underline{\text{Stokes theorem}}! \quad (\text{surface})\ ds^{\alpha\beta}$$

$$= -\frac{1}{2}\int \left(\partial_\alpha B^\nu_\beta - \partial_\beta B^\nu_\alpha \right) ds^{\alpha\beta} \qquad (2)$$

$$= -\frac{1}{2}\int \left[\partial_\alpha \left(\Gamma^\nu_{\beta\gamma} u^\gamma \right) - \partial_\beta \left(\Gamma^\nu_{\alpha\gamma} u^\gamma \right) \right] ds^{\alpha\beta}$$

$$= -\frac{1}{2}\int \left[\Gamma^\nu_{\beta\gamma,\alpha} u^\gamma + \Gamma^\nu_{\beta\gamma} \overset{(1)}{u^\gamma_{,\alpha}} - \Gamma^\nu_{\alpha\gamma,\beta} u^\gamma - \Gamma^\nu_{\alpha\gamma} \overset{(1)}{u^\gamma_{,\beta}} \right] ds^{\alpha\beta}$$

$$= -\frac{1}{2}\int \left[\Gamma^\nu_{\beta\gamma,\alpha} u^\gamma - \underset{\sigma \leftrightarrow \gamma}{\Gamma^\nu_{\beta\gamma} \Gamma^\gamma_{\alpha\sigma} u^\sigma} - \Gamma^\nu_{\alpha\gamma,\beta} u^\gamma + \underset{\sigma \leftrightarrow \gamma}{\Gamma^\nu_{\alpha\gamma} \Gamma^\gamma_{\beta\sigma} u^\sigma} \right] ds^{\alpha\beta}$$

$$= -\frac{1}{2}\int \Big(\underbrace{\Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\beta\sigma} \Gamma^\sigma_{\alpha\gamma} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\nu_{\alpha\sigma} \Gamma^\sigma_{\beta\gamma}}_{-R^\nu_{\gamma\alpha\beta}} \Big) u^\gamma ds^{\alpha\beta}$$

$$\boxed{du^\nu = \frac{1}{2} R^\nu_{\gamma\alpha\beta} u^\gamma ds^{\alpha\beta}} \qquad \boxed{R^\nu_{\gamma\alpha\beta} = \Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\nu_{\alpha\sigma} \Gamma^\sigma_{\beta\gamma} - \Gamma^\nu_{\beta\sigma} \Gamma^\sigma_{\alpha\gamma}}$$

⇒ R quantifies the change of a vector when parallely transported along a closed loop

- the change depends on the path

$$ds^{\alpha\beta} = dx^\alpha dx^\beta$$

$\div d\tau$ $\quad \dfrac{du^\nu}{d\tau} \equiv a^\nu$ → acceleration ∝ R geodesic eq. deviation $\quad a^\nu = \frac{1}{2} R^\nu_{\ \gamma\alpha\beta} u^\gamma u^\alpha dx^\beta$ ↑

From now on G.R. only!

$T=0$ and $\nabla_\mu g_{\alpha\beta}=0$

=> Riemannian connection

$R^i{}_{jkl}$ with a Riemann connection

- booth $R^i{}_{jkl}$ and $T^i{}_{kl}$ are based on connection Γ only! No metric was used! no reference to metric compatibility or Torsion free, super general!
- In G.R. we assume $T=0$ and $\nabla_\mu g_{\alpha\beta}=0$ => Γ^i_{jk} = Christoffel symbol (Riemann connection) (!)
- Explicit expression:

coord system such that in P $\Gamma=0$ and the 1° order $\partial_k g^{i\gamma}=0$ (local inertial frame)

$$\boxed{R^i{}_{jkl}} = \partial_k \overset{\downarrow}{\Gamma^i_{lj}} - \partial_l \overset{\downarrow}{\Gamma^i_{kj}} + \overset{=0}{\Gamma^i_{k\gamma}\Gamma^\gamma_{lj}} - \overset{=0}{\Gamma^i_{l\gamma}\Gamma^\gamma_{kj}}$$

$$= \frac{1}{2}\partial_k\left[g^{i\gamma}\left(\partial_l g_{\gamma j}+\partial_j g_{l\gamma}-\partial_\gamma g_{jl}\right)\right]-\frac{1}{2}\partial_l\left[g^{i\gamma}\left(\partial_k g_{\gamma j}+\partial_j g_{k\gamma}-\partial_\gamma g_{jk}\right)\right]$$

$$= \frac{1}{2}g^{i\gamma}\left(\cancel{\partial_k\partial_l g_{\gamma j}}+\partial_k\partial_j g_{l\gamma}-\partial_k\partial_\gamma g_{jl}-\cancel{\overset{\leftrightarrow}{\partial_l\partial_k} g_{\gamma j}}-\partial_l\partial_j g_{k\gamma}+\partial_l\partial_\gamma g_{jk}\right)+\frac{1}{2}\overset{=0}{\partial_k g^{i\gamma}}\left(\cdots\cdots\right)$$

$$= \boxed{\frac{1}{2}g^{i\gamma}\left(\partial_k\partial_j g_{l\gamma}-\partial_k\partial_\gamma g_{jl}-\partial_l\partial_j g_{k\gamma}+\partial_l\partial_\gamma g_{jk}\right)}$$

in local inertial frame

(1) $R^i{}_{j\underline{kl}}=-R^i{}_{j\underline{lk}}$ last 2 indeces

$$\boxed{R_{ijkl}} = g_{i\sigma}R^\sigma{}_{jkl} = \frac{1}{2}\overset{\delta_i^\gamma}{g_{i\sigma}g^{\sigma\gamma}}\left(\partial_k\partial_j g_{l\gamma}-\partial_k\partial_\gamma g_{jl}+\partial_l\partial_j g_{k\gamma}-\partial_l\partial_\gamma g_{jk}\right)$$

$$= \boxed{\frac{1}{2}\left(\partial_k\partial_j g_{li}-\partial_k\partial_i g_{jl}-\partial_l\partial_j g_{ki}+\partial_l\partial_i g_{jk}\right)}$$

Just swap the indeces to see it

(2) $R_{ijkl}=R_{\underline{kl}\,\underline{ij}}$ swap first 2 with last 2 indeces

(3) $R_{ijkl}=-R_{\underline{ji}kl}=-R_{ij\underline{lk}}$ swap first/last two indeces

(4) $R_{ijkl}+R_{iklj}+R_{iljk}=0$ cyclic indeces (anti-symmetric part of last 3 indeces)

(5) $R_{[ijkl]}=0$ Total antisymmetric part (1) (2) (3)

-> R_{ijkl} has $4^4=256$ components

-> The for conditions (1) (2) (3) (4) are not independent (they are linked by $g_{i\sigma}$) => 3 "constraints"

=> because of (2), (3) and (4): only 20 components are independent

(like a $\underbrace{6\times 6 \text{ symmetric matrix}}_{21 \text{ elements}}$ + condition (4) = 20 independent elements)

- **Bianchi identity**

choose Riemanian coordinates
(locally geodetic, $\Gamma = 0$)

$$\nabla_\beta R_{ijk\ell} = \partial_\beta R_{ijk\ell}$$

$$\boxed{\nabla_{[\beta} R_{ij]k\ell}} = \nabla_\beta R_{ijk\ell} + \nabla_i R_{j\beta k\ell} + \nabla_j R_{\beta i k\ell}$$

Cyclic indices

$$= \frac{1}{2}\partial_\beta\left(\cancel{\partial_k\partial_j} g_{\ell i} - \cancel{\partial_k\partial_i} g_{j\ell} + \cancel{\partial_\ell\partial_j} g_{ki} - \cancel{\partial_\ell\partial_i} g_{jk}\right)$$

$$+ \frac{1}{2}\partial_i\left(\cancel{\partial_k\partial_\beta} g_{\ell j} - \cancel{\partial_k\partial_j} g_{\beta\ell} + \cancel{\partial_\ell\partial_\beta} g_{kj} - \cancel{\partial_\ell\partial_j} g_{\beta k}\right)$$

$$+ \frac{1}{2}\partial_j\left(\cancel{\partial_k\partial_i} g_{\ell\beta} - \cancel{\partial_k\partial_\beta} g_{i\ell} + \cancel{\partial_\ell\partial_i} g_{k\beta} - \cancel{\partial_\ell\partial_\beta} g_{ik}\right) = \boxed{0}$$

(cancellations because of the symmetries of $g_{\alpha\beta}$ and anti-symmetry (3))

it holds in all frames

(with Torsion you have additional terms)

if a tensor is null in one frame it is null in all frames => general validity

↳ Related to Jacobi identity $[[\nabla_\alpha, \nabla_\beta], \nabla_\gamma] + [[\nabla_\beta, \nabla_\gamma], \nabla_\alpha] + [[\nabla_\gamma, \nabla_\alpha], \nabla_\beta] = 0$

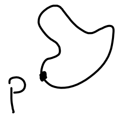
Appendix

Homology and homology group

- **Loop**

A closed curve passing through a point $P \in M$ $\quad \mathcal{C} : S^1 \to M$

$\lambda \in [0,1] \to P = \mathcal{C}(\lambda) \qquad \mathcal{C}(0) = \mathcal{C}(1)$



- **Homology**

- Given a connection, any $v \in M_pT$ can be parallely transported along $\mathcal{C}$ to give a new vector $v' \in M_pT$

- This defines a linear transformation $\quad M_{\mathcal{C}} : M_pT \to M_pT \quad M_{\mathcal{C}} : \vec{v} \to \vec{v}' \quad v'^{\beta} = (M_{\mathcal{C}})_{\alpha}{}^{\beta} \; v^{\alpha}$
  called homology at P of the connection for the loop $\mathcal{C}$

- Going around the loop in the opposite direction $\to$ inverse transformation

General, Linear group

dimension

$\Rightarrow M_{\mathcal{C}}$ is invertible $\quad M_{\mathcal{C}} \in GL(m, \mathbb{R})$

- $M_{\mathcal{C}_3} = M_{\mathcal{C}_2} M_{\mathcal{C}_3}$ is an homology (first you go along $\mathcal{C}_1$ and then $\mathcal{C}_2$, Two loops trough same P)

- **Homology group**

$\{M_{\mathcal{C}}\}$ for all curves $\mathcal{C}$ through P form a group H_p = homology group at P

is a sub group of $GL(m, \mathbb{R})$

···· diff. geom book

Existence of the curvature tensor

$f \in \mathcal{F} \quad \tilde{w} \in M_pT^* \quad \nabla_a\nabla_b(f w_c) \in \mathcal{T}\binom{0}{3}$

$$\nabla_a\nabla_b(f w_c) = \nabla_a(\nabla_b f \cdot w_c + f \nabla_b w_c) = (\nabla_a\nabla_b f) w_c + \nabla_b f\, \nabla_a w_c + \nabla_a f\, \nabla_b w_c + f\, \nabla_a\nabla_b w_c$$

$$\begin{aligned}\nabla_a\nabla_b(f w_c) - \nabla_b\nabla_a(f w_c) &= (\nabla_a \cancel{\nabla_b f}) w_c + \nabla_b \cancel{f\, \nabla_a} w_c + \nabla_a \cancel{f\, \nabla_b} w_c + f\, \nabla_a\nabla_b w_c \; - \\ &\quad (\nabla_b \cancel{\nabla_a f}) w_c + \nabla_a \cancel{f\, \nabla_b} w_c + \nabla_b \cancel{f\, \nabla_a} w_c + f\, \nabla_b\nabla_a w_c \\ &= f(\nabla_a\nabla_b - \nabla_b\nabla_a) w_c\end{aligned}$$

$(\nabla_a\nabla_b - \nabla_b\nabla_a) w_c|_p$ depends only on value of w_c in $P \in M$ => linear map

$$\boxed{R : M_pT^* \to \mathcal{T}\binom{0}{3} \qquad R(\tilde{w}) = R_{abc}{}^{d}\, w_d \equiv (\nabla_a\nabla_b - \nabla_b\nabla_a) w_c \qquad \underline{\text{Rieman curvature tensor}}}$$

- Expresses the commutation of double derivatives

- you can do the same by applying it to a vector

$$\begin{aligned}\nabla_a\nabla_b(f v^c) - \nabla_b\nabla_a(f v^c) &= (\nabla_a \cancel{\nabla_b f}) w_c + \nabla_b \cancel{f\, \nabla_a} w_c + \nabla_a \cancel{f\, \nabla_b} w_c + f\, \nabla_a\nabla_b w_c \; - \\ &\quad (\nabla_b \cancel{\nabla_a f}) w_c + \nabla_a \cancel{f\, \nabla_b} w_c + \nabla_b \cancel{f\, \nabla_a} w_c + f\, \nabla_b\nabla_a w_c \\ &= f(\nabla_a\nabla_b - \nabla_b\nabla_a) v^c\end{aligned}$$

$$\boxed{R : M_pT^* \to \mathcal{T}\binom{1}{2} \qquad R(\bar{v}) = R_{abc}{}^{d}\, v^a \equiv (\nabla_b\nabla_c - \nabla_c\nabla_b) v^d \qquad \underline{\text{Rieman curvature tensor}}}$$

Geodesic deviation equation

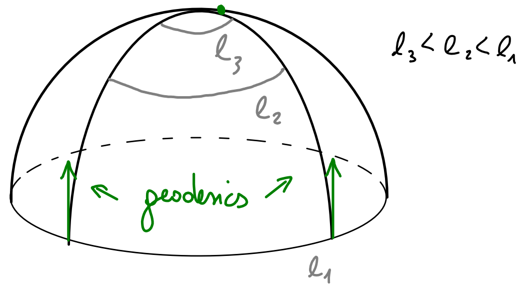
Last words on curvature, connection to physics

conceps of:

- "straight line" → geodesic eq.   OK
- Parallel lines in curved space?

  In flat space, 2 parallel straight lines never meet, this is not the case in curved space.

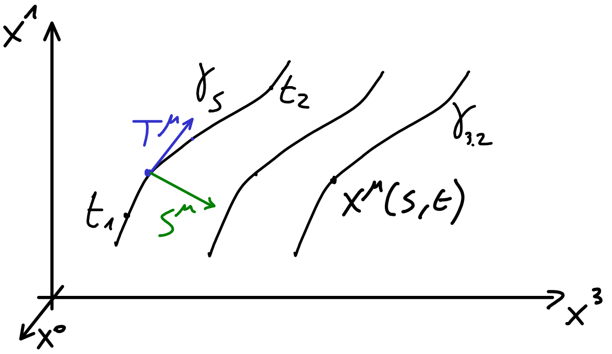
  points moving along geodesics can be drawn together/apart



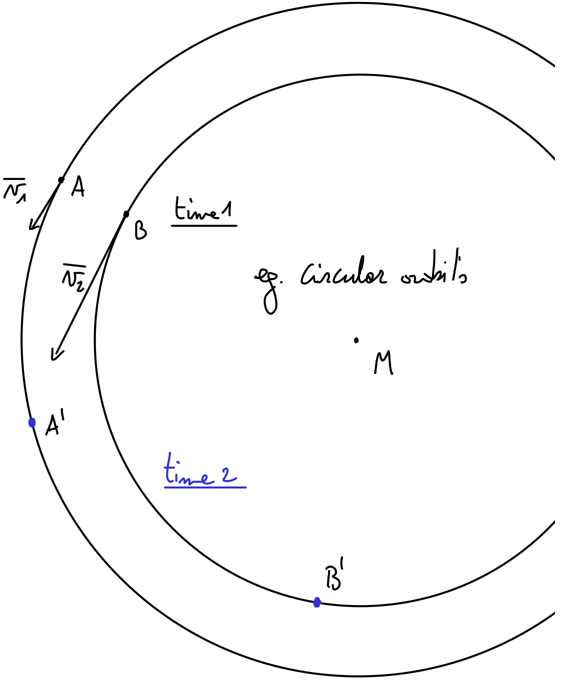
things moving along ≠ geodesics (free falling) are drawn togeter

Tydal forces

$$\bar{g}_1 \overset{1}{\rightarrow} \quad \bar{g}_2 \overset{2}{\longrightarrow} \quad \bullet M$$



we want to measure the acceleration along $\vec{S}$ of a free particle moving along the γ_s geodesic



Quantify the relative acceleration

- Define a sub-manifold "composed" by curves

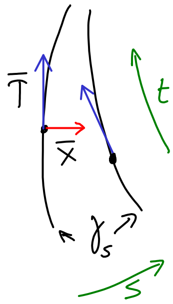
$\{\gamma_s(t)\}$ $s \in \mathbb{R}$ (continuous) One parameter set of non crossing geodesics $\gamma_s(t)$

$\mathcal{C}_s = \{P \in M \mid P = \gamma_s(t)|_s\}$ s-th geodesic curve (s fixed)

$\mathcal{C}_t = \{P \in M \mid P = \gamma_s(t)|_t\}$ t-th curve (t fixed)

$\gamma : \mathbb{R} \times \mathbb{R} \to M$ $\gamma : (t,s) \to \gamma_s(t)$ smooth map, one-to-one, smooth inverse

$\Sigma = \{P \in M \mid P = \gamma(s,t)\} \subset M$ submanifold spanned by the curves $\gamma_s(t)$



- Define a coordinate vector field based on the geodesics

- $\bar{T} = \frac{d}{dt}$ $T^a \nabla_a T^b = 0$ vector field, $T(t,s)$, tangent to the families of geodesics
- $\bar{X} = \frac{d}{ds}$ "deviation vector" " " , $X(t,s)$, gives displacement toward infinitesimally close geodesic
- you can reparametrize each curve γ_s $t \to t' = a(s) + b(s)t$ such that T and S orthogonal to each other

affine parameterization

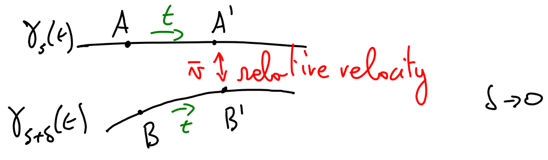
$\Rightarrow$ $g_{ab}T^aT^b = \text{const.}$ by varying s metric compatibility

$\Rightarrow$ convenient to use $\bar{T}, \bar{X}$ as a frame

- since T^a, X^a are coordinate vector fields $[\bar{T},\bar{X}] = 0$ $\Rightarrow$ $T^a\nabla_a X^b = X^a \nabla_a T^b - (\text{Torsion})^b$ $(*1)$

recall: $[\bar{T},\bar{X}]^b = T^a\nabla_a X^b - X^a\nabla_a T^b - \text{Torsion}^b$

- $\gamma_s(0)$ orthogonal to geodesics $\Rightarrow$ $X_aT^a = 0$ in $t=0$ $\models\!\!\Rightarrow$ $X_aT^a = 0$ everywhere



- "Geodesic velocity" $v^c \equiv \nabla_T X^c = T^b \nabla_b X^c$

rate of change along a geodesic (direction $\bar{T}$) of the displacement of an infinitesimally close geodesic

i.e. you have 2 points on infinitesimally close geodesics, $\bar{v}$ = they relative velocity as you run along geodesics

- "Geodesic acceleration" $a^c \equiv \nabla_T \overset{\downarrow}{v^c} = T^a \nabla_a (T^b \nabla_b X^c) = \ldots\ldots$

- *"Geodesic acceleration"*

$$
\begin{aligned}
a^c \equiv \nabla_T v^c &= T^a \nabla_a (\underbrace{T^b \nabla_b X^c}_{(*1)}) \\
&= T^a \nabla_a \left(X^b \nabla_b T^c + T_{or}{}^{b}(\bar T, \bar X) \right) \\
&= (T^a \nabla_a X^b) \nabla_b T^c + T^a X^b \underbrace{\nabla_a \nabla_b}_{(*2)} T^c + T^a \nabla_a T_{or}{}^{b}(\bar T, \bar X) \\
&= \quad '' \quad + T^a X^b R^c{}_{\gamma a b} T^\gamma + T^a X^b \nabla_b \nabla_a T^c \; (*3) + T^a \nabla_a T_{or}{}^{b}(\bar T, \bar X) \\
&= \cancel{''} \quad + \quad '' \quad + X^b \nabla_b (\underbrace{T^a \nabla_a T^c}_{\substack{=0 \\ \text{geodesic eq.}}}) - (T^a \cancel{\nabla_a X^b}) \nabla_b T^c + T^a \nabla_a T_{or}{}^{b}(\bar T, \bar X)
\end{aligned}
$$

$[T, S]^b = T^a \nabla_a S^b - S^a \nabla_a T^b - \text{Torsion}^b$

use commutation property (*1)

apply product rule

<u>define</u>: $R^c{}_{\gamma a b} T^\gamma \equiv \underbrace{\nabla_a \nabla_b}_{(*2)} T^c - \nabla_b \nabla_a T^c$

"inverse" (*3) product rule

$$\Rightarrow \boxed{a^c = R^c{}_{\gamma a b} T^\gamma T^a X^b} + T^a \nabla_a T_{or}{}^{b}(\bar T, \bar X)$$

<u>geodesic deviation eq.</u> (torsion free)

- relative acceleration (direction given by $\bar X$) between two points running along two infinitesimally close geodesics (T^γ and T^a) due to the curvature ($R^c{}_{\gamma a b}$)
- The <u>curvature tensor R</u> characterizes relative accelerations (due to "tydal forces")
- '' '' '' intrinsic curvature of space
- $\bar a = 0 \quad \forall$ family of geodesics $\iff$ $R^c{}_{\gamma a b} = 0$!
- derivatives of torsion affect accelerations
- tydal stress tensor $S^c{}_b \equiv R^c{}_{\gamma a b} T^\gamma T^a$
- in a frameless way: $a^c = R^c(\bar T, \bar T, \bar X) = S^c(\bar T, \bar T)\, \bar X$

- <u>This characterizes nearby geodesics (geometry)... now physical interpretation</u>

Consider $t = \tau$ proper time $\Rightarrow$ (time-like geodesics, where massive particles are)

$$T^\mu = \frac{dx^\mu}{d\tau} = u^\mu$$

$$\bar V = \nabla_T \bar X = \frac{d\bar X}{d\tau}$$

$$\bar A = \nabla_T \bar V = \frac{d^2 \bar S}{d\tau^2}$$

4-velocity along a geodesic

$$\boxed{\frac{d^2 x^c}{d\tau^2} = \underbrace{R^c{}_{\gamma a b} u^\gamma u^a}_{\equiv S^c{}_b \text{ tydal stress tensor}} X^b}$$

acceleration due to <u>tydal force</u>

4-acceleration toward the other geodesic

Contractions of the Rieman curvature tensor

- **Ricci tensor** $R_{\alpha\beta}$ contracion on 1° and 3° index $R = C^1_3 \bar{R}$

$\boxed{R_{j\ell} \equiv R^{k}{}_{jk\ell}}$ is the only non vanishing contraction of $R^{i}{}_{jk\ell}$ (because of Riemann connection!)

$R_{j\ell} = R_{\ell j}$ because $R_{j\ell} = R^{k}{}_{jk\ell} = g^{ik} R_{i\underset{\longleftrightarrow}{jk\ell}} = g^{ik} R_{i\ell kj} = R_{\ell j}$

Symmetry (2) (Torsion free)

(with other connections, eg. by having $T \neq 0$, you have other independent, non 0 contractions)

- **Ricci scalar** R: $R_{ij} \rightarrow R = R^{i}{}_{i}$

$\boxed{g^{ij} R_{ij} = R}$ trace of the Ricci tensor (double contraction of $R_{ijk\ell}$)

quantifies the local curvature

"alternative" : Kretschmann scalar $K = R^{\alpha\mu\nu\beta} R_{\alpha\mu\nu\beta}$

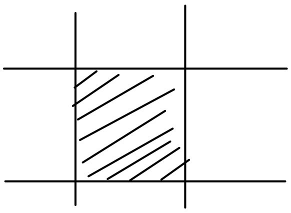
scalars ⇒ both are independent from coordinat choices

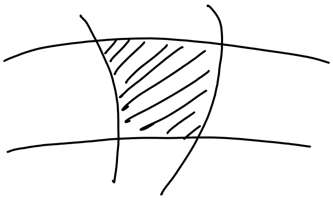
when moving

Geometrical interpretation

Ricci Tensor : how volums in curved space differ from flat space case
$R_{\alpha\beta}$ along a geodesics along the $\alpha\beta$ direction

(volume size, not shape) ⟵ shape change is given by Weyl tensor





Ricci scalar : $[R] = \frac{1}{m^2}$ (c=1)
R

Other related tensors

- **Weyl tensor**

- Is the symmetric part of the Riemann tensor

$$\boxed{R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha} - \frac{R}{3} g_{\alpha[\gamma} g_{\delta]\beta}}$$

- $C_{\alpha\beta\gamma\delta}$ is trace free on all its indeces
- Same symmetries of $R^{\alpha}{}_{\beta\gamma\delta}$: $C_{\alpha\beta\gamma\delta} = -C_{\beta\alpha\gamma\delta} = -C_{\alpha\beta\delta\gamma} = C_{\gamma\delta\alpha\beta}$
- Gives the diffeience in shape of volumes enclosed by geodesics with respect to flat space case

- **Einstein tensor** G_{ab}:

Double contraction of Bianchi identity $\nabla_{[\beta} \overset{\text{cyclic}}{R_{ij]}}{}_{k\ell} = 0$

$$g^{\beta k} g^{j\ell} \left(\nabla_\beta R_{ijk\ell} + \nabla_i R_{j\beta k\ell} + \nabla_j R_{\beta ik\ell} \right) = \nabla^k R_i{}^\ell{}_{k\ell} + \nabla_i R^{\ell k}{}_{k\ell} + \nabla^\ell R^k{}_{ik\ell} = \underline{\nabla^k R_{ik}} - \nabla_i R + \underline{\nabla^\ell R_{i\ell}} =$$

$$= 2\nabla^k R_{ik} - \nabla_i R = 2\nabla^k R_{ik} - g_{ik}\nabla^k R = \nabla^k \big(\underbrace{R_{ik} - \tfrac{1}{2} R g_{ik}}_{} \big) = 0$$

$$\boxed{\nabla^k G_{ik} = 0}$$

$$\boxed{G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}}$$

- $G_{\alpha\beta} = G_{\beta\alpha}$ because of symmetry of $R_{\alpha\beta}$ and $g_{\alpha\beta}$
- based on Curvature (Riemann tensor), i.e. 2° derivatives of the metric
- the only rank-2 tensor with vanishing covariant derivative (besides the metric because of metric compatibility condition)

- $\nabla^k G_{ik} = 0$ is also satisfied by $\nabla^k (G_{ik} + \Lambda g_{ik}) = 0$ $\Lambda \in \mathbb{R}$ (cosmological) constant because of metric compatibility $\nabla^k g_{ij} = 0$ more on that later....

↑

this property is super important !! We will see....

Summary: all our important tensors

g_{ab}	metric tensor: distances, scalar product
$Q_{cab} = \nabla_c g_{ab}$	non metricity tensor : change of scalar product with parallel transport
$T^a{}_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}$	torsion tensor: "closure of parallelograms" commutation $\nabla_{\bar{v}}\bar{u} - \nabla_{\bar{u}}\bar{v}$
$R^i{}_{jkl} = \delta_k \Gamma^i_{lj} - \delta_l \Gamma^i_{kj} + \Gamma^i_{k\gamma}\Gamma^\gamma_{lj} - \Gamma^i_{l\gamma}\Gamma^\gamma_{kj}$	Riemann tensor: "difference in parallely transport along a loop" commutation $(\nabla_{\bar{v}}\nabla_{\bar{u}} - \nabla_{\bar{u}}\nabla_{\bar{v}})\bar{s}$
$R_{jl} \equiv R^k{}_{jkl}$	Ricci tensor: "difference in volumes within geodesics"
$R^j{}_j = R$	Ricci scalar: "curvature" $[R] = \frac{1}{m^2}$
$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$	Einstein tensor: the one satisfying $\nabla^\kappa G_{i\kappa} = 0$

$C_{\alpha\beta\gamma\delta}$: $R_{\alpha\beta\gamma\delta} \equiv C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha} - \frac{R}{3} g_{\alpha[\gamma}g_{\delta]\beta}$ Weyl tensor:

"difference in shape of volumes"
whith respect to flat space

Conserved quantities

- $\boxed{\frac{\delta L}{\delta x^\mu}=0}$ => shift along x^μ direction no change in L (symmetry of the system)
  to each symmetry corresponds a conserved quantity

- eg. action of massive particle: $S=-mc\int(-g_{\alpha}u^\alpha u^\beta)^{1/2}d\tau$ $\quad L(\bar{x},\bar{u})=-mc(-g_{\alpha\beta}u^\alpha u^\beta)^{1/2}$ $\quad g_{\alpha\beta}(x^\mu)$ ↑ !

$$\Rightarrow \frac{\delta L}{\delta x^\mu}=-mc\frac{1}{2}(-g_{\alpha\beta}u^\alpha u^\beta)^{-1/2}(-\delta_\mu g_{\alpha\beta}u^\alpha u^\beta)=m\frac{1}{2}\delta_\mu g_{\alpha\beta}u^\alpha u^\beta=0$$

$$\Updownarrow$$

$$\frac{dP_\mu}{d\tau}=\frac{m}{2}\delta_\mu g_{\alpha\beta}u^\alpha u^\beta \qquad \boxed{\delta_\mu g_{\alpha\beta}=0}$$ i.e. metric does not depend on x^μ !

- Just coming out of intrinsic properties of space-time:

$$\frac{Du_\nu}{d\tau}=\frac{du_\nu}{d\tau}-\Gamma^\beta_{\alpha\nu}u^\alpha u_\beta$$ ↙ Torsion free + Metric compatibility

$$=\frac{du_\nu}{d\tau}-\frac{1}{2}g^{\beta\gamma}(\delta_\alpha g_{\gamma\nu}+\delta_\nu g_{\alpha\gamma}-\delta_\gamma g_{\nu\alpha})u^\alpha u_\beta$$

$$=\frac{du_\nu}{d\tau}-\frac{1}{2}(\cancel{\delta_\alpha g_{\gamma\nu}}+\delta_\nu g_{\alpha\gamma}-\cancel{\delta_\gamma g_{\nu\alpha}})u^\alpha u^\gamma$$

$$=\boxed{\frac{du_\nu}{d\tau}-\frac{1}{2}\delta_\nu g_{\alpha\gamma}u^\alpha u^\gamma=0}$$

Free particles (massive/massless) follow geodesics with tangent vector $\bar{u}$ given above
4-momentum and 4-frequency are tangents as well => obey same role of $\bar{u}$

- Eg. $\delta_0 g_{\mu\nu}=0$ => energy conservation (time symmetry)
- Eg. $\delta_\phi g_{\mu\nu}=0$ => angular momentum conservation [Polar coordinates $(x^\mu)=(ct,r,\theta,\varphi)$]

- Issues and solutions

1) You need a well suited coordinate system to identify the conserved quantities
   eg. spherical coord. if your system is spherically symmetric, very rigid approach ....

2) Define a "new" conserved quantity $P_\nu \to P_\nu\xi^\nu$ Killing vectors ξ (Killing vector field)

3) Use Lie derivatives $\mathcal{L}_K g_{\mu\nu}=0$

All these concepts are tightly related !!

Killing vectors

- We have seen: $\partial_\gamma g_{\alpha\beta} = 0 \Rightarrow$ conserve quantity
- Generalize this concept: identify arbitrary directions, $\bar{\xi}$, along which $g_{\alpha\beta} = \text{const.}$
- The projection of P_γ (u_γ!) on those direction is the conserved quantity

$$(P_\alpha \xi^\alpha)_{;\beta} \underset{\uparrow}{=} (P_\alpha \xi^\alpha)_{,\beta} = 0 \qquad P_\alpha \xi^\alpha = \text{component Projected along vector } \bar{\xi} \text{ is conserved}$$

because $P_\alpha \xi^\alpha \in \mathbb{R}$

ξ^α = Killing vector (Killing = person, not verb)

- <u>A trivial case</u>: if $g_{\alpha\beta,0} = 0$ (no dependency on x^0) $\Rightarrow P_0 = \text{const}$ i.e $P_{0,0} = 0$

$\bar{\xi} = \bar{\delta}_0$ killing vector $P_\alpha \xi^\alpha = P_\alpha \delta^\alpha_0 = P_0$ conserved

Killing vector field

Expand the concept to a vector field $\xi(\bar{x})$ to probe the entire space

$$p^\beta (p^\alpha \xi_\alpha)_{;\beta} = \underbrace{p^\beta p^\alpha_{;\beta}}_{=0} \xi_\alpha + p^\beta p^\alpha \xi_{\alpha;\beta} \overset{!}{=} 0 \quad \Rightarrow \quad \underbrace{p^\beta p^\alpha}_{} \xi_{\alpha;\beta} \overset{!}{=} 0$$

geodesic condition

symmetric $\Rightarrow \xi_{\alpha;\beta}$ anti-symmetric, i.e. $\xi_{\alpha;\beta} = -\xi_{\beta;\alpha}$

u^α! same for photons: $\bar{k}$

$$\Downarrow$$

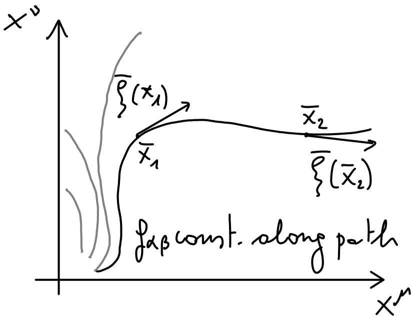
$$\boxed{\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0} \quad \underline{\text{Killing eq.}} \quad (1)$$

$$\boxed{K = \{\bar{\xi}(\bar{x}) \mid \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0\}} \quad \underline{\text{Killing vector field}}$$

- (1): Condition defyning the Killing vector field $\bar{\xi}(\bar{x})$ for a given metric: $g(P)$ determines $\xi(P)$

Γ contains the metric $g(x)$

- <u>Projection of $\bar{P}$ and $\bar{k}$ along a Killing vector field is conserved</u>



intuitive: particle moves from $\bar{x}_1$ to $\bar{x}_2$

$g_{\mu\nu}$ is always the same $\Rightarrow$ no "work" from gravity

$\Rightarrow$ conservation

<u>eg. Schwarzschild metric</u> : $g = \mathrm{diag}(-A(r), B(r), r^2, r^2 \sin^2\theta)$ (Polar coordinates) $(x^\mu) = (ct, r, \theta, \varphi)$

-> No dependency on t => P_t is conserved (energy) time symmetry

-> " " " φ => P_φ " " (angular momentum)

=> Killing vectors $\bar{\xi}_i$ are $\xi_\mu P^\mu = \text{const}$ => $P_\mu \xi_t^\mu = \text{const}$ $\bar{\xi}_t = \bar{K} = (1,0,0,0)$ for t

$P_\mu \xi_\varphi^\mu = \text{const}$ $\bar{\xi}_\varphi = \bar{\Phi} = (0,0,0,1)$ for φ

$$P_t = g_{tt} P^t = -A(r) P^0 = -A(r) \frac{E}{c} \equiv B = \text{const} \quad E = B' A(r) \text{ energy at } r$$

$$P_\varphi = \overset{\downarrow}{g_{\varphi\varphi}} P^\varphi = r^2 \underset{\uparrow}{\sin^2\theta}\, P^\varphi = r^2 m \frac{d\varphi}{d\tau} \equiv J = \text{const} \quad \text{angular momentum}$$

spherical symmetry => chose a convenient $\theta = \frac{\pi}{2}$ (no loss of generality)

Lie derivatives: introduction

- We have seen that if $g_{\mu\nu}$ = const along some direction => identify conserved quantity

  $\boxed{\partial_\nu g_{\mu\nu}=0}$

  Not general enough because $g_{\mu\nu}$ depend on coordinates

  => generalized by projecting on killing vectors $\bar{\xi}$

  => even more general by identifying Killing vector fields $\bar{\xi}(\bar{x})$ $\xi_{a;b}+\xi_{b;a}=0$

- Define a new type of derivatives to identify the Killing vector field ξ

  $\boxed{\mathcal{L}_\xi g_{ab}=0}$

  Derivative with respect to vector fields instead of coordinates!

  $\mathcal{L}_V T$: identifies the change of a tensor T along a vector field V

  Lie-derivatives are defined as $\boxed{\mathcal{L}_V T^{ab} = \lim_{t\to 0}\frac{\Phi_t^* T^{ab} - T^{ab}}{t}}$

- To define it we need:

  1) Pool-back / Push-forward Φ_t^*

     to move around tensors to quantify theyr change from one position to another

  2) Φ_t^* based on a family of diffeomorphysms $\{\phi_t \mid \phi_t : M \to M\}$

     to have at hands all possible "infinitesimal moving around"

  3) Associate a vector field to ϕ_t

     to express the newly defined Lie derivative in terms of vector fields

- See equivalent conditions

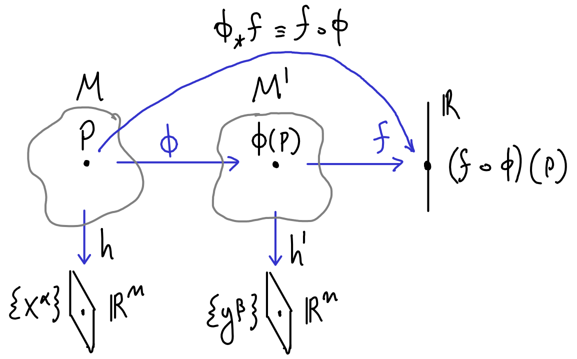
  $\mathcal{L}_\xi g_{ab}=0$ $\phi_t g = 0$ $\xi_{a;b}+\xi_{b;a}=0$

- They show the existance of conserved quantities for free particles $\xi_a \underset{\substack{\uparrow \\ \text{tangent to geodesics}}}{u^a} = 0$

Pullback / Pushforward

Idea: instrument allowing to evaluate a tensor in different points of the manifold

1) Link two manifolds M, M': pull-back, push-forward



$m = \dim(M)$ $\quad$ $m' = \dim(M')$

$\phi : M \to M'$ $\quad$ C^∞ map, $\phi(P) = P'$

$f : M' \to \mathbb{R}$ $\quad$ scalar smooth function

$\boxed{\phi_* f \equiv f \circ \phi : M \to \mathbb{R}}$ $\quad$ pull-back of f by ϕ

$\bar{v} \in M_p T$ $\quad$ vector in point $P \in M$

$\tilde{w}' \in M'_{\phi(P)} T$

$$\begin{tikzcd} M_p T & v \arrow[r, "\phi^* v"] & (\phi^* v) & M'_{\phi(P)} T \\ M_p T^* & \phi_* w' & w' \arrow[l, "\phi_* w"'] & M'_{\phi(P)} T^* \end{tikzcd}$$

- Pushforward of a vector

$$\boxed{\begin{array}{ll} \text{define:} & \phi^* : M_p T \longrightarrow M'_{p'} T \qquad \bar{v} \longrightarrow \phi^* \bar{v} \in M'_{p'} T \quad \text{linear map} \\ \text{by requiring:} & (\phi^* \bar{v})(f) \overset{!}{=} \bar{v}(f \circ \phi) = v(\phi_* f) \end{array}}$$

in matrix form
$$(\phi^* \bar{v})^\beta = (\phi^*)^\beta{}_\alpha v^\alpha$$

↳ vector $\bar{v}$ applied to a scalar functions $(\phi_* f)$ in $p \in M$

→ vector $(\phi^* v)$ applied to scalar functions f in $\phi(P) \in M'$

it satisfies properties of tangent vectors

- Pullback of one form

$$\boxed{\begin{array}{ll} \text{define:} & \phi_* : M'_{p'} T^* \to M_p T^* \qquad \tilde{w}' \to \phi_* \tilde{w}' \in M_p T^* \quad \text{linear map} \\ \text{by requiring:} & (\phi_* \tilde{w}')\bar{v} \overset{!}{=} \tilde{w}'(\phi^* \bar{v}) \quad \langle \phi_* \tilde{w}', \bar{v} \rangle = \langle \tilde{w}', \phi^* \bar{v} \rangle \end{array}}$$

in matrix form
$$(\phi_* \tilde{w}')_\mu = (\phi_*)_\mu{}^\alpha \tilde{w}'_\alpha$$

↳ consistency of 1-forms and vectors in M and M' "moved by ϕ^*, ϕ_*"

- This can be extended to all tensors of type $\binom{m}{0}$ and $\binom{0}{m'}$ not to mixed ones

- Push-forward of tensor $\binom{r}{0}$ by ϕ:

$$\boxed{\phi^* : M_p T^r_0 \to M'_{\phi(P)} T^r_0 \quad T \mapsto \phi^* T = T \circ \phi_*}$$
$$(\phi^* T)(\tilde{v}_1, \dots \tilde{v}_r) = T(\phi_* \tilde{v}_1, \dots \phi_* \tilde{v}_r)$$ as for vectors

- Pull-back of tensor $\binom{0}{r}$ by ϕ:

$$\boxed{\phi_* : M'_{\phi(P)} T^0_r \to M_p T^0_r \quad T \mapsto \phi_* T = T \circ \phi^*}$$
$$(\phi_* T)(\bar{v}_1, \dots \bar{v}_r) = T(\phi^* \bar{v}_1, \dots \phi^* \bar{v}_r)$$ as for 1-forms

- Pull-back, push-forward for tensors $\binom{r}{s}$ by ϕ:

possible only if $\dim(M) = \dim(M')$

ϕ diffeomorphysm : $\phi : M \to M'$ C^∞ map, one-to-one with inverse C^∞

$\phi^* = (\phi^{-1})_* \quad \phi_* = (\phi^{-1})^*$

$\Rightarrow$ can use ϕ^{-1} to extend ϕ^* to tensors of all types

$$\boxed{\begin{array}{l} \phi^* : M_p T^r_l \to M'_{\phi(p)} T^r_l \\ \phi_* : M'_{\phi(p)} T^l_r \to M_p T^l_r \end{array}}$$

$$(\phi^* T)(\tilde{v}_1, \dots \tilde{v}_r, \bar{v}_1, \dots \bar{v}_l) = T(\phi_* \tilde{v}_1, \dots \phi_* \tilde{v}_r, (\phi^{-1})^* \bar{v}_1, \dots, (\phi^{-1})^* \bar{v}_l)$$

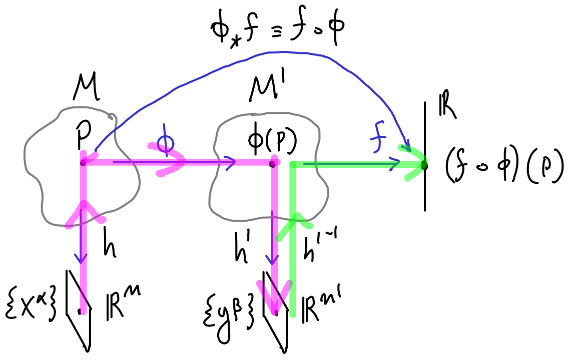
$$(\phi_* T)(\tilde{v}_1, \dots \tilde{v}_r, \bar{v}_1, \dots \bar{v}_l) = T((\phi^{-1})_* \tilde{v}_1, \dots, (\phi^{-1})_* \tilde{v}_r, \phi^* \bar{v}_1, \dots, \phi^* \bar{v}_l)$$

ϕ^* with charts

- Express $(\phi^* v)(f) \equiv v(\phi_* f)$ with respect to the charts:

Chart (U, h) on M $\quad P \in U \quad$ coord $\{x^\alpha\}$ $\quad$ basis $\frac{\delta}{\delta x^\alpha}$ $\quad v = v^\alpha \delta_\alpha \big|_p \in M_p T$

" (U', h') " M' $\quad \phi(P) \in U'$ $\quad$ " $\{y^\beta\}$ $\quad$ " $\frac{\delta}{\delta y^\beta}$ $\quad (\phi^* v) = (\phi^* v)^\beta \delta_\beta \big|_{\phi(p)} \in M'_{\phi(p)} T$



$$\begin{pmatrix} M_p T \\ M_p T^* \end{pmatrix} \quad \langle \phi_* \tilde{w}', \bar{v} \rangle = \langle \tilde{w}', \phi^* \bar{v} \rangle \quad \begin{pmatrix} M'_{\phi(p)} T \\ M'_{\phi(p)} T^* \end{pmatrix}$$

ϕ^*

ϕ_*

$(\phi^* v)(f) \equiv v(\phi_* f)$

$$(\phi^* v)^\beta \frac{\delta}{\delta y^\beta}(f \circ h'^{-1}(y)) \Big|_{y=(h' \circ \phi)(P)} = v^\alpha \delta_\alpha (f \circ \phi \circ h^{-1}(x)) \Big|_{x=h(P)}$$

$$(f \circ \phi \circ h^{-1})(x) = (f \circ h'^{-1}) \circ \underbrace{(h' \circ \phi \circ h^{-1})(x)}_{y(x)}$$

$$= v^\beta \delta_\beta (f \circ h'^{-1} \circ y(x)) \Big|_{x=h(P)}$$

$$= v^\beta \frac{\delta(f \circ h'^{-1}(y))}{\delta y^\alpha} \frac{\delta y^\alpha(x)}{\delta x^\beta} \Big|_{y=(h' \circ \phi)(P)} \quad \Rightarrow \quad \boxed{(\phi^*)_\beta^{\ \alpha} = \frac{\delta y^\alpha}{\delta x^\beta}}$$

$\Rightarrow$ ϕ^* is a generalization of a coordinate transformations

Dimension of M and and N might differ and ϕ might not be invertible

- View (1)

  ϕ changes the relationship between Tensor fields and the position on the manifold

  ϕ^*T : T moved by ϕ in another position of M

  Tensors at P to tensors at $\phi(P)$ => They are "sort of distorted" with respect to position

- View (2)

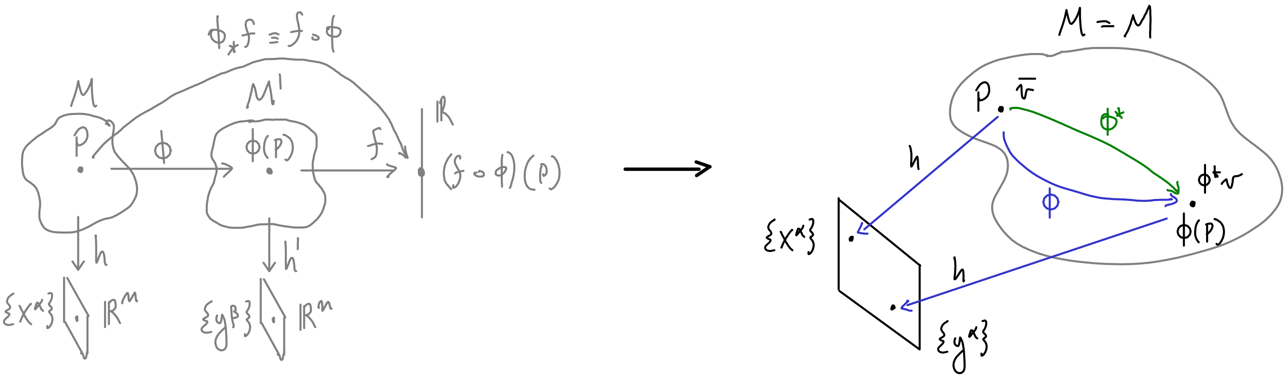
  when you have frames, you can see the effect of ϕ as a coordinate transformation

  $(\phi^*)_\beta{}^\alpha$ expresses a (even more general) coord transformation once a chart is induced

  => Tensors are unchanged, you change their relationship to coordinates

Diffeomorphysms and symmetries of a system

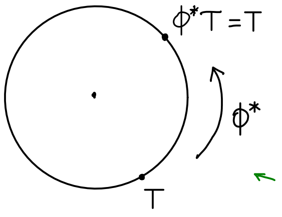
- $\phi : M \to M$ diffeomorphysm Diff(M) ↗ invertible
  i.e. isomorphysm (1 to 1 invertible mapping) between differentiable (smooth) manifold



- ϕ^* and ϕ_* used "within" the same manifold
- can compare T with $\phi^* T$ T tensor field in M

- $\boxed{\text{If } \phi^* T = T}$ => ϕ is a symmetry (transformation) of T (you push T somewhere else and you get the same properties)

e.g. if you have a sperically symmetric system



You "move" the tensor to an "equivalent" point

↖ do you see the link between simmetries and conserved quantities?

- $\boxed{\text{If } (\phi^* g)_{ab} = g_{ab}}$ => ϕ = isometry of M continuous symmetries of a metric g on M
  e.g. the Lorentz transforms are isometries in manifold with $\mathbb{M}$: $\Lambda^T \eta \Lambda = \eta$

Equivalence of physical theories

- $\phi : M \to N$ diffeomorphysm

$G_{\mu\nu} = \alpha T_{\mu\nu}$ (G.R.)
$\uparrow$

- If M, N identical manifold structure

  $(M, \{T\})$: theory describing nature in terms of a manifold M and tensors $\{T\}$

  $(N, \{\phi^* T\})$: another theory

  related by a Diffeo. ϕ

  $\Rightarrow$ solutions of $(M, \{T\})$ and $(N, \{\phi^* T\})$ have same physical properties

- If M, N not related by a diffeo ϕ $\Rightarrow$ different physical results

  $\Rightarrow$ $(M, \{T\})$ and $(N, \{\phi^* T\})$ physically distinguishable

$\models\!\!\Rightarrow$ Diffeomorphisms comprise the gauge freedom of any theory based on tensor fields on a manifold (GR included)

$\Downarrow$

is a diffeomorphysm invariant theory

Vector fields, diffeomorphisms and coord. transformations: flows

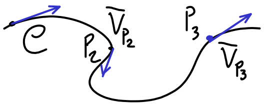
- Idea: Convenient to generate all ϕ with vector fields

- Vector field : smooth map assigning a vector to each point P $\bar{V}: M \to M_P T$ $P \to \bar{V}(P) = \bar{V}_P$
  with a chart $h(P) = \bar{x}$: $\bar{V}(\bar{x})$

$\bar{V}(P)$ generates integral curves: curves whose tangent vectors at any P are given by V_P

- $\mathcal{C}$ integral curve of $\bar{V}$: $\mathcal{C}: \mathbb{R} \to M$ $\lambda \to P(\lambda)$ whose tangentvectors at any P are given by $\bar{V}(P) = \frac{d}{d\lambda}\Big|_P$

with a chart: $\frac{d}{d\lambda} = \frac{dx^\mu(\lambda)}{d\lambda} \partial_\mu \Big|_{P(\lambda)}$ for $\boxed{\frac{dx^\mu}{d\lambda} = V^\mu(x(\lambda))}_{(1)}$ (flow equation)

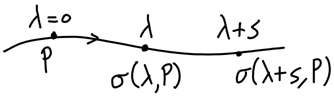


(1): ODE to be integrated to get $x^\mu(\lambda)$ of curve $\Rightarrow$ $\begin{cases} \text{unique solution} & x^\mu(\lambda) = \sigma_V^\mu(\lambda, x_{(0)}) \\ \text{given init. condition} & x^\mu(0) = x^\mu_{(0)} \end{cases}$

- Flow of $\bar{V}$: to have integral curves starting at any point P

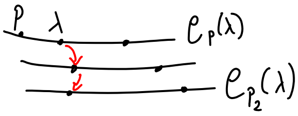
smooth map $\sigma_V : \mathbb{R} \times M \to M$ $(\lambda, P) \to \sigma_V(\lambda, P)$

- $\sigma_{cV}(\lambda, P) = \sigma_V(c\lambda, P)$ $c \in \mathbb{R}$ $V \to cV$ scaling the vector field = reparameterization $\frac{dx}{d\lambda} = c\frac{dx}{d(c\lambda)}$
- $\sigma_V(\lambda+s, P) = \sigma_V(\lambda, \sigma_V(s,P)) = \sigma_V(\lambda, P) \circ \sigma_V(s,P)$



For fixed P $\Rightarrow$ gives a curve passing through P : $\mathcal{C}_P(\lambda): \mathbb{R} \to M$

For fixed λ $\Rightarrow$ gives a diffeomorphism $\sigma(\lambda, P)|_\lambda = \phi_\lambda : M \to M$ $P \to (\sigma_V(\lambda))(P)$



(2) $\{\phi_\lambda\}$: 1 parameter continuous group with Abelian structure (paramer = λ):

given V !

$\phi_\lambda \circ \phi_s = \sigma_V(\lambda+s)$	multiplication (they commute)	because $\lambda+s = s+\lambda$
$\phi_0 = Id_M$	identity	$\phi_\lambda \circ \phi_0 = \phi_{\lambda+0} = \phi_\lambda$
$(\phi_\lambda)^{-1} = \phi_{-\lambda}$	inverse	$\phi_\lambda \circ \phi_{-\lambda} = \phi_0 = Id_M$

$\Rightarrow$ "active" coord. transformations generated by V

(b) Transformation for infinitesimal $\lambda = \varepsilon$: $P' = \phi_\varepsilon(P)$
inverse : $P = \phi_{-\varepsilon}(P')$

in a frame

Taylor expand solution of (1) : $x^{\mu'} = \phi^\mu_\varepsilon(x^\nu) = x^\mu + \overbrace{V^\mu(x)\cdot\varepsilon}^{dx^\mu} + O(\varepsilon^2)$
inverse : $x^\mu = \phi^\mu_{-\varepsilon}(x^\nu) = x^{\mu'} - V^\mu(x)\cdot\varepsilon + O(\varepsilon^2)$

$\Rightarrow \forall f$ $f(x') = f(x) + V^\alpha(x)\left(\partial_\alpha f(x)\right) + O(\varepsilon^2)$

Lie derivatives

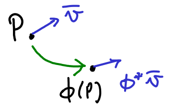
Define a new types of derivatives ~ $\delta f = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

rate of change of scalar function at P along an integral curve through P given by $\sigma_V(\lambda, \underset{\substack{\uparrow \\ \text{fixed}}}{P})$
(directional derivative)

given a vector field => $V_P[f] = \left.\frac{df}{d\lambda}\right|_P \equiv \lim_{\varepsilon \to 0} \frac{f(P') - f(P)}{\varepsilon}$ $P' = \sigma_V(\varepsilon)(P)$

The Lie derivative is "that" with on top the pull-back to manage tensors

1) Need to evaluate a tensor in different locations to compute the difference
Idea: use pull-back/forward to "transport" tenssors in different points



eg. $T_{ab}(\bar{x})$ tensor field: $T_{ab}(P)$ evaluated in P

$(\Phi_* T_{ab})(\phi(P))$ "T_{ab}" evaluated in $\phi(P)$ Diffeo. $\phi: M \to M$

$\Delta T = \left[(\Phi_* T_{ab})(\phi(P)) - T_{ab}(P)\right]$ Direct comparison can be done

ϕ provides the "$x + \Delta x$" ~ $P \longrightarrow \phi(P)$

Φ_* allows to evaluate T_{ab} in $\Phi(P)$ because $(\Phi_* T) \in M_{\phi(P)} \mathcal{T}(0,2)$

with one ϕ we get the change of ϕ along a certain curve only

2) One ϕ alone is not sufficient to define a derivative, you need a set of Φ_t
to move smoothly from one point to another (need infinitesimal shifts)

take a vector field: $V: M \to M_p T$

define the flow of its integral curves: $\sigma_V: \mathbb{R} \times M \to M$ $(\lambda, P) \to \sigma_V(\lambda, P)$

to buil a continuous set of diffeomorphysms: $\{\Phi_\lambda\}$ $\Phi_\lambda: M \to M$ $\Phi_\lambda \equiv \sigma_V(\lambda, P)$

3) Define the Lie derivative of a tensor field T with respect to a vector field V

$\mathcal{L}_V: M_p \mathcal{T}^q_r \longrightarrow M_p \mathcal{T}^q_r$
$T \to \mathcal{L}_V T$

$$\boxed{\mathcal{L}_V T\Big|_P \equiv \lim_{\lambda \to 0} \frac{\Phi_\lambda^* T\big|_{P'} - T\big|_P}{\lambda}}$$

$P' = \Phi_\lambda(P)$ Φ_λ generated by V
$T \in \mathcal{T}(q, r)$

like a directional derivative defined we have seen... but in all $P \in M$
direction given by $\{\Phi_t\}$ / vector field

- Components of the Lie derivatives (applied to a vector fields)

eg. $\mathcal{L}_V W$: $\bar{V} = V^\mu(x)\partial_\mu$ $\bar{W} = W^\mu(x)\partial_\mu$ transf: $x^\mu(p') = x^\mu(p) + \varepsilon V^\mu(p) + O(\varepsilon^2)$

use definition => $\mathcal{L}_V W\big|_p = \left(V^\mu(x)\partial_\mu W^\nu(x) - W^\mu \partial_\mu V^\nu(x)\right)\partial_\nu\big|_{x=x(p)} = \underbrace{[\bar{V},\bar{W}]^\nu}_{\text{Lie brackets}} \partial_\nu\big|_{x=x(p)}$

- Lie brackets of two vector fields

$[\cdot,\cdot] : \mathcal{T}^1_0 \times \mathcal{T}^1_0 \to \mathcal{T}^1_0$ $(X,Y) \to [X,Y]$ $[X,Y](f) \equiv X[Y(f)] - Y[X(f)]$

satisfy Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Properties of the Lie-derivatives

- valid for all derivative operators! Being based on push-forward, it does not depend on the connection Γ!
- $\mathcal{L}_V(aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S$ Linear $a,b \in \mathbb{R}$, $T \in \mathcal{T}^\ell_k$ ↙ this is because we imposed $\phi_t \circ \phi_s = \phi_{s+t}$
- $\mathcal{L}_V(T \otimes S) = S \otimes (\mathcal{L}_V T) + T \otimes (\mathcal{L}_V S)$ obey Leibnitz rule (i.e. product rule)
- $C(\mathcal{L}_V T) = \mathcal{L}_V(CT)$ commutes with contractions
- $\mathcal{L}_{\bar{X}}\bar{Y} = [\bar{X},\bar{Y}]$ $\mathcal{L}_{\bar{X}} Y^a = [\bar{X},\bar{Y}]^a = X^b\partial_b Y^a - Y^b\partial_b X^a = X^b\nabla_b Y^a - Y^b\nabla_b X^a - T^a(\cancel{\bar{X},\bar{Y}})$ $\bar{X},\bar{Y}$ vector fields (Lie brackets) consider Torsion free

=> convenient to use $\partial_a \to \underline{\nabla_a}$ (clear meaning in our context)

For a scalar: $\mathcal{L}_V f = V^\nu \nabla_\nu f$ ↙ "Lie-brackets"

" " vector: $\mathcal{L}_V W^\mu = [V,W]^\mu = V^\nu\nabla_\nu W^\mu - W^\nu\nabla_\nu V^\mu$ (torsion free)

" " 1-form: $\mathcal{L}_V w_\mu = V^\alpha\nabla_\alpha w_\mu \underset{\circledast}{+} w_\alpha\nabla_\mu V^\alpha$ $\circledast$ ↰

" " Tensor: $\mathcal{L}_V T_{\mu\nu} = V^\gamma\nabla_\gamma T_{\mu\nu} + T_{\mu\gamma}\nabla_\nu V^\gamma + T_{\gamma\nu}\nabla_\mu V^\gamma$ ↜ with metric compatibility

" " Metric: $\mathcal{L}_V g_{\mu\nu} = V^\gamma\nabla_\gamma g_{\mu\nu} + g_{\mu\gamma}\nabla_\nu V^\gamma + g_{\gamma\nu}\nabla_\mu V^\gamma = \nabla_\nu V_\gamma + \nabla_\gamma V_\nu = 2\nabla_{(\nu}V_{\gamma)}$

in general: $\mathcal{L}_v T^{a_1\dots a_k}{}_{b_1\dots b_\ell} = v^c\nabla_c T^{a_1\dots a_k}{}_{b_1\dots b_\ell} - \sum_{i=1}^{k} T^{a_1\dots a_k}{}_{b_1\dots b_\ell}\nabla_c v^{a_i} + \sum_{i=1}^{k} T^{a_1\dots a_k}{}_{b_1\dots b_\ell}\nabla_{c_i} v^{a_i}$

- $\mathcal{L}_{\bar{v}}\bar{\partial}_i = [\bar{v},\bar{\partial}_k] = \bar{v}\bar{\partial}_k - \bar{\partial}_k\bar{v} = \left(v^i\bar{\partial}_i\,\partial^j_k\bar{\partial}_j - \partial^i_k\bar{\partial}_i\, v^j\bar{\partial}_j\right) = \left(v^i\underbrace{\bar{\partial}_i\partial^j_k}_{=0} - \partial^i_k\bar{\partial}_i v^j\right)\bar{\partial}_j = -\left(\bar{\partial}_k v^j\right)\bar{\partial}_j$
- Many other properties ....

↱ $\circledast$ $\mathcal{L}_v(w_a u^a) \overset{w_a u^a \in \mathbb{R}}{=} v(w_a u^a) = v^b\partial_b(w_a u^a) \overset{\text{Leibnitz}}{=} v^b u^a\partial_b w_a + \cancel{v^b w_a\partial_b u^a}$ ↔

$\mathcal{L}_v(w_a u^a) \underset{\text{Leibnitz}}{=} u^a\mathcal{L}_v w_a + w_a\mathcal{L}_v u^a = u^a\mathcal{L}_v w_a + w_a[v,u]^a \underset{\text{Lie brackets}}{=} u^a\mathcal{L}_v w_a + \cancel{w_a v^b\partial_b u^a} - w_a u^b\partial_b v^a$

$\Rightarrow \cancel{u^a}\mathcal{L}_v w_a = v^b\cancel{u^a}\partial_b w_a + w_a\cancel{u^b}\partial_b v^a$ ✓

Lie-derivatives and killing vector fields

- Take $T_{ab} = g_{ab}$ $\Rightarrow$ if $\boxed{\mathcal{L}_{\xi} g = 0}$ advantage : no dependency on coordinates

$\Rightarrow$ $\underline{\bar{V} = \bar{\xi}}$ is the Killing vector field (!)

vector field along which g_{ab} do not vary

defines a symmetry transformation ϕ (isometry) : $\phi^* g = g$

it tells you about conserved quantities

- Killing equation

$$\mathcal{L}_{K} g_{ab} = \xi^c \underset{* = 0}{\nabla_c g_{ab}} + g_{cb} \nabla_a \xi^c + g_{ac} \nabla_b \xi^c = \boxed{\nabla_a \xi_b + \nabla_b \xi_a = 0}$$

* Metric compatibility

- Conserved quantities

ξ^a : Killing vector field

γ : geodesic line with tangent u^a

$\Rightarrow$ Projection of u on Killing vectors $\boxed{\xi_a u^a = \text{const.}}$ along γ

4-momentum $p^a = m u^a$

4-frequency k^a

□ directional derivative along γ : $\nabla_{\bar{u}}(\xi_a u^a) = u^b \nabla_b (\xi_a u^a) = \overset{(1)}{u^b \xi_a \nabla_b u^a} + \overset{(2)}{u^b u^a \nabla_b \xi_a} = 0$ ✓

(1) = 0 because geodesic eq.

(2) = 0 because Killing eq. : $u^b u^a$ symmetric and $\nabla_b \xi_a$ antisymmetric

- Equivalent conditions to identify ξ / $\{\phi_t\}$ / coord transf. $x^{\nu}_{,\nu'}$ (symmetries)

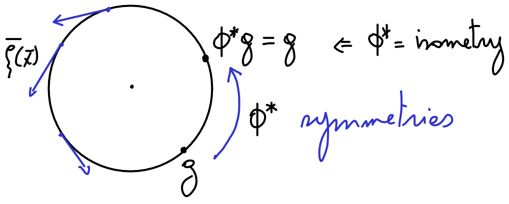
$\xi_{b;a} + \xi_{a;b} = 0$ Killing eq.

$\Updownarrow$

$\mathcal{L}_{\xi} g_{\mu\nu} = 0$ ξ = Killing v. field

$\Updownarrow$

$\phi_* g_{\mu\nu} = g_{\mu\nu}$ ϕ = isometry



$\Downarrow$

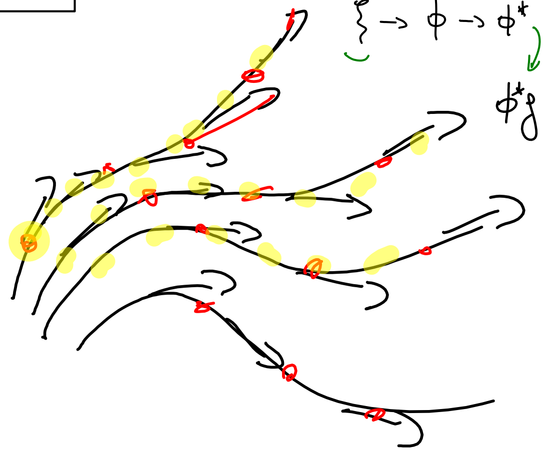
$\xi_a u^a$ = const. Conserved quantities for free particles (massive and massless)

because they follow geodesics (with tangent $\bar{u}$)

- In general, for any tensor T, if $\mathcal{L}_V T = 0$ T = const along vectors $\bar{V}(P)$

eg. Energy-Momentum tensor, 4-momentum

Appendix



$\bar{\xi} \to \phi \to \phi^*$

$\phi^* g = g \longleftrightarrow \mathcal{L}_{\bar{\xi}} g = 0$

Tensor field

- Definition: Is a map that smoothly assigns a tensor to each point in M,

  $A : M \to \mathcal{T}(r,q)_P \qquad P \to A_P = A(P)$ with a chart $h(P) = x$: $A(x)$

- Vector fields are in $\mathcal{T}(1,0)$ $\quad \bar{v} : M \to \mathcal{T}(1,0)_P \quad P \to \bar{v}_P \quad P \in M \quad \bar{v}_P \in M_P T = \mathcal{T}(1,0)_P$

- Define "smooth" for t. fields $\quad$ =)

  - $\mathcal{F} = \{ f \mid f : M \to \mathbb{R} , \text{smooth} \}$

    $\forall f \in \mathcal{F}$ we have directional derivative $\bar{v}_P[f]$ gives a scalar at each P

    $\bar{v}[f] : M \to \mathbb{R} \quad P \to \bar{v}_P[f]$ $\qquad$ same for vector fields ?

  - action of vector fields on functions gives a scalar function of P

    $\mathcal{T}(1,0) \times \mathcal{F} \to \mathcal{F} \qquad (\bar{v}, f) \to \bar{v}[f]$ $\qquad$ (which is the dir. deriv of f)

  $\Rightarrow$ $\bar{v}(P)$ is smooth on M if $\bar{v}[f](P)$ is a smooth function on M $\forall f \in \mathcal{F}$

- In a chart: $\bar{v}(x) = v^\mu(x)\, \partial_\mu$ smooth if $\bar{v}[f](x) = \underset{\uparrow}{v^\mu(x)}\, \partial_\mu f(x)$ is smooth

  are themselves smooth functions

- Tensor fields are linear $\quad C(P) = a(P) A(P) + b(P) B(P) \qquad C, A, B \in \mathcal{T}(r,q) \ \forall P \in M$

  (inherited by the linearity of tensors) $\qquad a, b \in \mathcal{F}(M)$

  A, B smooth

- covector fields are in $\mathcal{T}(0,1)$ .... same...

Strong equivalence principle

- In flat space-times free particles move along straight lines, i.e. $u^a \partial_a u^b = 0$
- " " " " " " " geodesic ", i.e. $u^a \nabla_a u^b = 0$

This is a Postulate! ∼ 1° Axiom of Newtonian mechanics
↓
we defined the action $S = -mc \int ds$

- Equivalence principle had to do with non inertial frames, not really gravity

- Strong equivalence principle (SEP) is really about gravity:
  every gravitational law that in Special Relativity can be expressed as a tensor eq.
  has the same form in a local inertial frame (free fall) in GR

  (in fact we wanted to deal with a Lorentzian metric which is η in a local inertial frame)

- more in general one would have $u^a \nabla_a u^b = R^{\mu\nu}$ but SEP would be broken
  ↑
  no need for it given the experiments

Electrodynamics

Special Relativity → General Relativity

$\eta_{\mu\nu}$ indices are raised lowered with → $g_{\mu\nu}$

∂_ν careful with double derivatives! → ∇_ν

} role, not law!

- $u^a \partial_a u^b = 0 \longrightarrow u^a \nabla_a u^b = 0$

- $\boxed{F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}}$ Tensor of E.M. field $\longrightarrow$

$$\boxed{F_{\mu\nu}} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - \Gamma^\alpha_{\nu\mu} A_\alpha - A_{\mu,\nu} + \Gamma^\alpha_{\mu\nu} A_\alpha = \boxed{A_{\nu,\mu} - A_{\mu,\nu} + \cancel{T^\alpha_{\mu\nu}} A_\alpha}$$

(as in S.R.!)

For Torsion free

- $\boxed{m \frac{du^\mu}{d\tau} = \frac{q}{c} F^\mu{}_\alpha u^\mu}$ "Lorentz force" $\longrightarrow \boxed{m \frac{Du^\mu}{d\tau} = \frac{e}{c} F^{\mu\alpha} u_\alpha}$

eq. of motion of a particle

↳ as in S.R.

→ NOT as in special relativity

$$\frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$$

Contains the effect of gravity!

- 1° couple of Maxwell's equations (homogeneous) ← eq. of motion of motion of the field (in vacuum)

$$F_{[\mu\nu,\gamma]} \equiv F_{\mu\nu,\gamma} + F_{\nu\gamma,\mu} + F_{\gamma\mu,\nu} = 0 \longrightarrow F_{[\mu\nu;\gamma]} \equiv F_{\mu\nu;\gamma} + F_{\nu\gamma;\mu} + F_{\gamma\mu;\nu} = 0$$

$$\nabla_{[\lambda} F_{\mu\nu]} = \partial_{[\lambda} F_{\mu\nu]}$$

because F antisymmetric, Γ symmetric (torsion free)

- 2° couple of Maxwell's equations (4-current)

$$\boxed{\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu} \longrightarrow \boxed{\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu}$$

$\partial_\mu A^\mu = 0$ Lorentz gauge

$$\boxed{\Box A^\mu = -\frac{4\pi}{c} j^\mu}$$

$$\nabla_\nu F^{\mu\nu} = \nabla_\nu (\nabla^\mu A^\nu - \nabla^\nu A^\mu) = \underline{\nabla_\nu \nabla^\mu A^\nu} - \nabla_\nu \nabla^\nu A^\mu = R^\mu{}_\nu A^\nu - \nabla_\nu \nabla^\nu A^\mu$$

$\nabla_\nu \nabla^\mu A^\nu \neq \nabla^\mu \nabla_\nu A^\nu$! ⇒ can not apply gauge

curvature: $R^\nu{}_{\beta\mu\nu} A^\beta = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A^\nu$

$$\nabla_\nu \nabla^\mu A^\nu = \underset{\substack{=0 \\ \text{gauge}}}{\nabla^\mu \nabla_\nu A^\nu} + R_\nu{}^\mu A^\nu$$

$$\boxed{\Box A^\mu - \underline{R^\mu{}_\nu A^\nu} = -\frac{4\pi}{c} j^\mu}$$

! This arise from non commutation of ∇_μ!

Do you see?! Careful when applying $\partial_\nu \to \nabla_\nu$ rule (rule, not law!)

Charge conservation

j^μ : charge density current

covariant divergence

$$\partial_\mu j^\mu = 0 \quad (\text{S.R.}) \longrightarrow (\text{G.R.}) \quad \nabla_\mu j^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, j^\mu \right) = 0$$

- Let us check! where is it coming from and if it holds up in G.R.:

As in S.R. it should be ensured by Maxwels equations

$$\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \quad (\nabla_\mu \cdot) \Rightarrow \nabla_\mu \nabla_\nu F^{\mu\nu} = \frac{4\pi}{c} \nabla_\mu j^\mu$$

(*1) because $F^{\mu\nu}$ is antisymmetric

$$\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\, F^{\mu\nu} \right)$$

$$g \equiv \det(g_{\alpha\beta})$$

(*2) $\partial_\nu v^\nu = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, v^\nu)$

$$\nabla_\mu \underbrace{\nabla_\nu F^{\mu\nu}}_{(*1)} \overset{(*2)}{=} \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\, \nabla_\mu F^{\mu\nu} \right)$$

$$= \frac{1}{\sqrt{-g}} \partial_\nu \left[\cancel{\sqrt{-g}} \frac{1}{\cancel{\sqrt{-g}}} \partial_\mu \left(\sqrt{-g}\, F^{\mu\nu} \right) \right]$$

$$= \frac{1}{\sqrt{-g}} \underbrace{\partial_\nu \partial_\mu}_{\text{symmetric}} \left(\sqrt{-g}\, \underbrace{F^{\mu\nu}}_{\text{anti-symmetric}} \right) \underline{= 0} \quad \text{identically}$$

$$\Rightarrow \boxed{\nabla_\mu j^\mu = 0} \quad ✓$$

because of the antisymmetry of $F^{\mu\nu}$

For electrodynamics, so good so far... what about energy-momentum conservation?

$$\partial_\nu T^{\mu\nu} = 0 \longrightarrow \nabla_\nu T^{\mu\nu} = 0$$

Not just a rule (Strong equivalence principle)

↑ valid because G.R. is a diffeomorphyc theory, more later...

Comments on torsion : Torsion couples with "spin"

- Lorentz force:

$$m \frac{Du^\mu}{d\tau} = \frac{e}{c} F^{\mu\alpha} u_\alpha = \frac{e}{c} \left(A^{\mu,\alpha} - A^{\alpha,\nu} \right) u_\alpha + \underline{\frac{e}{c} T^{\gamma\mu\alpha} A_\gamma u_\alpha} \quad \Rightarrow \quad \text{Eq. of motion of charged part.}$$

- Maxwell's eq.

$$\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \qquad \nabla_\nu \left(A^{\nu,\mu} - A^{\mu,\nu} + T^{\alpha\mu\nu} A_\alpha \right) = \nabla_\nu \left(A^{\nu,\mu} - A^{\mu,\nu} \right) + \underline{\nabla_\nu \left(T^{\alpha\mu\nu} A_\alpha \right)}$$

Part IV

Einstein Field equations

Energy-Momentum tensor

- **Particle**: characterized by $(p^\mu) = \left(\frac{E}{c}, \bar{p}\right)$
- **Fields**: " " energy-momentum tensor $\overset{\substack{\text{Source of gravity}\\ \downarrow}}{T^{\mu\nu}}$ (good for covariant theory)
  ↳ e.g. matter field (continuous matter distribution), electro-magnetic field

- **Consider flat geometry**: $g = \eta$ $\qquad q(x^\mu)$ scalar field

- **Action** $\qquad S = \int L\, dt = \int \mathcal{L}\, dt\, dV = \frac{1}{c}\int \mathcal{L}\, d\Omega \qquad d\Omega \equiv dx^0 dx^1 dx^2 dx^3$

- **Euler-Lagrange eq.** $\qquad \partial S = 0 \quad \rightarrow \quad \frac{\delta}{\delta x^\nu}\left(\frac{\delta \mathcal{L}}{\delta q_{,\nu}}\right) - \underbrace{\frac{\delta \mathcal{L}}{\delta q^\alpha}}_{(1)} = 0$ $\qquad$ eq. of motion
  evolution of the system

- **identify conserved quantities** $\qquad \frac{\delta \mathcal{L}}{\delta x^\mu} = 0 \quad \Rightarrow$ corresponds to a symmetry of the system (Noethern theorem)

Consider $\mathcal{L}(q^\alpha, q^\alpha_{,\nu})$:

$$\underset{\downarrow}{\frac{\delta \mathcal{L}}{\delta x^\mu}} = \overset{\substack{(1)\\ \downarrow}}{\frac{\delta \mathcal{L}}{\delta q^\alpha}}\delta_\mu q^\alpha + \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}} \overset{\substack{\delta_\mu\delta_\nu \text{ commute}\\ \downarrow}}{\delta_\mu q^\alpha_{,\nu}} = \frac{\delta}{\underset{\smile}{\delta x^\nu}}\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\right)\delta_\mu q^\alpha + \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\underset{\smile}{\delta_\nu} \overset{\substack{\text{product rule}\\ \downarrow}}{q^\alpha_{,\mu}} = \delta_\nu\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\, \delta_\mu q^\alpha\right)$$

$$\left.\begin{aligned} \delta^\nu_\mu \delta_\nu \mathcal{L} &= \eta^{\sigma\nu}\underset{\smile}{\eta_{\sigma\mu}}\underset{\smile}{\delta_\nu}\mathcal{L} \\ \delta_\mu q^\alpha &= \underset{\smile}{\eta_{\sigma\mu}}\delta^\sigma q^\alpha \end{aligned}\right\} \Rightarrow \quad \delta_\nu \underbrace{\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\delta^\sigma q^\alpha - \eta^{\sigma\nu}\mathcal{L}\right)}_{\equiv T^{\sigma\mu}} \eta^{\sigma\nu} = 0$$

$\equiv T^{\sigma\mu}$ Energy-momentum tensor
(this is the quantity which is conserved!)

$$\boxed{T^{\sigma\nu} = \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\delta^\sigma q^\alpha - \eta^{\sigma\nu}\mathcal{L}} \quad \text{and} \quad \boxed{\delta_\mu T^{\sigma\mu} = 0}$$

Energy-Momentum conservation

- **Note: $T^{\mu\nu}$ is not unique**

$$\tilde{T}^{\mu\overset{\frown}{\nu}} = T^{\mu\overset{\frown}{\nu}} + \delta_\sigma \psi^{\mu\overset{\frown}{\nu\sigma}} \quad \text{with } \psi^{\mu\overset{\frown}{\nu\sigma}} = -\psi^{\mu\overset{\frown}{\sigma\nu}}$$

antisymmetric with respect to $\sigma\nu$

$$\delta_\mu \tilde{T}^{\mu\nu} = 0 \quad \text{still satisfied because of the antisymmetry of } \psi: \quad \underbrace{\delta_\nu\delta_\sigma}_{\text{Symmetric}}\underbrace{\psi^{\mu\nu\sigma}}_{\text{antisymmetric}} = 0$$

- What is the meaning of $T^{\mu\nu}$?

$$\boxed{T^{00}} = \dot{q}\frac{\delta \mathcal{L}}{\delta \dot{q}} - \mathcal{L} = \varepsilon$$ Hamiltonian density (local inertial frame) (Legendre transform of $\mathcal{L}$)

$$\left(\frac{T^{0\mu}}{c}\right) = \left(\frac{T^{00}}{c}, \frac{T^{0i}}{c}\right) \leftrightarrow (p^\mu) = \left(\frac{E}{c}, \bar{p}\right) \Rightarrow \frac{T^{0\mu}}{c} \sim \text{4-momentum density of system}$$

$$= \left(\frac{\varepsilon}{c}, \frac{\bar{S}}{c^2}\right)$$

for a particle: $p^i = \frac{E}{c^2} v^i \Rightarrow c^2 p^i = E v^i \ [E \frac{m}{s}]$ energy flux

$$\Rightarrow \cancel{c^2}\frac{T^{0i}}{\cancel{c}} \equiv S^i \quad \text{" density flux}$$

$$c^2 \underset{\wr}{p^i} = S^i$$

$\partial_\nu T^{0\mu} = \partial_0 T^{0\mu} + \partial_i T^{i\mu} = 0$ is a continuity eq.: $T^{0\mu}$ change related to the flux $T^{i\mu}$

$$\partial_\nu T^{\mu\nu} = 0 \begin{cases} \underline{\mu = 0}: & \frac{\delta T^{00}}{c\,\delta t} + \partial_i T^{0i} = 0 \quad \frac{\delta \varepsilon}{\delta t} + c\,\partial_i T^{0i} = 0 \Rightarrow c T^{0i} \equiv S^i \quad \text{energy density flux} \\ \underline{\mu = i}: & \frac{\delta T^{i0}}{c\,\delta t} + \partial_j T^{ij} = 0 \quad \overset{p^i}{\frac{\delta S^i}{c^2 \delta t}} + \partial_j T^{ij} = 0 \Rightarrow T^{ij} = \sigma^{ij} \quad \text{momentum " "} \end{cases}$$

$$(T^{\mu\nu}) = \begin{pmatrix} \boxed{\varepsilon} & \boxed{S_x/c \quad S_y/c \quad S_z/c} \\ \cdot & \sigma_{xx} \quad \sigma_{xy} \quad \sigma_{xz} \\ \cdot & \cdot \quad \sigma_{yy} \quad \sigma_{yz} \\ \cdot & \cdot \quad \cdot \quad \sigma_{zz} \end{pmatrix}$$

$\bar{S}$ = energy-density flux $\left[\frac{d\varepsilon}{dt\,dA}\right]$

σ_{ij} = stress tensor $\left[\frac{dP}{dt\,dA}\right]$ "pressure"

- **Note:** in general T is not diagonal but it can be diagonalized

• What about G.R?!

- In a local inertial frame (free fall) the above holds

careful with interpretation

- Strong equivalence principle: $\partial_\alpha \to \nabla_\alpha$, $\eta_{\mu\nu} \to g_{\mu\nu}$ $\Rightarrow$ $\boxed{\nabla_\mu T^{\mu\nu} = 0}$

actually not just strong equiv. principle... there is something more fundamental, more on that later....

Energy momentum tensors of a perfect relativistic fluid

perfect = No heat conduction

$\boxed{T = (\rho + \frac{P}{c^2})\bar{u} \otimes \bar{u} + P g^{-1}}$ $\quad T^{ab} = (\rho + \frac{P}{c^2}) u^a u^b + P g^{ab}$ (components)

ρ : density $\in \mathbb{R}$
P : pressure of the fluid $\in \mathbb{R}$
$\}$ in rest frame

$\rho(x^\nu)$ $P(x^\nu)$

$\bar{u}$: time-like vector representing the 4-velocity of the fluid volume element

- note: $T^{\mu\nu} g_{\mu\nu} = (\rho + \frac{P}{c^2}) u^\mu u_\mu + 4P = -\rho c^2 - P + 4P = 3P - \rho$

$\Rightarrow P = \frac{1}{3}(T^{\mu\nu} g_{\mu\nu} - \rho)$ $\underline{\in \mathbb{R}}$ because $T^{\mu\nu} g_{\mu\nu}$ is a scalar

i.e. P and ρ are scalar functions

- In a local inertial frame (isotropic)

$g \to \eta$, $\bar{u} = (c, 0, 0, 0)^T$

$T^{00} = (\rho + \frac{P}{c^2}) c^2 - P = \rho c^2$ rest energy density

$T^{ii} = P$ isotropic pressure

$T = \mathrm{diag}(\rho c^2, P, P, P)$

- Pressureless fluid (dust) $P = 0$ $\quad T^{\mu\nu} = \rho u^\mu u^\nu$

- Hydrodynamics "for free":

$\Rightarrow \nabla_a T^{ab} = 0$ hydrodynamic equations!
eq. of motion of fluid

$$\begin{cases} \nabla_\mu T^{0\mu} = 0 & \\ \nabla_\mu T^{i\mu} = 0 & \text{Euler's eq.} \end{cases}$$

$\nabla_\mu T^{0\mu} = 0$ energy conservation
$\nabla_\mu T^{i\mu} = 0$ Euler's eq. momentum conservation
$\partial_\mu \rho = 0$ continuity eq. matter conservation
$P = P(\rho, T)$ eq. of state $\quad \rho$ = scalar field

Energy-momentum tensor of electromagnetic fields

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} F_{\alpha\beta} F_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta} \Rightarrow \boxed{T^{\mu\nu} = \frac{1}{4\pi}\left[-F^{\mu\alpha} F^{\nu}{}_{\alpha} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right]}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ electromagnetic field $\quad A^\mu$ = 4-potential

Again but now on an arbitrary manifold

1) Euler-Lagrange equation for fields in a curved space

- E-L. eq. gives you the eq. of motion of a system described by a Lagrangian
- $\mathcal{L}(\psi^i, \partial_\nu \psi^i, \eta) = \mathcal{L}(\psi^i, \partial_\nu \psi^i) \longrightarrow \mathcal{L}(\psi^i, \nabla_\nu \psi^i, g_{\mu\nu})$ Lagrangian density
- E.g. scalar field : matter density $\rho(x^\mu)$
- E.g. vector field : 4-potential in electrodynamics $A^\mu(x^\nu)$
- $\sqrt{-\det(g)}\, d\Omega$ = invariant volume element under coord. transf.

Least action principle ⟶ Euler-Lagrange eq. (for a given $g_{\mu\nu}$)

$$\partial S = \partial \int_V \mathcal{L} \sqrt{-g}\, d\Omega$$

$$= \int_V \left(\frac{\delta \mathcal{L}}{\delta \psi^i} \partial \psi^i + \frac{\delta \mathcal{L}}{\delta \psi^i_{;\nu}} \partial \psi^i_{;\nu} \right) \sqrt{-g}\, d\Omega$$

$$= \int_V \left(\frac{\delta \mathcal{L}}{\delta \psi^i} \partial \psi^i + \cancel{\nabla_\nu \left(\frac{\delta \mathcal{L}}{\delta \psi^i_{;\nu}} \partial \psi^i \right)} - \partial \psi^i \nabla_\nu \frac{\delta \mathcal{L}}{\delta \psi^i_{;\nu}} \right) \sqrt{-g}\, d\Omega \overset{!}{=} 0$$

Gauss theorem

$$\int_V \nabla_\nu \left(\frac{\delta \mathcal{L}}{\delta \psi^i_{;\nu}} \psi^i_{;\nu} \right) \sqrt{-g}\, d\Omega = 0$$

$$\forall\, \partial \psi^i \Rightarrow \boxed{\frac{\delta \mathcal{L}}{\delta \psi^i} - \nabla_\nu \frac{\delta \mathcal{L}}{\delta \psi^i_{;\nu}} = 0}$$

Euler-Lagrange eq. in curved space

Example: neutral scalar field:

$$\mathcal{L} = -\frac{1}{2} g(\nabla^\mu \psi, \nabla^\mu \psi) - \frac{1}{2} m^2 \psi^2 = -\frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi - \frac{1}{2} m^2 \psi^2 = 0$$

$$(\nabla_\mu \nabla^\mu \psi + m^2 \psi)\psi = 0 \qquad \nabla_\mu \nabla^\mu \psi + m^2 \psi = 0 \qquad (\Box + m^2)\psi = 0$$

Klein-Gordon eq.
for particle with mass m

2) Energy-momentum tensor

Within the same context of the leastactionprinciple, we can identify $T^{\mu\nu}$

Here we consider scalar field $\phi(x^\mu)$ as an example and $\mathcal{L}(\phi, \phi_{;\beta})$

Leibnitz rule

$$1)\ \underset{(a)}{\partial \mathcal{L}} = \frac{\delta \mathcal{L}}{\delta \phi}\partial\phi + \frac{\delta \mathcal{L}}{\delta \phi_{;\beta}}\partial \phi_{;\beta} = \frac{\delta \mathcal{L}}{\delta \phi}\partial\phi + \nabla_\beta\left(\frac{\delta \mathcal{L}}{\delta \phi_{;\beta}}\partial\phi\right) - \partial\phi\, \nabla_\beta \frac{\delta \mathcal{L}}{\delta \phi_{;\beta}}$$

$$= \overset{=0}{\left(\frac{\delta \mathcal{L}}{\delta \phi} - \nabla_\beta \frac{\delta \mathcal{L}}{\delta \phi_{;\beta}}\right)}\partial \phi_\alpha + \nabla_\beta\left(\frac{\delta \mathcal{L}}{\delta \phi_{;\beta}}\underset{(b)}{\partial\phi}\right) \quad (1) \quad (=0 \text{ Euler-Lagrange eq.})$$

2) Variation with respect to infinitesimal translation δx^μ

$\mathcal{L}$ does not depend on position x^ν => its variation can only be caused by a variation of $\phi(x^\nu)$

$$\left.\begin{array}{l}(a)\ \partial \mathcal{L} = \nabla_\beta \mathcal{L}\, \partial x^\beta = \nabla_\beta \mathcal{L}\, \delta^\beta_\gamma \partial x^\gamma \\ (b)\ \partial\phi = \nabla_\gamma \phi(x^\mu)\partial x^\gamma\end{array}\right\} \Rightarrow \nabla_\beta \mathcal{L}\, \delta^\beta_\gamma \partial x^\gamma - \nabla_\beta\left(\frac{\delta \mathcal{L}}{\delta \phi_{;\beta}} \nabla_\gamma \phi(x^\mu)\partial x^\gamma\right) = 0$$

$$\nabla_\beta\left(\mathcal{L}\, \overset{(*)}{\delta^\beta_\gamma} - \frac{\delta \mathcal{L}}{\delta \phi_{;\beta}}\nabla_\gamma \phi(x^\mu)\right)\partial x^\gamma = \nabla_\beta\left(\mathcal{L}\, g_{\alpha\gamma} g^{\alpha\beta} - \frac{\delta \mathcal{L}}{\delta \phi_{;\beta}} g_{\alpha\gamma}\nabla^\alpha \phi(x^\mu)\right)\partial x^\gamma = 0 \qquad (*)\ \delta^\beta_\gamma = g_{\alpha\gamma} g^{\alpha\beta}$$

$$\Rightarrow \boxed{T^{\alpha\beta} \equiv \frac{\delta \mathcal{L}}{\delta \phi_{;\beta}}\nabla^\alpha \phi(x^\mu) - \mathcal{L}\, g^{\alpha\beta}} \quad \text{and} \quad \boxed{\nabla_\beta T^{\alpha\beta} = 0}$$

$T^{\mu\nu}$ covariantly conserved regardless the nature of the "substance" described by $\mathcal{L}$

$T^{\mu\nu}$ is the source of gravity

=> G.R. is the gravity theory of systems with conserved energy-momentum as electrodynamics is the theory of electromagnetism of any system with conserved charge

change of ϕ due to coord shift: $\phi(x^\nu + \delta x^\nu) = \phi(x^\nu) + \nabla_\alpha \phi(x^\nu)\delta x^\alpha + \ldots$

$$\partial\phi = \phi(x^\nu + \delta x^\nu) - \phi(x^\nu) = \nabla_\alpha \phi(x^\nu)\delta x^\alpha$$

Einstein's fields equations

- What do we know by now:

1) the field eq. of Newtonian gravity

$\bar{\nabla}^2 \varphi = 4\pi G \rho$ Poisson eq. $\Delta = \partial_i \partial^i = \delta^{ij} \partial_i \partial_j$ $i = 1, 2, 3$

$g u^\alpha u^\beta = \gamma^2 c^2$

$\varphi = -\frac{GM}{r}$ M = mass of the source of gravity (case of point mass)

2) the weak-field limit

- Static : $\partial_0 g_{\mu\nu} = 0$
- Weak : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $|h_{\mu\nu}| \ll 1$
- Non relat : $|u^i| \ll |u^0|$

$\Rightarrow$ $g_{00} = -\left(1 + \frac{2\varphi}{c^2}\right)$ $\Rightarrow$ $\bar{\nabla}^2 g_{00} = \bar{\nabla}^2 h_{00} = -2 \frac{\bar{\nabla}^2 \varphi}{c^2}$ (1)

$g_{00} = \underset{\sim}{\eta_{00}} + \underset{\sim}{h_{00}}$

(dust) $T_{00} = -\rho c^2$ (2)

$\overset{(1)}{\bar{\nabla}^2 \varphi} = 4\pi G \overset{(2)}{\rho}$ $\Rightarrow$ $\boxed{\bar{\nabla}^2 h_{00} = \frac{8\pi G}{c^4} T_{00}}$ field eq. for the metric (the metric $g_{\alpha\beta}(x^\mu)$ is the field)

- "Einstein's original approach", what are we lloking for are:

- Look for simplest generalization of Poisson eq.
- It should return the weak field limit result when the same approximations are used
- An expression between tensors (covariant theory)

$\underset{\text{tensor}}{\boxed{\text{geometry}}} = \underset{\text{tensor}}{\boxed{\text{source}}}$

1) $\rho \rightarrow T^{\mu\nu}$ energy-momentum tensor (rank-2, symmetric) (2)

2) $(\bar{\nabla}^2 g_{00}) \rightarrow A_{\mu\nu}$ some rank-2 symmetric tensor (because of $T_{\mu\nu}$, consistency between two sides of eq.)

3) $\bar{\nabla}^2 \longrightarrow$ linear in (up to) at least 2° derivatives of $g_{\mu\nu}$ (as in weak field limit)

$\downarrow$

(coordinates such to have at 1° order $g_{\mu\nu} = \eta_{\mu\nu} + O(d\bar{x}^2)$, $\partial_\alpha g_{\mu\nu} = 0$
$\Rightarrow$ you (at least) need 2° derivatives of $g_{\mu\nu}$)

4) $\nabla_\nu T^{\mu\nu} = 0 \longrightarrow$ must be $\nabla_\nu A^{\mu\nu} = 0$

$\mapsto$ $\boxed{A_{\mu\nu}(g, \partial g, \partial^2 g) = K T_{\mu\nu}}$ $K \in \mathbb{R}$ const Now... look for the right $A_{\mu\nu}$ tensor

Let's try out various $A_{\mu\nu} = ?_{\mu\nu}$ tensors fulfilling the conditions above

Identifying the candidate symmetric rank-2 in the field equations

(1) The simplest object:

$\boxed{\Box g_{\mu\nu} = T_{\mu\nu}}$? • no because $\Box = \nabla^\alpha \underbrace{\nabla_\alpha g_{\mu\nu}}_{=0} = 0$ (metric compatibility)

Something related to the Riemann tensor? Intrinsic property of space-time
Proportional to accelerations (geodesic deviation eq.)

(2) The ricci tensor?

$\boxed{R_{\mu\nu} = \kappa T_{\mu\nu}}$? • rank-2, symmetric tensor containing double derivatives of $g_{\mu\nu}$ ✓

• but... issues with energy conservation:

$\nabla_\mu T^{\mu\nu} = 0$ energy-momentum conservation

$\Rightarrow$ we should have

$\nabla_\mu R^{\mu\nu} = 0$ but this is not true in generic curved space-time, in fact:

$$\left.\begin{aligned} \nabla^\mu T_{\mu\nu} &= g^{\mu\nu} \nabla_\nu T_{\mu\nu} \\ \nabla^\mu R_{\mu\nu} &= \frac{1}{2} \nabla_\nu R \leftarrow \text{from Bianchi identity} \end{aligned}\right\} \frac{1}{2}\nabla_\nu R = \kappa g^{\mu\nu} \nabla_\nu T_{\mu\nu}$$

integrate $\Rightarrow$ $R = 2\kappa g^{\mu\nu} T_{\mu\nu} = 2\kappa T \qquad T^\nu{}_\nu = T_{\text{scalar}}$

$$\nabla_\nu T = \partial_\nu T = 0 \Rightarrow T = \text{const.}$$

this can not be! $T=0$ in vacuum, $T>0$ in matter

(3) The Einstein tensor?

$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}$ • $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ rank-2, symmetric tensor containing double derivatives of $g_{\mu\nu}$

it does look good!

• $\nabla_\mu G^{\mu\nu} = 0$ $\Rightarrow$ no issues with energy conservation

• 1° and 2° derivatives of the metric

They can also be reshaped as follow:

$$g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \kappa g^{\mu\nu} T_{\mu\nu} \qquad R - \frac{1}{2} 4R = \kappa T \qquad \boxed{R = -\kappa T}$$

$$R_{\mu\nu} - \frac{1}{2} \overset{\swarrow}{R} g_{\mu\nu} = R_{\mu\nu} + \frac{\kappa}{2} T g_{\mu\nu} = T_{\mu\nu} \qquad \Rightarrow \qquad \boxed{R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)}$$

convenient for vacuum $(T=0)$ solutions $R_{\mu\nu} = 0$

Weak static non-relativistic field with $R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=T_{\mu\nu}$

- Here it is easier to use: $R_{\mu\nu}=\kappa\left(T_{\mu\nu}-\frac{1}{2}Tg_{\mu\nu}\right)$

- Weak field: $|\varphi|\ll 1 \Rightarrow |h_{\mu\nu}|\ll 1 \quad g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$
- Static: $\partial_0 g_{\mu\nu}=0$
- Non relativistic matter $|u^i|\ll \overset{u^0}{\downarrow}c \quad T_{00}\gg$ all other $T_{\mu\nu}$ terms $\quad \overset{\text{eg. dust}}{T^{\mu\nu}=\rho u^\mu u^\nu}$ $\begin{cases} T^{00}=\rho c^2 & \leftarrow c^2 \\ T^{0i}=\rho c u^i & \leftarrow c \\ T^{ij}=\rho u^i u^j & \leftarrow 1 \end{cases}$

$\Rightarrow$ focus on $\mu=0=\nu$ terms non-relativistic

$$(1)_{00}=\kappa\left(T_{00}-\frac{1}{2}\overset{\swarrow}{T}g_{00}\right)\simeq\kappa\left(T_{00}+\frac{1}{2}T_{00}(-1)\right)=\boxed{\frac{\kappa}{2}T_{00}}$$

$$g_{00}=-\left(1+\frac{2\varphi}{c^2}\right)\qquad g^{00}=-\left(1+\frac{2\varphi}{c^2}\right)^{-1}\simeq\left(-1+\frac{2\varphi}{c^2}\right)$$

$$\nwarrow\quad T=g^{\mu\nu}T_{\mu\nu}\simeq g^{00}T_{00}=-T_{00}+\frac{2\varphi}{c^2}T_{00}\simeq -T_{00}$$

$$(2)_{00}=R_{00}=R^{\alpha}{}_{0\alpha0}=\partial_\alpha\overset{\downarrow}{\Gamma}{}^{\alpha}_{00}-\underset{\substack{=0\\ \text{Static}}}{\partial_0\overset{\downarrow}{\Gamma}{}^{\alpha}_{\alpha0}}+\underbrace{\Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{00}-\Gamma^{\alpha}_{0\gamma}\Gamma^{\gamma}_{\alpha0}}_{2^\circ\text{ order in }\delta g}\quad(\Gamma\ 1^\circ\text{ order in }\delta g)$$

$$=\partial_\alpha\left[\frac{1}{2}\underset{\sim}{g^{\alpha\gamma}}\left(\underset{=0}{\partial_0 g_{\gamma0}}+\underset{=0}{\partial_0 g_{0\gamma}}-\partial_\gamma g_{00}\right)\right]\quad(g^{i\gamma}\simeq\eta^{i\gamma})$$

static

$$\simeq-\frac{1}{2}\eta^{\alpha\gamma}\partial_\alpha\partial_\gamma g_{00}$$

$$=-\underset{\substack{=0\\ \text{Static}}}{\eta^{0\gamma}\partial_0\partial_\gamma g_{00}}-\frac{1}{2}\vec{\nabla}^2\underset{\substack{\uparrow\\ g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}}}{g_{00}}=\boxed{-\frac{1}{2}\vec{\nabla}^2 h_{00}}$$

$$(1)=(2)\Rightarrow\quad\vec{\nabla}^2 h_{00}=-\kappa T_{00}\quad\text{yes! with}\quad\kappa=-\frac{8\pi G}{c^4}\quad\leftarrow\overset{\substack{\text{non relat. fluid}\\ \text{weak field}}}{\left(\vec{\nabla}^2 h_{00}=\frac{8\pi G}{c^4}T_{00}\right)}$$

$\Downarrow$

$$\boxed{R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu}}$$

Einstein's field equations (Nov. 25th 1915)

↳ The space time reacts to the presence of a $T_{\mu\nu}$!
Matter/fields eq. of motion is affected by the curvature of space time!

↳ $T_{\mu\nu}$ refers to all forms of energy and momentum (matter, fields)
$\Rightarrow$ charged and neutral bodies produce different space-time distorsions

The cosmological constant

- Recall: the argument we used: $A_{\mu\nu} = k T_{\mu\nu}$
  - -> Must respect energy-momentum conservation $\nabla_\nu T_{\mu\nu} = 0 \Rightarrow \nabla_\nu A^{\mu\nu} = 0$
  - -> $A_{\mu\nu}$ a rank-2 symmetric tensor linear in (up to) at least 2° derivatives of $g_{\mu\nu}$

=> we are free to add an additional term proportional to $g_{\mu\nu}$

↳ 0^{th} derivative of $g_{\mu\nu}$

↳ $\underset{=0}{\nabla_\nu G^{\alpha\beta}} + \underset{=0}{\nabla_\nu g^{\alpha\beta}} = 0$ (ok!) because of metric compatibility assumption

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

2 universal constants in the theory: G, Λ!

attraction repulsion

cosmological constant term $\Lambda \in \mathbb{R}$ fixed, const

it allows -> Static universe (what Einstein looked for) } according to value of Λ

-> Accelerated expansion

Remember: R is not a constant, it is $R^\mu{}_\mu(x^\nu)$ whereas: $\Lambda \in \mathbb{R}$ const.

- Reinterpreting Λ: not part of geometry, but a source term

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{c^4 \Lambda}{8\pi G} g_{\mu\nu}\right) \Rightarrow \boxed{T^{(\Lambda)}_{\mu\nu} = \frac{c^4 \Lambda}{8\pi G} g_{\mu\nu}}$$

$T^{(\Lambda)}_{\mu\nu}$ = energy and momentum of vacuum predicted by quantum field theories

- Quantum mechanics: harmonic oscillator (frequency ω) -> minimum classical energy $E_0 = 0$
  - -> quantized ground state: $E_0 = \frac{1}{2}\hbar\omega$
  - -> quantum field ~ collection of infinite number of oscillators
  - -> each mode contributes to the ground state energy $= \infty \Rightarrow$ cut off at high ω
  - -> natural scale $\Lambda \sim m_p^4$ $m_p \simeq 10^{19}\,\mathrm{GeV} \simeq 10^{-5}\,g$
  - -> cosmological observation: $\Lambda > 0$ but it is 10^{120} times smaller !!!
  - => possible solution: $\Lambda = 0$ but there is fluid = dark energy

Are Einstein's equations sufficient to constrain the metric?

- $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ (1) 2° order differential equations for $g_{\mu\nu}$

  $\mu,\nu = 0,1,2,3 \Rightarrow 4^2 = 16$ equations but they are not all independent

  1) $T_{\mu\nu}$, $G_{\mu\nu}$ symmetric $\Rightarrow$ 10 independent equations

  2) Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ are 4 constraints on $R_{\mu\nu}$

     $\Rightarrow$ only $10-4=6$ independent equations

- How many deg. of freedom has the metric?

  1) $g_{\mu\nu}$ symmetric: 10 independent components

  2) tensorial eq. (1): solution $g_{\mu\nu}$ must satisfy (1) ∀ coordinate system $x^{\mu'}$ $g_{\mu'\nu'} = \frac{dx^\mu}{dx^{\mu'}}\frac{dx^\nu}{dx^{\nu'}} g_{\mu\nu}$

     4 non physical combination of coefficients

  $\Rightarrow$ $g_{\mu\nu}$ has $10-4=6$ possible (physical) independent coefficients

$$\Downarrow$$

6 indepentent equations for 6 d.of.f. in g. The system of equations is closed!!

Isn't $G_{\mu\nu} = kT_{\mu\nu}$ a cute simple equation?

- NO!! they are a nightmer to solve! Think about it: $G(R_{\mu\nu}(g^{\mu\nu} R_{---}(\Gamma(g))) = kT(g)$

  Like a russian doll

- There are few analytical solutions for simple cases in which on can exploit symmetries ore rely on approximations

- Einstein at first thought that they can not be solved... but we will!

  ↳ linearization → gravitational waves
  Spherical symmetry → Schwarzschild
  Axially symmetric → Kerr
  Homogeneity, hisotropy → FLRW (Cosmology)

Can we have gravity (curved space time) with a 2 or 3 dimensional space?

look at the number of constraints to fix the degrees of freedom in R_{ijkl}

# dimensions	$G_{\mu\nu}$			# R_{ijkl} components
$n=2$	$\begin{pmatrix} \cdot & \cdot \\ & \cdot \end{pmatrix}$	$\Rightarrow$ 3 Einstein equations	$\downarrow$ 6 indep. eq.	1
$n=3$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix}$	$\Rightarrow$ 6 " "		6
$n=4$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}$	$\Rightarrow$ 10 " "	$-$ 4 Bianchi identities $= 6$ independent eq.	20

$\Rightarrow$ for $n=2$, $n=3$ field equations in empty space guarantee $R_{ijkl}=0$ (flat)
$3>1$ $6=6$

$\Downarrow$
no gravity is possib
in vacuum

for $n \geq 4$: $6 < 20$ $\Rightarrow$ can have $R_{\mu\nu}=0$ but some $R_{ijkl} \neq 0$

$\Rightarrow$ can have gravity in vacuum !

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Einstein field equation: variational approach

- Hilbert was not satisfied by Einstein's approach -> least action approach

- Action for the field $g_{\mu\nu}$ only

$\mathcal{L}_H = R\sqrt{-g}$ lagrangian density

$S_H = \int R\sqrt{-g}\, d\Omega$ Hilbert action $g = \det(g_{\mu\nu})$ $R = g^{ab}R_{ab}$ simplest scalar containing the curvature

- Least action -> find equation of motion for $g_{\mu\nu}$: field equations in cavuum

$$\partial S_H = \int \partial\left(g^{\ell j} R_{\ell j}\sqrt{-g}\right) d\Omega$$

$$= \int R_{\ell j}\sqrt{-g}\,\partial g^{\ell j} + \underbrace{\partial R_{\ell j}\, g^{\ell j}}_{(a)}\sqrt{-g} + \overbrace{R_{\ell j}\, g^{\ell j}}^{R}\,\underbrace{\partial\sqrt{-g}}_{(b)}\, d\Omega$$

a) $$g^{\ell j}\overset{\downarrow}{\partial} R_{\ell j} = g^{\ell j}\partial\left(\delta_\kappa \Gamma^\kappa_{ej} - \delta_\ell \Gamma^\kappa_{\kappa j}\right) \overset{(1)}{=} g^{\ell j}\left(\delta_\kappa \partial\Gamma^\kappa_{ej} - \delta_e \partial\Gamma^\kappa_{\kappa j}\right) \overset{(2)}{=} g^{\hat{\ell} j}\overset{\ell \Leftrightarrow \kappa}{\left(\nabla_\kappa \partial\Gamma^\kappa_{ej} - \nabla_{\breve{e}} \partial\Gamma^{\hat{\kappa}}_{\breve{\kappa} j}\right)}$$

$$= g^{\ell j}\nabla_\kappa \partial\Gamma^\kappa_{ej} - g^{\kappa j}\nabla_\kappa \partial\Gamma^\ell_{\ell j} \overset{(3)}{=} \nabla_\kappa\left(g^{\ell j}\partial\Gamma^\kappa_{ej} - g^{\kappa j}\partial\Gamma^\ell_{\ell j}\right) \equiv \nabla_\kappa A^\kappa$$ A^κ some vector

(1) frame in which $\Gamma = 0$ $R_{je} \equiv R^\kappa{}_{j\kappa e} = \delta_\kappa \Gamma^\kappa_{ej} - \delta_\ell \Gamma^\kappa_{\kappa j} + \cancel{\Gamma^\kappa_{\kappa\gamma}\Gamma^\gamma_{ej} - \Gamma^\kappa_{e\gamma}\Gamma^\gamma_{\kappa j}}$

(2) $\partial \to \nabla$ anyway $\Gamma = 0$ and $\partial\Gamma$ = tensor ($\partial\Gamma$ is the difference between 2 Γ)

↖ like torsion

(3) Metric compatibility $\nabla_\kappa g^{\ell j} = 0 = \nabla_\kappa g^{\kappa j}$

$\partial R_{\ell j}$: Palatini identity, tensorial expression => valid in all frames every where

b) $$\overset{\swarrow}{\partial}\sqrt{-g} = -\frac{1}{2}(-g)^{-1/2}\,\partial \overset{g(g_{ej})}{g} = -\frac{1}{2}(-g)^{-1/2}\overset{\downarrow \text{Kramer's role}}{\frac{\delta g}{\delta g_{ej}}}\partial g_{ej} = -\frac{1}{2}(-g)^{+1/2}\cdot \cancel{g}\, g^{\ell j}\partial g_{ej} = \frac{1}{2}(-g)^{1/2} g^{\ell j}\partial g_{ej}$$

$$= -\frac{1}{2}(-g)^{1/2} g_{ej}\,\partial g^{\ell j} \quad *$$

$$= \int \sqrt{-g}\,\overset{☆}{\nabla_\kappa} A^\kappa d\Omega + \int R_{ej}\sqrt{-g}\,\partial g^{\ell j} - R\,\frac{1}{2}(-g)^{1/2} g_{ej}\partial g^{ej}\, d\Omega$$

$\left(* : \partial(\overset{4}{g_{\mu\nu}g^{\mu\nu}}) = \partial g_{\mu\nu}\, g^{\mu\nu} + g_{\mu\nu}\,\partial g^{\mu\nu} = 0\right)$

$$= \int\left(R_{\ell j} - R\,\frac{g_{ej}}{2}\right)\partial g^{\ell j}\sqrt{-g}\, d\Omega \overset{!}{=} 0 \quad \forall\, \partial g^{\ell j}$$, $\sqrt{-g}\, d\Omega$ invariant volume element

$$\Rightarrow \quad R_{ej} - \frac{1}{2}R g_{ej} = \boxed{G_{ej} = 0}$$ Fields equation in vacuum , G = Einstein tensor

☆ integral of a divergence => Stokes theorem => boundary term, does not vanish in general but at ∞ no effect on action => ignore

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- Including the source term (Total action)

$S = S_H + \alpha_M S_M$ $\quad \alpha_M = \text{const.} \in \mathbb{R}$ "strength of coupling" $\quad S_M = \int \mathcal{L}_M \sqrt{-g}\, d\Omega \quad \mathcal{L}_M(g_{\mu\nu}, \phi)$

$\mathcal{L}_M$ function of g, not its derivatives, and field ϕ (eg. Matter field)

must be in the variation

$$\delta S = \delta(S_H + \alpha_M S_M) = \delta S_H + \alpha_M \delta S_M \qquad \delta S_M = \int \delta(\mathcal{L}_M \sqrt{-g})\, d\Omega = \int \frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g^{\mu\nu}}\, \delta g^{\mu\nu}\, d\Omega + \cancel{\int \underbrace{\frac{\delta \mathcal{L}_M}{\delta \phi}}_{=0} \delta\phi\, d\Omega}$$

satisfies eq. of motion

$$= \int \left[G_{\mu\nu} + \alpha_M \frac{\delta(\mathcal{L}_M\sqrt{-g})}{\sqrt{-g}\,\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g}\, d\Omega \overset{!}{=} 0 \qquad \forall\, \delta g^{\mu\nu}$$

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g^{\mu\nu}}}$$ source(s) energy-momentum tensor!

$$\Rightarrow \boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$ Coupled Einstein field equations!

↳ $T_{\mu\nu}$ automatically symmetric because $g^{\mu\nu}$ is symmetric

↳ " " Gauge invariant

↳ " always applicable in curved space-time

↳ " very arbitrary, we want physical 4i

$\boxed{T_{\mu\nu} v^\mu v^\nu \geq 0}$ $\forall$ time-like vector (inside light-cone) $\quad$ eg. $\rho = T_{00} \geq 0$ (in local inertial frame)

Weak energy condition

not super solid condition but it holds for "non extreme cases"

- As a func of action instead

$\delta S = \delta S_G + \alpha_M \delta S_M \qquad \alpha_M = \text{const} \in \mathbb{R} \qquad \delta S_M = \int \frac{\delta S_M}{\delta g^{ab}} \delta g^{ab} + \cancel{\overset{=0}{\frac{\delta S_M}{\delta\phi}} \delta\phi}\, dx^m \qquad S_M(g^{\mu\nu}, \phi)$

eq. of motion

$\Rightarrow \int \left(G_{ab} + \frac{\alpha_M}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}} \right) \delta g^{ab} \sqrt{-g}\, d\Omega \qquad G_{ab} = -\frac{\alpha_M}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}} \qquad T_{ab} = -\frac{\overset{*}{1}}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}}$

$\alpha_M \overset{!}{=} \alpha_{KG}$ Klein-Gordon (matter)

$\alpha_M \overset{!}{=} \alpha_{EM}$ Electro-Magnetic field

are sum of all sources

* careful, you find definitions with different factors in T_{ab}

Energy momentum conservation and diffeomorphysms

- G.R. is a "diffeomorphism invariant" theory

i.e. G.R. is a coordinate invariant theory, $\phi_* g = g$

i.e. universe $= (\mathcal{M}, g_{\mu\nu}, \Psi)$ $\Rightarrow$ $(\mathcal{M}, \phi_* g, \phi_* \Psi)$ same physical system

$\uparrow$ matter field

$\phi =$ diffeomorphysm

$\neq$ configurations of $g_{\mu\nu}$ and Ψ might be just the same, related by a differ ϕ

($\hookrightarrow$ free of "prior geometry" + no preferred coord system for space-time

$\downarrow$

$g_{\mu\nu}$ is a dynamical variable $\Rightarrow$ nothing is give from the start like in S.R.)

- Energy-momentum conservation in G.R. comes from that

– Complete action of gravity $S = S_H(g_{\mu\nu}) + \alpha_M S_M(g_{\mu\nu}, \Psi^i)$ $\Psi^i =$ matter fields

$S_H =$ Hilbert action (gravity) $\rightarrow$ differ invariant (i.e. invariant under coord. transf.)

$S_M =$ matter action $\rightarrow$ " " because S must be so

– Variation of S_M under diffeomorphism (only!) given by Lie derivatives $\mathcal{L}_V$,

$V =$ vector field generating the Diffeo. : $\delta g_{ab} = \mathcal{L}_V g_{ab} = \nabla_a V_b + \nabla_b V_a = 2\nabla_{(a}V_{b)}$ $*(1)$

$\Rightarrow$ $\delta S_M = 0$ variation under differ because $S_M[g^{ab}, \Psi] = S_M[\phi^*_\lambda g^{ab}, \phi^*_\lambda \Psi]$

$$\delta S_M = \int \frac{\delta S_M}{\delta g_{ab}} \underset{\nwarrow *(1)}{\delta g_{ab}}\, dx^m + \cancel{\int \frac{\delta S_M}{\delta \Psi^i} \delta\Psi^i}_{(2)}\, dx^m \overset{!}{=} 0$$

(2)=0 because of matter eq. of motion, see Kline-Gordon <137

$$= 2\int \frac{\delta S_M}{\delta g_{ab}} \underset{*}{\nabla_{(a}V_{b)}}\, dx^m \cdot \frac{\sqrt{-g}}{\sqrt{-g}}$$

$*$ $2A^{ab}\nabla_{(a}V_{b)} = A^{ab}\nabla_a V_b + A^{\overset{a\leftrightarrow b}{\bar{b}\bar{a}}}\nabla_{\underline{b}} V_{\underline{a}} = 2A^{ab}\nabla_a V_b$ because A simmetric

$\uparrow$ $= \frac{\delta S_M}{\delta g_{ab}}$

$$= 2\int \left[\cancel{\nabla_a\left(\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{ab}} V_b\right)} - V_b \nabla_a\left(\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{ab}}\right)\right]\sqrt{-g}\, dx^m$$

1° term $= 0$ (border term)

$$= -2\int V_b \nabla_a\left(\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{ab}}\right)\sqrt{-g}\, dx^m = 0$$

valid $\forall\, V_a$ $\Rightarrow$ $\nabla_a \underbrace{\left(\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{ab}}\right)}_{T^{ab}} = 0$

$\Rightarrow$ $\boxed{\nabla_a T^{ab} = 0}$ because GR is diffeomorphism invariant!

$(\partial_a T^{ab} = 0 \rightarrow \nabla_a T^{ab} = 0$ Not just the equivalence principle$)$

- Energy-momentum conservation and gravity

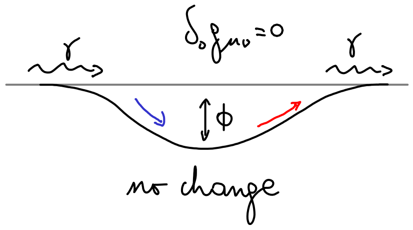
$$\nabla_\nu T^{\mu\nu} \overset{*}{=} \boxed{\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\,T^{\mu\nu}\right) + \Gamma^\mu_{\nu\gamma}T^{\gamma\nu} = 0}$$

* expression for divergence

$\hookrightarrow \partial_\nu\left(\sqrt{-g}\,T^{\mu\nu}\right) = -\sqrt{-g}\,\Gamma^\mu_{\nu\gamma}T^{\gamma\nu}$ but in general $\Gamma^\mu_{\nu\gamma}T^{\gamma\nu} \neq$ because $T^{\gamma\nu}$ is symmetric

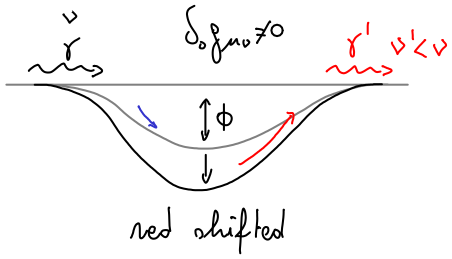
↑ exchange with the gravitational field
it acts as a source term

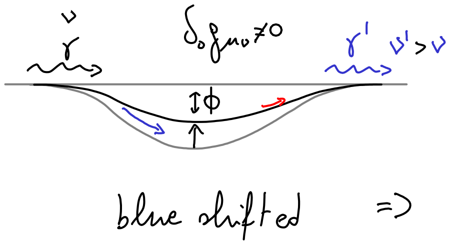
- Example, photon going through a varying gravitational field



$\partial_0 g_{\mu\nu}=0 \Rightarrow$ red-blue shift cancel each other

Time evolution while a photon is going through

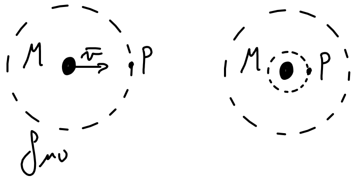




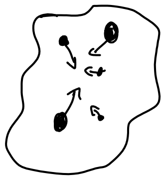
$\Rightarrow$ Energy of γ is not conserved!

→ Rees-Sciama effect

→ Sachs-Wolfe ″



Shift orthogonal to line of sight



Collapse of a cosmic structure

You get these signatures in the photons of the Microwave Cosmic Background

Appendix

- Palatini identity

- Looking at variation of Ricci tensor, $\partial R^a{}_{bad}$, e.g. needed in variational approach of Hilbert action

- Riemann curvature is a function of connection $\Gamma^\alpha{}_{\mu\nu}$ and $\delta_\rho\Gamma^\alpha{}_{\mu\nu}$

  1) In local cartesian frame: $\Gamma=0$, $\delta\Gamma\neq0$ in general
  $\Rightarrow$ $D_\mu\to\delta_\mu$ and Riemann curvature depends on $\delta\Gamma$ only

  2) Variation of Γ, i.e. $\partial\Gamma$, is a tensor (the non tensorial part of Γ drops out)
  $$\Rightarrow \partial R^\alpha{}_{\beta\mu\nu}=\delta_\mu\partial\Gamma^\alpha{}_{\beta\nu}-\delta_\nu\partial\Gamma^\alpha{}_{\beta\mu}=D_\mu\partial\Gamma^\alpha{}_{\beta\nu}-D_\nu\partial\Gamma^\alpha{}_{\beta\mu}$$

- For Ricci tensor: $\boxed{\partial R^\alpha{}_{\rho\alpha\nu}=D_\alpha\partial\Gamma^\alpha{}_{\rho\nu}-D_\nu\partial\Gamma^\alpha{}_{\rho\alpha}}$: Palatini identity

- By keeping torsion: $\partial R^\alpha{}_{\rho\alpha\nu}=D_\alpha\partial\Gamma^\alpha{}_{\rho\nu}-D_\nu\partial\Gamma^\alpha{}_{\rho\alpha}+\underline{T^\sigma{}_{\rho\gamma}\,\partial\Gamma^\gamma{}_{\sigma\nu}}$

- Example: Klein-Gordon scalar field ϕ in η (eg. matter field)

□ $\mathcal{L}_{KG}=-\frac{1}{2}(\delta_a\phi\,\delta^a\phi+m^2\phi^2)$

a by part

$$\frac{dS_{KG}}{d\lambda}\Big|_{\lambda=0}=-\int\delta_a\phi_0\,\delta^a(\partial\phi)+m^2\phi_0\,\partial\phi=-\int\delta^a(\delta_a\phi_0\,\partial\phi)-\partial\phi\,\delta_a\delta^a\phi_0+m^2\phi_0\,\partial\phi=\int(\delta_a\delta^a\phi_0-m^2\phi_0)\,\partial\phi$$

Stokes (Border term =0)

$\Rightarrow$ $\Box\phi_0-m^2\phi_0=0$ $(*)$ eq. of motion

□ From general expression $T_{ab}(\mathcal{L})$: $T_{ab}=\delta_a\phi\,\delta_b\phi-\frac{1}{2}\eta_{ab}(\delta_c\phi\,\delta^c\phi+m^2\phi^2)$

it satisfy conservation law:

$$\delta^a T_{ab}=\underset{\underset{m^2\phi_0\delta_b\phi}{\circledast\uparrow}}{\cancel{\Box\phi\,\delta_b\phi}}+\overset{a\to c}{\cancel{\delta_a\phi\,\delta^a\delta_b\phi}}-\frac{1}{2}\cancel{(\delta_b\delta_c\phi)\,\delta^c\phi}-\frac{1}{2}\cancel{\delta_c\phi\,\delta_b\delta_c\phi}-\cancel{\frac{1}{2}\eta_{ab}\,m^2\,2\phi\,\delta^a\phi}=0\quad\checkmark$$

- Example of other sources

$\mathcal{L}_{KG}=-\frac{1}{2}\sqrt{-g}\,(g^{ab}D_a\phi D_b\phi+m^2\phi^2)$ matter field $\Rightarrow$ Klein-Gordon eq.

$\mathcal{L}_{EM}=-\frac{1}{4}\sqrt{-g}\,g^{ac}g^{bd}F_{ab}F_{cd}$ electro-magnetic field $\Rightarrow$ Maxwell's eq.s $F_{ab}=D_aA_b-D_bA_a$

$\mathcal{L}=\mathcal{L}_G+\alpha_M\mathcal{L}_M$ $\alpha_M\in\mathbb{R}$ const Coupled Einstein-matter

- Palatini action

- Another action leading to Einstein equations

$\mathcal{L}_G[g^{ab}, \nabla_a] = \sqrt{-g}\, R_{ab}\, g^{ab}$ $R_{ab}(g_{cd}, \nabla_e)$ g^{ab} and ∇_a as independent variables

$\uparrow$ i.e. $C^a{}_{bc}$ because $\nabla_a = \hat{\nabla}_a + C^c{}_{ab}$ $\hat{\nabla}$ arbitrary and fixed

choose ∇_a compatible with g^{ab} in $\lambda=0$ => "Christoffel" => $\dot{R}_{ac} = -2\, {}^{\circ}\nabla_{[a} \dot{C}^b{}_{b]c}$ W 7.5.10

$$\delta \mathcal{L}_G = -2 \int g^{ab}\, \overset{\swarrow}{\nabla}_{[a} \delta C^c{}_{c]b} \sqrt{-g}\, d\Omega + \int \left(R_{ab} - \frac{1}{2} R g_{ab}\right) \delta g^{ab} \sqrt{-g}\, d\Omega$$

$$= \int \left(C^{bd}{}_d\, \delta^a_c \cancel{+ C^d{}_{dc}\, g^{ab}} - 2 C^b{}_c{}^a\right) \delta C^c{}_{ab} \sqrt{-g}\, d\Omega + \int \left(R_{ab} - \frac{1}{2} R g_{ab}\right) \delta g^{ab} \sqrt{-g}\, d\Omega$$

$\hookrightarrow$ =0 after symmetrizing with respect to ab

=> Einstein's equations

Uniqueness of G.R.: Lovelock's theorem

(1934)

If we assume for gravity:

1) 4 dimensional space-time
2) Gravity depends on the metric
3) any Energy-momentum contribution source gravity
4) $A(g)$ function of g and its derivatives up to 2° order
5) Energy-Momentum conservation

$\left.\begin{matrix} 2) \\ 3) \end{matrix}\right\} \; A(g) = T$

$\Rightarrow \boxed{A_{\mu\nu}(g) = \alpha G_{\mu\nu} + \beta g_{\mu\nu}}$ $\alpha, \beta \in \mathbb{R}$ const (i.e. A is linear combination of G and g)

unique solution!

↳ Einstein eq. with cosmological constant! $\alpha = 1$, $\beta = \Lambda$

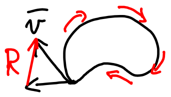
Other theories of gravity?

- We found field equations such to reproduce Newton's gravity by construction
- Are there other possible ways?
- Of course, just change and/or drop some of the assumptions above
- We have already seen one.... Λ: the cosmological constant is a generalization of GR

- Here I will mention few (classical) theories among the zillions available on the market

Equivalent (extended) theories

1) **Curvature** $R^i_{\ jk\ell} \neq 0 \quad (T^i_{\ jk} = 0)$



difference of $\bar{v}$ along closed loops

Pseudo-Riemanian manifold Γ = Christoffel symbols $\leftarrow \delta^2 g$

General relativity

$\boxed{g_{\mu\nu}}$ is our tensor field

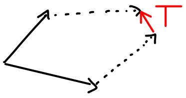
$\mathcal{L}_H = R\sqrt{-g}$ (Hilbert lagrangian density)

f(R) theories:

$$G_{\mu\nu} \to E_{\mu\nu} \equiv \overbrace{f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}}^{\text{non linear in } R_{\mu\nu}} \overbrace{- f'(R)_{;\mu\nu} + g_{\mu\nu}g^{\alpha\beta}f'(R)_{;\alpha\beta}}^{\text{Higher order derivatives}} \qquad f'(R) \equiv \frac{df(R)}{dR}$$

if $f(R) = R \Rightarrow f' = 1 \quad f'(R)_{;\mu\nu} = 1_{;\mu\nu} = 0 \Rightarrow E_{\mu\nu} = G_{\mu\nu}$

2) **Torsion** $T^i_{\ jk} \neq 0$ (Einstein-Cartan theories)



difference of 2 vectors parallely transporte one along the other

manifold based on torsion (Einstein-Cartan theories)

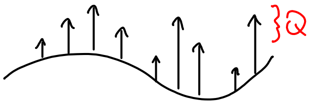
Teleparallel equivalent of gravity

$\boxed{T_{jk}^{\ \ i}} \leftarrow K^i_{\ jk} = \frac{1}{2}T^i_{\ jk} + T_{(j\ \ k)}^{\ \ \ i}$ Contorsion tensor

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(c_1 T_i^{\ jk} T^i_{\ jk} + c_2 T_i^{\ jk} T_j^{\ i}{}_k + c_3 T_i T^i\right) + \lambda_i^{\ \ell jk} R^i_{\ \ell jk} + \tilde{\lambda}_i^{\ jk} Q_i^{\ jk}$$

if $\left(\sum_{i=1}^{3} c_i T_i^2\right) - 2D_\alpha T^\alpha = R \Rightarrow$ equivalent to "standard GR"

3) **Non-metricity** $Q^i_{\ jk} \neq 0$



difference in modulus of $\bar{v}$ when moving along a path

manifold based on non-metricity

$\boxed{g_{\mu\nu}, \Gamma^\alpha_{\mu\nu}} \leftarrow L^i_{\ jk} = \frac{1}{2}Q^i_{\ jk} - Q_{(j\ \ k)}^{\ \ \ i}$ disformation tensor

$$\mathcal{L} = \sqrt{-g}\sum_{i=1}^{5} c_i Q_i^2 + \lambda_i^{\ jk\ell} R^i_{\ jk\ell} + \tilde{\lambda}_i^{\ k\ell} T^i_{\ k\ell}$$

if $c_1 = -\frac{1}{2}c_2 = -c_3 = -\frac{1}{2}c_5 = -\frac{1}{4} \quad c_4 = 0 \Rightarrow$ equivalent to "standard GR"

Brans-Dicke theory

-> Theories can be based on 3 types of fields: scalar, vector, tensor or their combinations

-> GR is based on a rank-2 tensor

-> Based on a vector field only? NO! You would get that 2 massive particles would repel one another

-> Mixed ....

- The idea:

-> Started from equivalence principle => gravity as space-time curvature

-> Scalar-tensor theory assuming

1) G not a constant,
scalar field $\phi(x^\mu)$ determining G, i.e. setting the coupling strength of $T^{\mu\nu}$ to gravity

2) ϕ is determined by matter only, with coupling constant λ T_M (matter only)

$$\boxed{\Box^2\phi = -4\pi\lambda (T_M)^{\nu}_{\ \nu}} \quad (1)$$

3) Curvature is related to the energy-momentum tensors of scalar field ϕ and matter

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi}{c^4\phi}\left[(T_M)_{\mu\nu} + (T_\phi)_{\mu\nu}\right]} \quad (2)$$

- (1) and (2) solved simultaneously, T_ϕ is an horror show
- for $\lambda \to 0$, ϕ not affected by matter, $\phi = G^{-1}$ constant => same as G.R.
- ϕ can also evolve with time

dropping conditions 4 C 113

Linearized field equations

- Perturbative approach: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ⊛ $\quad |h_{\mu\nu}| << 1 \quad h_{\mu\nu} = h_{\nu\mu}$

$h_{\mu\nu} \sim \frac{\phi}{c^2} \simeq 10^{-6}$ Sun $\qquad h_{\mu\nu} \simeq 10^{-4}$ Galaxy

small perturbation $h_{\mu\nu}$ does not mean small matter density fluctuations

⊛ see also 23b

- In all equations you keep only 1° order terms → linearize equations

=> When lowering / rising indeces you can use $\quad g_{\mu\nu} \simeq \eta^{\mu\nu}$

=> Inverse of the metric : $\quad g^{\mu\nu} \simeq \eta^{\mu\nu} + h^{\mu\nu}$

- Christoffel symbols:

$$\Gamma^{\alpha}_{ij} = \frac{1}{2} \overset{\downarrow}{g}{}^{\alpha\gamma}(\partial_i g_{\gamma j} + \partial_j g_{i\gamma} - \partial_\gamma g_{ji})$$
$$\simeq \frac{1}{2}(\eta^{\alpha\gamma} + h^{\alpha\gamma})(\partial_i h_{\gamma j} + \partial_j h_{i\gamma} - \partial_\gamma h_{ji})$$
$$= \frac{1}{2}\eta^{\alpha\overset{\frown}{\gamma}}(\partial_i h_{\underset{\smile}{\gamma} j} + \partial_j h_{i\underset{\smile}{\gamma}} - \partial_{\underset{\smile}{\gamma}} h_{ji})$$
$$= \frac{1}{2}(\partial_i h^{\alpha}{}_{j} + \partial_j h^{\alpha}{}_{i} - \partial^{\alpha} h_{ji})$$

$\partial_j g_{\ell k} = \partial_j(\eta_{\ell k} + h_{\ell k}) = \partial_j h_{\ell k}$

$h^{\mu\nu}\partial_j h_{\gamma k}$ 2° order term => drop

- Ricci tensor

$$R_{ij} = R^{\alpha}{}_{i\alpha j} = \partial_\alpha \overset{\downarrow}{\Gamma}{}^{\alpha}_{ij} - \partial_j \overset{\downarrow}{\Gamma}{}^{\alpha}_{\alpha i} + \overbrace{\Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{ji} - \Gamma^{\alpha}_{j\gamma}\Gamma^{\gamma}_{\alpha i}}^{2° \text{ order, neglect}}$$
$$= \frac{1}{2}(\partial_\alpha\partial_i h^{\alpha}{}_{j} + \cancel{\partial_\alpha\partial_j h^{\alpha}{}_{i}} - \overset{\Box}{\partial_\alpha\partial^{\alpha}} h_{ji} - \cancel{\overset{\leftrightarrow}{\partial_j\partial_\alpha} h^{\alpha}{}_{i}} - \partial_j\partial_i \overset{h}{h^{\alpha}_{\alpha}} + \partial_j\partial^{\alpha} h_{i\alpha}) \qquad \Box \equiv \partial_\alpha\partial^\alpha \qquad h \equiv h^{\alpha}{}_{\alpha}$$
$$= \frac{1}{2}(\partial_\alpha\partial_i h^{\alpha}{}_{j} + \partial_j\partial_\alpha h^{\alpha}{}_{i} - \Box h_{ji} - \partial_j\partial_i h)$$

- Ricci scalar

$$R = g^{ij}R_{ij} \simeq \frac{1}{2}(\partial_\alpha \overset{\downarrow}{\partial^j} h^{\alpha}{}_{\overset{\uparrow}{j}} + \partial_j\partial_\alpha h^{\alpha j} - \Box h^{j}_{j} - \overset{\Box}{\partial_j\partial^j} h)$$
$$= \partial_{\underset{\smile}{j}}\partial_\alpha h^{\alpha \overset{\frown}{j}} - \Box h \qquad j \to \beta$$

j → β to avoid confusion later

- Einstein tensor

$$G_{ij} = \overset{\swarrow}{R_{ij}} - \frac{1}{2}\overset{\swarrow}{R} g_{ij} \simeq \frac{1}{2}(\partial_\alpha\partial_i h^{\alpha}{}_{j} + \partial_j\partial_\alpha h^{\alpha}{}_{i} - \Box h_{ji} - \partial_j\partial_i h) - \frac{1}{2}(\partial_\beta\partial_\alpha h^{\alpha\beta} - \Box h)\eta_{ij}$$
$$= \frac{1}{2}(\partial_\alpha\partial_i h^{\alpha}{}_{j} + \partial_j\partial_\alpha h^{\alpha}{}_{i} - \Box h_{ji} - \partial_j\partial_i h - \eta_{ij}\partial_\beta\partial_\alpha h^{\alpha\beta} + \eta_{ij}\Box h) = \kappa T_{ij}$$

linearized E. equations

- Contracted bianchi identity

$$\nabla^j G_{ij} = 0 = g^{j\beta} D_\beta G_{ij} = g^{j\beta}\Big(\partial_\beta G_{ij} - \underbrace{\overset{\propto h_{ij}}{\Gamma^\gamma_{\beta i}} G_{\gamma j} - \overset{\propto h_{ij}}{\Gamma^\gamma_{\beta j}} G_{i\gamma}}_{\text{"}\Gamma\cdot G\text{" terms of 2° order : } |h_{ij}|^2}\Big) \simeq \boxed{\partial^j G_{ij} = 0} \quad \overset{G_{ij}=\kappa T_{ij}}{\Rightarrow} \quad \boxed{\partial^j T_{ij} = 0}$$

Example : pressureless ($P=0$) , incompressible ($\rho = \text{const}$) :

$T^{ij} = \rho u^i u^j \Rightarrow \rho\, \partial_j u^i u^j = 0 \quad u^j \partial_j u^i = 0$ straight trajectory for volume of fluid element

- Remember: $T_{ij}(g_{\alpha\beta}) \Rightarrow$ to evaluate T you need g but g is determined by T ....

need iterative approach:

use $T_{ij}(\eta_{\alpha\beta})$ to evaluate $h^{(1)}_{\alpha\beta} \Rightarrow g^{(1)}_{\alpha\beta} = \eta_{\alpha\beta} + h^{(1)}_{\alpha\beta}$

use $T_{ij}(g^{(1)}_{\alpha\beta})$ to evaluate $h^{(2)}_{\alpha\beta} \Rightarrow g^{(2)}_{\alpha\beta} = \eta_{\alpha\beta} + h^{(2)}_{\alpha\beta}$

use $T_{ij}(g^{(2)}_{\alpha\beta})$ to evaluate $h_{\alpha\beta}$ ...... till convergence $h_{\alpha\beta}$

You need that because $g_{\alpha\beta}$ affects eq. of motion of matter ($T_{\alpha\beta}$) and $T_{\alpha\beta}$ affects $g_{\alpha\beta}$

This works if the back-reaction of g on T is small

- Simplify the equations, define:

$$\boxed{\gamma_{ij} \equiv h_{ij} - \frac{1}{2}\eta_{ij} h} \qquad \boxed{\gamma} = \gamma^i{}_i = h^i{}_i - \frac{1}{2}\overset{=4}{\eta^i{}_i} h = \boxed{-h}$$

trace-reversed perturbation

$$G_{ij} \simeq \frac{1}{2}\Big(\underbrace{\partial_\alpha\partial_i h^\alpha{}_j}_{①} + \underbrace{\partial_j\partial_\alpha h^\alpha{}_i}_{②} - \underbrace{\Box h_{ji}}_{③} - \underbrace{\partial_j\partial_i h}_{①} - \underbrace{\eta_{ij}\partial_\beta\partial_\alpha h^{\alpha\beta}}_{④} + \underbrace{\eta_{ij}\Box h}_{③}\Big)$$

plug $h_{ij} = \gamma_{ij} - \frac{1}{2}\eta_{ij}\gamma$ in G_{ij}

$$① \quad \partial^\alpha\partial_i\Big(h_{\alpha j} - \frac{1}{2}\eta_{\alpha j} h - \frac{1}{2}\eta_{\alpha j} h\Big) = \partial^\alpha\partial_i\gamma_{\alpha j} - \frac{1}{2}\cancel{\partial_j\partial_i h}$$

$$② \quad \partial_j\partial_\alpha\Big(\gamma^\alpha{}_i - \frac{1}{2}\eta^\alpha{}_i\gamma\Big) = \partial_j\partial^\alpha\gamma_{\alpha i} - \cancel{\partial_j\partial_i\frac{1}{2}\gamma}$$

$h = -\gamma$

$$③ \quad -\Box\Big(h_{ji} - \frac{1}{2}\eta_{ij} h - \frac{1}{2}\eta_{ij} h\Big) = -\Box\gamma_{ij} + \cancel{\frac{1}{2}\eta_{ij}\Box h}$$

$$④ \quad -\eta_{ij}\partial_\beta\partial_\alpha\Big(\gamma^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\gamma\Big) = -\eta_{ij}\partial^\alpha\partial^\beta\gamma_{\alpha\beta} + \cancel{\frac{1}{2}\eta_{ij}\Box\gamma}$$

$h = -\gamma$

$$= \boxed{\frac{1}{2}\Big(\partial_j\partial^\alpha\gamma_{\alpha i} + \partial^\alpha\partial_i\gamma_{\alpha j} - \eta_{ij}\partial^\alpha\partial^\beta\gamma_{\alpha\beta} - \Box\gamma_{ij}\Big)} = \kappa T_{ij}$$

linearized field eq. (wave eq.)

- Applying Gauge transformation to furhter simplify (Hilbert gauge = Lorentz gauge)

- Gauge transformation = infinitesimal coordinate transformation
(diffeomorphysm invariance => "preserve physics")

- Convenient to use Lie-derivatives (change with respect to a coord transformation)

$\Phi_t : M \to M'$ diffeomorphysm   M and M' are physically equivalent (just coordinat transf.)

$V \to$ vector field generating ϕ metric g in $M \to \Phi_* g$ in M' (Φ_t = pool back)

$\phi_t \in$ 1-parameter group

- Transformed g at 1° order: ⊛ see also 23b

$$g' = \Phi_* g \simeq g + \mathcal{L}_V g \cdot t$$

($\mathcal{L}_V$ = Lie derivative, V generator of Φ => "change" in g because of ϕ)

$$= g + \mathcal{L}_\xi g \qquad \xi \equiv \bar{v} t \quad \bar{v} \in V$$

ξ infinitesimal vector

$g = \eta + h \to$

$$= \eta + h + \mathcal{L}_\xi \eta + \underset{\sim}{\mathcal{L}_\xi h}$$

negligible: "infinitesimal change of small thing"

$$\simeq \eta + \underbrace{h + \mathcal{L}_\xi \eta}_{h' \simeq \Phi_* h}$$

- Components:

analogous to electrodynamics in which $A_i' = A_i + \partial_i \lambda$

$$h_{ij} \Rightarrow h'_{ij} = h_{ij} + (\mathcal{L}_\xi \eta)_{ij} \overset{\text{explicit}}{=} h_{ij} + \xi^\gamma \underbrace{\partial_\gamma \eta_{ij}}_{=0} + \eta_{i\gamma} \partial_j \xi^{\gamma} + \eta_{\gamma j} \partial_i \xi^{\gamma} = h_{ij} + \partial_j \xi_i + \partial_i \xi_j$$

$$\gamma_{ij} \rightarrow \gamma'_{ij} = h'_{ij} - \frac{1}{2} \eta_{ij} h' = h_{ij} + \partial_j \xi_i + \partial_i \xi_j - \frac{1}{2} \eta_{ij} (h + \partial_\alpha \xi^\alpha + \partial_\alpha \xi^\alpha)$$

$$= \underline{h_{ij}} + \partial_j \xi_i + \partial_i \xi_j \underline{- \frac{1}{2} \eta_{ij} h} - \eta_{ij} \partial_\alpha \xi^\alpha$$

$$= \gamma_{ij} + \partial_j \xi_i + \partial_i \xi_j - \eta_{ij} \partial_\alpha \xi^\alpha$$

$$\partial_j \gamma^{ij} \longrightarrow \partial_j \gamma^{ij'} = \partial_j \gamma^{ij} + \overset{\square}{\partial_j \partial^j} \xi^i + \partial_j \partial^i \xi^j - \partial_j (\eta^{ij} \partial_\alpha \xi^\alpha)$$

$\partial_j \eta^{ij} = 0$, $\eta^{ij} \partial_j = \partial^i$

$$= \partial_j \gamma^{ij} + \square \xi^i + \cancel{\partial_j \partial^i \xi^j} - \overset{\alpha \to j}{\cancel{\partial^i \partial_\alpha \xi^\alpha}} \overset{!}{=} 0$$

i.e. use transformation satisfying $\square \xi^i \overset{!}{=} -\partial_j \gamma^{ij}$

"harmonic condition"

$$\Rightarrow \boxed{\partial_\nu \gamma'^{\mu\nu} \overset{!}{=} 0}$$ Hilbert gauge (Lorentz gauge / harmonic gauge) $\rightarrow G^{\mu\nu} = -\square \gamma^{\mu\nu}$

$$\boxed{\square \gamma'^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}}$$ Linearized field equations in Hilbert gauge

↖ G = Newton's grav. constant

- Now for simplicity I drop the prime γ^{*}, i.e. γ is γ in Hilbert gauge

- Homogeneous linearized field equation

$\Box \gamma^{\mu\nu} = 0$ $\Rightarrow$ wave equation

$\hookrightarrow$ gravitational radiation (grav. waves!)

more on that later...

- Solving the inhomogeneous linearized field equations

$\Box A^{\mu} = -\frac{4\pi}{c} j^{\mu}$ Recall Maxwell's equation in Lorentz gauge ($\partial_\mu A^\mu = 0$)

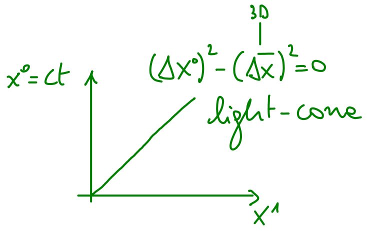
$\Box \gamma^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}$ Wow! $\gamma^{\mu\nu}$ plays the role of the 4-potential A^μ in electrodynamics

$\Rightarrow$ same techniques to solve them

– Green's function G of d'Alambert operator $\Box$

$$\Box G(x^\mu, x^{\mu'}) = \Box G(t-t', \bar{x}-\bar{x}') = -4\pi \delta_D(t-t', \bar{x}-\bar{x}')$$

$$G(x^\mu, x^{\mu'}) = \frac{1}{|\bar{x}-\bar{x}'|} \delta_D(\underbrace{x^0 - x'^0 - |\bar{x}-\bar{x}'|}_{\text{light-cone}}) \qquad x^0 = ct$$



$\Rightarrow$ Solution

$$\gamma^{\mu\nu}(x^\alpha) = \frac{\overset{4}{\cancel{16\pi}} G}{\cancel{4\pi} c^4} \int \frac{T^{\mu\nu}(x^{0'}, \bar{x}')}{|\bar{x}-\bar{x}'|} \delta_D(x^0 - x'^0 - |\bar{x}-\bar{x}'|)\, d^3x'$$

$$= \frac{4G}{c^4} \int \frac{T^{\mu\nu}(x^0 - |\bar{x}-\bar{x}'|, \bar{x}')\, d^3x'}{|\bar{x}-\bar{x}'|}$$

$$\gamma_{ij} \equiv h_{ij} - \frac{1}{2}\eta_{ij} h$$

$\Rightarrow$ retardation: $\delta_D(x^0 - x'^0 - |\bar{x}-\bar{x}'|)$ contribution from the past light cone

$\Rightarrow$ perturbed metric given by matter point like elements $\delta_D(x^0 - x'^0 - |\bar{x}-\bar{x}'|)$

$\Rightarrow$ Sum of $\int \cdots d^3x$: superposition principle is valid in the linear regime

$\Rightarrow$ perturbation $\propto r^{-1}$

$h_{\mu\nu} = \overset{\downarrow}{\gamma_{\mu\nu}} - \frac{1}{2}\eta_{\mu\nu} \overset{\downarrow}{\gamma}$ $\Rightarrow$ $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ you have the metric!

(perturbation at 1° order)

Gauge transformation and perturbative approach

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ · does not fully specify the coord. system on space-time

· decomposition in background η and perturbation h is not unique

· Consider

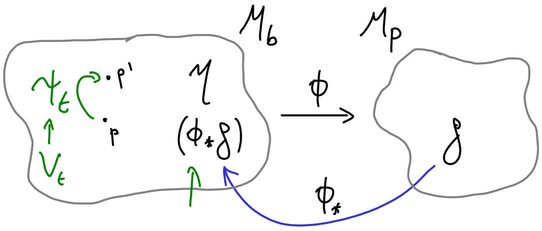
M_p	Physical space-time	with $g_{\mu\nu}$ obaying Einstein's eq.s
M_b	Background space-time	with $\eta_{\mu\nu}$
$\phi : M_b \to M_p$	Diffeomorphysm	

=> M_b and M_p are the same manifold because they are diffeomorphic but they posses different tensors fields e.g. η ang g

=> perturbation as $\boxed{h_{\mu\nu} \equiv (\phi^* g)_{\mu\nu} - \eta_{\mu\nu}}$ in M_b

general: no reason for $h_{\mu\nu}$ to be small

ϕ can pool-back Einstein eq.s on M_b



This represent the full metric g in M_b

· weak field: $|h_{\mu\nu}| \ll 1$ (for some ϕ that now we consider)

=> $h_{\mu\nu}$ obays linearized Einstein equations because $g_{\mu\nu}$ does

· Vector fields $V_t^\mu(x^\nu)$ in M_b: generate $\{\psi_t\}$ 1 param. continuous set of diffeomorphysms

$\psi_t : M_b \to M_b$ $t \ll 1$ for which $|h_{\mu\nu}| \ll 1$ => $\phi \to \phi \circ \psi_t$ ($\eta \sim$ const => $\eta \to \eta$)

$$h_t \equiv (\phi \circ \psi_t)_* g - \eta \quad \text{also small}$$

$$= \psi_{t*}(\phi_* g) - \eta = \psi_{t*}(\eta + h) - \eta = \psi_{t*}\eta + \psi_{t*} h - \eta$$

$$= \psi_{t*} h + t\left(\frac{\psi_{t*}\eta - \eta}{t}\right) = \psi_{t*} h + t \cdot \mathcal{L}_{V_t}\eta \qquad t \text{ infinitesimal}$$

$$\simeq h + \mathcal{L}_{V_t}\eta \cdot t = h + \mathcal{L}_{\xi_t}\eta \qquad \xi_t \equiv t V_t$$

to lower order: $\psi_{t*} h \simeq h$ because small transformation

components: $h^t_{\mu\nu} = h_{\mu\nu} + 2\, \partial_{(\mu}\xi_{(t)\nu)}$ Gauge tranformation in linearized theory transformation giving physically equivalent spacetime

Nearly Newtonian regime

- Assumptions

$T_{00} \gg |T_{0j}|$ non relativistic mean velocity of the fluid $\alpha,..=0,1,2,3$ $i,j=1,2,3$

$T_{00} \gg |T_{ij}|$ rest mass energy dominates over kinetic energy

- Solution for dust

$T_{\mu\nu}=\rho u_\mu u_\nu$ $\gamma_{\mu\nu}(x^\alpha) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(x^0-|\bar{x}-\bar{x}'|,\bar{x}')\,d^3x'}{|\bar{x}-\bar{x}'|}$

$T_{00}=\rho c^2$ $\gamma_{00}(x^\alpha) = \frac{4G}{c^2}\int \frac{\rho(\bar{x}')\,d^3x'}{|\bar{x}-\bar{x}'|} = -4\frac{\varphi}{c^2}$ $\varphi \equiv -G\int \frac{\rho(\bar{x})\,d^3(\bar{x})}{|\bar{x}-\bar{x}'|}$ far from source: $\varphi = -\frac{GM}{r}$

↑ monopole dominates

$T_{0i}=\rho c u_i$ $\gamma_{0i} \simeq 0$

$T_{ij}=\rho u_j u_i$ $\gamma_{ij} \simeq 0$

$$\gamma = \eta^{\alpha\beta}\gamma_{\alpha\beta} = -\gamma_{00}+\dots \simeq -\gamma_{00}$$

- Resulting metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \left(\gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma\right) \quad \therefore \quad g_{00} \simeq -1 - \frac{4\varphi}{c^2} + 2\frac{\varphi}{c^2} = -\left(1+\frac{2\varphi}{c^2}\right)$$

$$g_{0i} \simeq 0$$

$$g_{ii} \simeq 1 - \frac{1}{2}\frac{4\varphi}{c^2} = \left(1-\frac{2\varphi}{c^2}\right) \quad (!)$$

- 4-interval

$$ds^2 = -\left(1+\frac{2\varphi}{c^2}\right)c^2dt^2 + \left(1-\frac{2\varphi}{c^2}\right)(dx^2+dy^2+dz^2)$$

$$ds^2 = -\left(1-\frac{2GM}{c^2}\right)c^2dt^2 + \left(1-\frac{2GM}{c^2}\right)(dx^2+dy^2+dz^2) \quad \text{for any source}$$

- Gravitational lensing

photons: $ds^2=0$ $\left(1+\frac{2\varphi}{c^2}\right)c^2dt^2 = \left(1-\frac{2\varphi}{c^2}\right)d\bar{x}^2$

$$c' = \frac{|d\bar{x}|}{dt} = c\left(1+\frac{2\varphi}{c^2}\right)^{1/2}\left(1-\frac{2\varphi}{c^2}\right)^{-1/2} \simeq c\left(1+\frac{\varphi}{c^2}\right)\left(1+\frac{\varphi}{c^2}\right) = c\left(1+\frac{2\varphi}{c^2}+\overset{\text{neglect}}{\cancel{\frac{\varphi^2}{c^4}}}\right)$$

Refraction index $n \equiv \frac{c}{c'} = \left(1+\frac{2\varphi}{c^2}\right)^{-1} \simeq \left(1-\frac{2\varphi}{c^2}\right)$ for weak gravitational lensing

$\varphi \leq 0$, $\varphi \to 0$ at ∞ $\Rightarrow$ $\underline{c' \leq c}$ → time delay (Shapiro delay)

Going to an higher order: gravitomagnetic field

Keep next higher order in c^{-1}

$$\cdot\ T_{\mu\nu} = \rho c^2 u_\mu u_\nu \quad (\text{dust},\ P=0) \quad \Longrightarrow \quad \overset{(1)}{T_{00}},\ \overset{(c^{-1})}{T_{0i}},\ \underbrace{\overset{(c^{-2})}{T_{ij}}}_{\text{neglect}} \simeq 0 \quad i,j=1,2,3 \quad (\text{no stress terms})$$

$$\cdot\ \Box \gamma_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad \begin{cases} \boxed{\Box \gamma_{ij} \simeq 0} & i,j=1,2,3 \\ \Box \gamma_{0\mu} = -\frac{16\pi G}{c^4} T_{0\mu} \end{cases} \qquad \underset{\mu=0,1,2,3}{\boxed{\Box A_\mu = -\frac{4\pi}{c^2} j_\mu}}$$

$A_\mu \equiv \frac{\gamma_{0\mu}}{4}$ 4-vector potential

$j_\mu \equiv \frac{G}{c^2} T_{0\mu}$ matter 4-current

Gravitomagnetic potential

A_μ fully determine the field like in electrodynamics because:

quasi static case: $\Box = \left(-\frac{\delta}{c^2\delta t} + \vec{\nabla}^2\right) \longrightarrow \vec{\nabla}^2$ (i.e. neglect retardation)

$\Box \gamma_{ij} \simeq \vec{\nabla}^2 \gamma_{ij} = 0$ together with $\gamma_{ij} \simeq 0 \Rightarrow \gamma_{ij} = 0$ everywhere

For dust

$A_0 = -\frac{\varphi}{c^2}$ (from previous result) scalar potential

$A_i = \frac{G}{c^4} \int \frac{T_{0i}(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|} = \frac{G}{c} \int \frac{\rho(\vec{x}')v(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|}$ vector potential

$T_{0i} = \rho c v_i$

=> Matter currents create a "magnetic" gravitational potential

=> we expect to find a Lorentz force like component in the eq. of motion of matter in a gravitational field (Wow!)

The resulting metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \left(\gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma\right):$$

$\gamma_{00} = 4A_0 \qquad \gamma = -\gamma_{00} + \underbrace{\gamma_{11} + \dots}_{\simeq 0} \simeq -4A_0 \quad \Rightarrow \quad g_{00} = -1 + 2A_0$

$\gamma_{0i} = 4A_i \qquad$ " $\quad \Rightarrow \quad g_{0i} = \gamma_{0i} = 4A_i$

$\gamma_{ij} \simeq 0 \qquad$ " $\quad \Rightarrow \quad g_{ij} = (1 + 2A_0)\delta_{ij}$

$$ds^2 = -\Big(1 + \overset{-2A_0}{\frac{2\varphi}{c^2}}\Big)c^2dt^2 + \Big(1 - \overset{+2A_0}{\frac{2\varphi}{c^2}}\Big)(dx^2 + dy^2 + dz^2) + \underbrace{4A_i c(dt\,dx^i + dx^i dt)}_{= 8A_i c\,dt\,dx^i} \quad (!)$$

Frame dragging: the geodesics are dragged by the motion of matter

- Equation of motion of free particle in grav. field

- Action ⇒ Lagrangian ⇒ Euler-Lagrange eq.

$$S = -m_0 c \int \sqrt{-g_{\mu\nu} u^\mu u^\nu}\, d\tau$$

non relat. motion ⇒ $d\tau \simeq dt$ $\gamma \simeq 1$: $u^\mu = \dot{x}^\mu$ $\cdot = \frac{d}{dt}$ $\dot{x}^0 = \frac{c\,dt}{dt} = c$

$$= -m_0 c \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, dt$$

$g_{00} = -1 + 2A_0$, $g_{0i} = g_{i0} = 4A_i$, $g_{ij} = (1 + 2A_0)\delta_{ij}$

$$= -m_0 c \int \left[(1 - 2A_0)c^2 - 8A_i c \dot{x}^i - (1 + 2A_0)\delta_{ij} \dot{x}^i \dot{x}^j \right]^{1/2} dt$$

$$= -m_0 c \int \left(c^2 - 2A_0 c^2 - 8c\bar{A}\bar{v} - \bar{v}^2 - \underbrace{2A_0 \bar{v}^2}_{\text{negligible} = 2\frac{\varphi}{c^2}\bar{v}^2} \right)^{1/2} dt$$

$$L = -m_0 c \left(c^2 - \underbrace{2A_0 c^2 - 8c\bar{A}\bar{v} - \bar{v}^2}_{\ll c^2} \right)^{1/2} \simeq +m_0 c \left(-c^2 + A_0 c^2 + 4c\bar{A}\bar{v} + \frac{1}{2}\bar{v}^2 \right)$$

$$A_0 = -\frac{\varphi}{c^2}$$

$$A_i = \frac{G}{c} \int \frac{\rho(\bar{x}') v(\bar{x}')}{|\bar{x} - \bar{x}'|} d^3x$$

$$\frac{d}{dt}\frac{\delta L}{\delta \bar{v}} - \frac{\delta L}{\delta \bar{x}} = 0 \quad \Rightarrow \quad \boxed{\frac{d\bar{v}}{dt} = \underbrace{c^2 \bar{\nabla} A_0}_{(1)\ (\text{Newtonian})} + \underbrace{4c\bar{v} \times (\bar{\nabla} \times \bar{A})}_{(2)\ \text{"Magnetic"}}} = \frac{\bar{f}}{m}$$

⇒ Gravitomagnetic field

(1) (Newtonian)

$A_0 = \varphi / c^2$

$\bar{\nabla}\varphi$

$\left(\cdot \xrightarrow{\dot{\bar{v}}} \cdot M \right)$

(2) "Magnetic"

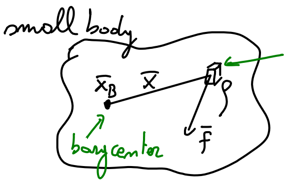
$\bar{B} \equiv \bar{\nabla} \times \bar{A}$

$\ddot{x}$ ⊥ to "magnetic" field $\bar{B} \equiv \bar{\nabla} \times \bar{A}$

$\ddot{x}$ directly proportional to $\bar{v}$ particle

We will see frame dragging again in the Kerr metric

- Analize torque excerted on a small body



$\bar x_B \overset{!}{=} (0,0,0)$

$\bar f = \rho \frac{d\bar v}{dt}$

$$\bar M = \int \bar x \times (\rho \dot{\bar v})\, d^3x = \int \bar x \times \left[\rho\left(c^2 \bar\nabla A_0 + 4c\bar v \times \bar B\right)\right] d^3x$$

$$= c^2 \int \bar x \times (\underbrace{\rho}_{\text{scalar}} \bar\nabla A_0)\, d^3x + 4c\int \bar x \times (\underbrace{\rho \bar v} \times \bar B)\, d^3x$$

$\bar a \times \bar b = -\bar b \times \bar a$, $\bar j \equiv \rho \bar v$ = matter current

$$= -c^2 \int (\underbrace{\bar\nabla A_0}) \times (\rho \cdot \bar x)\, d^3x + 4c \int \bar x \times (\bar j \times \bar B)\, d^3x$$

$\bar\nabla A_0 \sim$ const across small body

$$\simeq -c^2 \bar\nabla A_0 \times \underbrace{\int \bar x \rho\, d^3x} + 4c\int \bar x \times (\bar j \times \bar B)\, d^3x$$

$\int \bar x \rho\, d^3x = \bar x_B \overset{!}{=} \bar 0$ barycenter

$$= 0 + 2c\Big(\underbrace{\int \bar x \times \bar j\, dx^3}_{\bar S}\Big) \times \bar B$$

just math $\bar x \times \bar j \times \bar B = (\bar x \bar B)\bar j - (\bar x \bar j)\bar B$

$\bar S$ = intrinsic angular momentum of the body (spin) $\bar S = \int \bar x \times (\rho \bar v)\, dx^3$

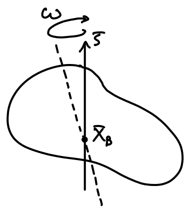
– Precession of "orbiting" object

angular momentum: $\bar L = \bar r \times \bar p$

torque : $\bar\tau \equiv \bar x \times \bar F$ $d\bar L = \bar \tau dt$ => $\dot{\bar L} = \bar\tau$ change in direction of $\bar L$

$\bar M = 2c\bar S \times \bar B$ => $\dot{\bar S} = 2c\bar S \times \bar B$ ⟵ $\dot{\bar S} = \bar\Omega \times \bar S$, $\bar\Omega = -2c\bar B$

Orient coordinates such that : $\bar B = B\bar e_3$ => $\bar B = (0,0,B)^T$



$\underbrace{\dot S_1 = 2cBS_2 \quad \dot S_2 = -2cS_1 B}$ $\dot S_3 = 0$ for convenience: $\sigma \equiv S_1 + iS_2$

=> $\dot\sigma = -2cBi\sigma$ (1 eq. to capture the evolution of $\bar S$)

Ansatz $\sigma = \sigma_0 e^{i\omega t}$ $\boxed{\omega = -2cB}$ => $\bar\omega = -2c\bar B = -2c\bar\nabla \times \bar A$

ω = spin precession frequency experienced by a spinning body in a gravitational field

Lens-Thirring effect

$$\bar B \equiv \bar\nabla \times \bar A = \frac{G}{c}\int \frac{\overset{\text{Source!}}{\rho_s(\bar x')}\, \bar v(\bar x)}{|\bar x - \bar x'|}\, d^3x$$

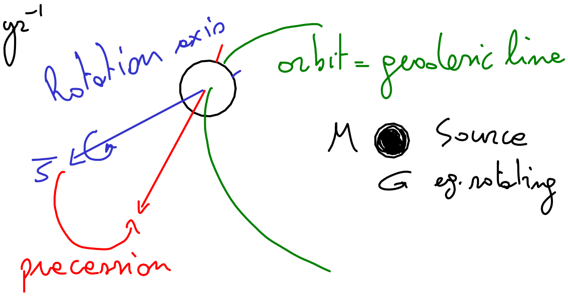
– Experiment: satellite Gravity probe B (2011)

GR prediction: geodetic precession $-6606{,}1$ mas yr^{-1} mas = milly arcseconds

Lens-Thirring precession $-39{,}2$ mas yr^{-1}

Measure: $(-6601{,}8 \pm 18{,}3)$ mas/yr

$(-37{,}2 \pm 7{,}2)$ mas/yr



Gravitational waves

- Homogeneous linearized field equations (Hilbert gauge condition $\partial_\nu \gamma^{\mu\nu}=0$)

$\boxed{\Box \gamma^{\mu\nu}=0}$ ⇒ d'Alambert equation : γ is a dynamical field
⇒ vacuum solution : plane wave, gravitational radiation : GW

$\boxed{\gamma_{\mu\nu}=Re(\varepsilon_{\mu\nu}e^{ik_\alpha x^\alpha})}$ $\varepsilon_{\mu\nu}=\varepsilon_{\nu\mu}=const\in\mathbb{R}$ polarization tensor, setting the amplitude
$Re(-)$ because only real solutions are physical

(1) $\underline{\Box\gamma^{\mu\nu}} = \partial_\alpha\partial^\alpha\gamma^{\mu\nu} \Rightarrow Re(-k_\alpha k^\alpha \varepsilon^{\mu\nu}e^{ik_\alpha x^\alpha})=0 \Rightarrow \boxed{k_\alpha k^\alpha=0}$

k^α = null vector ⇒ propagat at speed of light (along the light-cone)

(2) $\underline{k_\alpha x^\alpha} = \hbar\left(\frac{\omega}{c},\bar{k}\right)\begin{pmatrix}x^0\\ \bar{x}\end{pmatrix} \simeq \hbar\left(-\frac{\omega}{\cancel{c}}\cancel{c}t+\bar{k}\bar{x}\right) = -\hbar(\omega t-\bar{k}\bar{x})$ $(g_{\mu\nu}\simeq\eta_{\mu\nu})$

$\bar{k}$ set direction of propagation , ω: frequency of oscillations

(3) $\underline{\partial_\nu\gamma^{\mu\nu}=0} = Re\left(ik_\nu\varepsilon^{\mu\nu}e^{ik_\alpha x^\alpha}\right) \Rightarrow \boxed{k_\nu\varepsilon^{\mu\nu}=0}$ (in Hilbert gauge)

$\varepsilon^{\mu\nu}$ orthogonal to k_ν

(4) linear eq. ⇒ any linear combination of solutions is a solution: GW are "policromatic"

- In Hilbert gauge, we can further require : $\gamma=\gamma^\mu{}_\mu=0$ with an appropriate choice of ξ^α

$\gamma'_{\mu\nu}=\gamma_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu-\eta_{\mu\nu}\partial_\alpha\xi^\alpha$ ← recall generic gauge transf.

1) $\partial_\nu\gamma'^{\mu\nu}=\partial_\nu\gamma^{\mu\nu}+\Box\xi^\mu+\cancel{\partial_\nu\partial^\mu\xi^\nu}-\eta^{\mu\nu}\cancel{\partial_\nu\partial_\alpha\xi^\alpha}=0$ Hilbert gauge satisfied if $(\Box\xi^\nu\overset{!}{=}0)\overset{i.e.}{\Rightarrow}\xi^\nu=A^\nu e^{ik_\beta x^\beta}$

2) $\gamma'=\gamma^\alpha{}_\alpha+\partial_\alpha\xi^\alpha+\partial_\alpha\xi^\alpha-\eta^{\mu\nu}\eta_{\mu\nu}\partial_\alpha\xi^\alpha=\gamma+2\partial_\alpha\xi^\alpha-4\partial_\alpha\xi^\alpha=\gamma-2\partial_\alpha\xi^\alpha\overset{!}{=}0$

$\boxed{\gamma'=0}$ if we set A^α such that : $2\partial_\alpha\xi^\alpha=2\partial_\alpha A^\alpha e^{ik_\beta x^\beta}\overset{!}{=}\gamma$

↑ wave dependent gauge transf.

- Hilbert gauge (giving $k_\nu\varepsilon^{\mu\nu}=0$) + Traceless gauge $(\gamma=0)$

⇒ $h'_{\mu\nu}\equiv\gamma'_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\overset{\overset{!}{=}0}{\gamma'}$ ⟶ $\boxed{h_{\mu\nu}=\gamma_{\mu\nu}}$ $h_{\mu\nu}=Re(\varepsilon_{\mu\nu}e^{ik_\alpha x^\alpha})$

$\gamma'=\gamma'^\alpha{}_\alpha=Re(\varepsilon'^\mu{}_\mu e^{ik_\alpha x^\alpha})=0$ ⟶ $\boxed{\varepsilon^\mu{}_\mu=0}$ ε is traceless

here I dropped the prime "'"

- Identifying free components of $\varepsilon_{\mu\nu}$

- Consider GW along z direction (no loss of generality)

$$(k^\mu) = \hbar\left(\frac{\omega}{c}, \vec{k}\right)^T = \hbar\left(\frac{\omega}{c}, 0, 0, k\right)^T = \hbar\underset{\omega>0}{\frac{\omega}{c}}(1,0,0,1)^T$$

observer moving with 4-velocity u^μ sees $\omega = -k_\mu u^\mu$

1) Symmetry of metric $\Rightarrow$ $\boxed{\varepsilon^{\mu\nu} = \varepsilon^{\nu\mu}}$

2) $\underline{\varepsilon^{\mu\nu} k_\nu = 0} = \frac{\hbar\omega}{c}\left(-\varepsilon^{\mu 0} + \varepsilon^{\mu 3}\right) = 0 \Rightarrow \boxed{\varepsilon^{\mu 0} = \varepsilon^{\mu 3}}$ * $\longrightarrow$

Hilbert gauge

$$\varepsilon^{10} = \varepsilon^{13} = \varepsilon^{31}$$
$$\varepsilon^{20} = \varepsilon^{23} = \varepsilon^{32}$$
$$\varepsilon^{30} = \varepsilon^{33} = \varepsilon^{03}$$
$$\varepsilon^{00} = \varepsilon^{03} \quad (2^*)$$

because of (2^*)

3) $\underline{\varepsilon = \varepsilon^\alpha{}_\alpha = 0} = -\cancel{\varepsilon^{00}} + \varepsilon^{11} + \varepsilon^{22} + \cancel{\varepsilon^{33}} \Rightarrow \boxed{\varepsilon^{11} = -\varepsilon^{22}}$ *

Traceless

4) $\underline{\partial_\mu \xi^\mu \overset{!}{=} 0}$ $\quad \xi_\mu = A_\mu e^{ik_\alpha x^\alpha}$ $\quad \partial_\mu\left(A^\mu e^{ik_\alpha x^\alpha}\right) = ik_\mu A^\mu e^{ik_\alpha x^\alpha} = 0$ $\quad k_\mu A^\mu = 0 = \frac{\hbar\omega}{c}\left(-A^0 + A^3\right)$

$A^0 = A^3$

Transverse
purely spatial

- we again $\gamma'_{\mu\nu} = \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha$ with ξ^ν given by this further constrain

$$\Rightarrow \underset{*}{\boxed{\varepsilon^{00} = \varepsilon^{01} = \varepsilon^{02} = 0}} = \varepsilon^{03} = \varepsilon^{33} \; (2^*)$$

$\Rightarrow$ free coefficients : $(1),(2) \longrightarrow \underbrace{\varepsilon^{00}, \overset{(4)}{\varepsilon^{01}}, \varepsilon^{02}}_{=0}, \underline{\varepsilon^{11}}, \overset{(3)}{\varepsilon^{12}}, \underline{\varepsilon^{22}}$

$$\text{Degrees of freedom} = 2 = 10\,\underbrace{[h_{\mu\nu}]}_{\text{metric symmetric}} - 4\,\underbrace{[\partial_\nu h^{\mu\nu} = 0]}_{\text{Hilbert gauge}} - 1\,\underbrace{[h^\alpha{}_\alpha = 0]}_{\text{Traceless}} - 3\,\underbrace{[\partial_\mu \xi^\mu = 0]}_{\text{Transverse}}$$

Hilbert-Transverse-Traceless gauge (TT)

$$(\varepsilon^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon^{11} & \varepsilon^{12} & 0 \\ 0 & \varepsilon^{12} & -\varepsilon^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\varepsilon^{11}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\varepsilon^+} + \underbrace{\varepsilon^{12}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\varepsilon^\times} = \varepsilon^+ + \varepsilon^\times$$

2 degrees of freedom
=
2 polarization states

oscillations in x-y plane $\perp$ z

- Perturbed metric in TT gauge seen by observer at rest $\vec{u} = (c,0,0,0)^T$

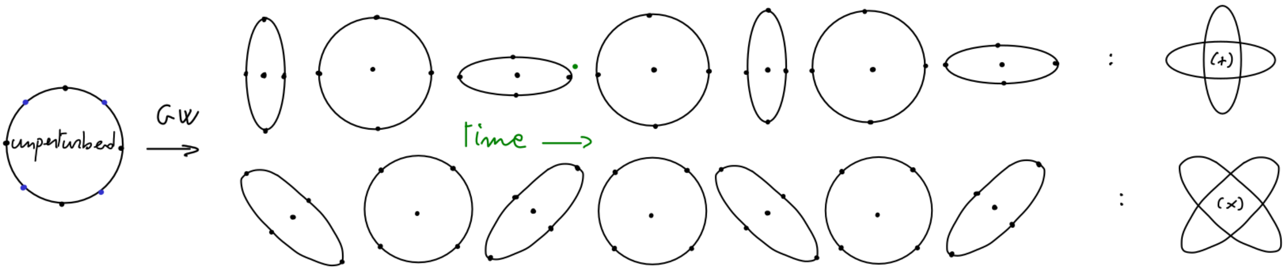
$$\left(h^{TT}_{\mu\nu}\right) = \mathrm{Re}\left(\varepsilon_{\mu\nu} e^{ik_\alpha x^\alpha}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(ct-z)]$$

e.g. binary system (2 black holes) $\begin{cases} \lambda = \frac{2\pi}{\omega} \sim \text{orbital radious} \\ h_{xx} \approx 10^{-21} \quad \text{i.e. } 10^{-9}\,\mu m \text{ over a distance of } 1\,Km \end{cases}$

- Polarization states of GW

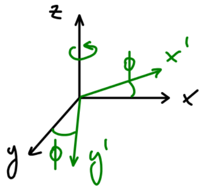
$\Rightarrow$ 2 Gauge invariant polarization states $\varepsilon_{\mu\nu} = \varepsilon^{+}_{\mu\nu} + \varepsilon^{\times}_{\mu\nu} = \begin{pmatrix} \varepsilon_{11} & 0 \\ 0 & -\varepsilon_{11} \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon_{12} \\ \varepsilon_{12} & 0 \end{pmatrix}$

Generic GW = sum of 2 GWs states (orthogonal and cross polarizations)



- GW are spin 2 (from a tensor field) v.s. Electromagnetic waves spin 1 (from a vector field)

- Another way to look at the two polarization states



- Transformation of ε under rotation about z axis by angle ϕ

$$\varepsilon^{\mu'\nu'} = R^{\mu'}{}_{\mu} R^{\nu'}{}_{\nu}\, \varepsilon^{\mu\nu} \qquad R(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{aligned} \varepsilon'^{11} &= \varepsilon^{11}\cos(2\phi) + \varepsilon^{12}\sin(2\phi) \\ \varepsilon'^{12} &= -\varepsilon^{11}\sin(2\phi) + \varepsilon^{12}\cos(2\phi) \end{aligned}$$

Rotation matrix

$$\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\varepsilon'_{\pm}} \equiv \varepsilon'^{11} \pm i\varepsilon'^{12} = \varepsilon^{11}\underline{\cos 2\phi} + \varepsilon^{12}\sin 2\phi \mp i\varepsilon^{11}\sin 2\phi \pm i\varepsilon^{12}\underline{\cos 2\phi}$$

$$= \left(\varepsilon^{11} \pm i\varepsilon^{12}\right)\cos 2\phi + \left(\varepsilon^{12} \mp i\varepsilon^{11}\right)\sin 2\phi$$

$$= \left(\underbrace{\varepsilon^{11} \pm i\varepsilon^{12}}_{*}\right)\cos 2\phi + \left(\underbrace{-i\varepsilon^{12} \mp \varepsilon^{11}}_{*}\right) i\sin 2\phi \qquad * \equiv \varepsilon_{\pm} \equiv \varepsilon^{11} \pm \varepsilon^{12}$$

$$= \boxed{\varepsilon_{\pm}\, e^{i2\phi}}$$

$\varepsilon_{\pm}$ has helicity ± 2 $\Rightarrow$ 2 polarization states : right, left-handed circular polarization

Motion of particles in presence of GW

- Consider test particles moving slowly $\Rightarrow$ $\bar{u} = (c,0,0,0)^T$ spatial components $\bar{u}$ are negligible

- 1 particle initially at rest :

$$\frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta}\frac{du^\alpha}{d\tau}\frac{du^\beta}{d\tau} = 0 \qquad \frac{du^\nu}{d\tau} = -\Gamma^\nu_{00}\, c^2 = -\frac{1}{2}\eta^{\nu\gamma}\big(\underbrace{\partial_0 h_{\gamma 0}}_{=0} + \underbrace{\partial_0 h_{\gamma 0}}_{=0} - \underbrace{\partial_\gamma h_{00}}_{=0}\big)c^2 = 0 \quad \leftarrow \text{TT gauge}$$

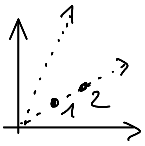
$\Rightarrow$ In the particle frame, the GW is not perceived $h_{\mu\nu}$ fluctuates but the particle is "free-falling" $\rightarrow$ lives in $\mathbb{M}$

- 2 particles separated by Δx :

$\bar{x}_1 = (0,0,0,0)$

$\bar{x}_2 (0, \Delta x, 0, 0)$

$\Delta t = 0$



Space like interval $\sqrt{+g\, dx dx}$ $\uparrow$

$$\text{proper distance: } ds = \sqrt{g_{\mu\nu}dx^\nu dx^\mu} = \sqrt{g_{00}0^2 + g_{11}\Delta x^2 + g_{22}0^2} = \Delta x\big(\eta_{11} + h_{11}\big)^{1/2} \simeq \Delta x\big(1 + \tfrac{1}{2}h_{11}\big)$$

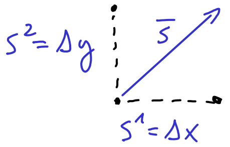
$\Rightarrow$ The GW is perceived as a change of proper distance!

the space-time is "squized" and "stretched" harmonically

e.g. binary system (2 black holes) $h_{11} \approx 10^{-21}$ i.e. $10^{-9}\mu m$ over a distance of 1 Km

- 3 particles displaces as an L :

$\bar{s} = (ct, \Delta x, \Delta y, 0)$ components represent the separation between the particles



$\frac{d^2 s^\nu}{d\tau^2} = \eta^{\alpha\nu} R_{\alpha\beta\gamma\mu} u^\beta u^\gamma s^\mu$ geodesic deviation eq. to see how a vector s^α changes as a function of time

$u^i \to 0 \quad i=1,2,3$

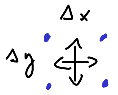
$$\text{We need only } R_{\gamma 00\alpha}: \quad R_{\alpha 00\mu} = \frac{1}{2}\big(\partial_0\partial_0 h^{TT}_{\alpha\mu} + \overset{=0}{\cancel{\partial_\mu\partial_\alpha h^{TT}_{00}}} - \overset{=0}{\cancel{\partial_\mu\partial_0 h^{TT}_{\alpha 0}}} - \overset{=0}{\cancel{\partial_\alpha\partial_0 h^{TT}_{\mu 0}}}\big) = \frac{1}{2}\partial_0\partial_0 h^{TT}_{\alpha\mu} \qquad h^{TT}_{\mu 0} = 0$$

ε_+ : set $h^{TT}_{12} = 0 = h^{TT}_{21}$

$$\boxed{\begin{aligned}\frac{d^2 s^1}{d\tau^2} &= \frac{1}{2}\frac{\partial^2 h^{TT1}{}_1}{\cancel{c^2}\partial t^2}\cancel{c^2} s^1 \\ \frac{d^2 s^2}{d\tau^2} &= \frac{1}{2}\frac{\partial^2 h^{TT2}{}_2}{\partial t^2} s^2\end{aligned}}$$

$\ddot{s}^1 \to \Delta x$

$\ddot{s}^2 \to \Delta y$



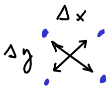
(+) mode

$\varepsilon_\times$: set $h^{TT}_{11} = 0 = h^{TT}_{22}$

$$\boxed{\begin{aligned}\frac{d^2 s^1}{d\tau^2} &= \frac{1}{2}\frac{\partial^2 h^{TT1}{}_2}{\partial t^2} s^2 \\ \frac{d^2 s^2}{d\tau^2} &= \frac{1}{2}\frac{\partial^2 h^{TT1}{}_2}{\partial t^2} s^1\end{aligned}}$$

$\ddot{s}^1 \to \Delta y$

$\ddot{s}^2 \to \Delta x$



(x) mode

$\Rightarrow$ 2 linear polarization modes

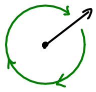
$\Rightarrow$ right- and left-handed circular polarized modes
$$\begin{cases} h_R = \frac{1}{\sqrt{2}}(\varepsilon_{11} + i\,\varepsilon_{12}) \\ h_L = \frac{1}{\sqrt{2}}(\varepsilon_{11} - i\,\varepsilon_{12}) \end{cases}$$
(particles move in little epicicles)

- Antennas are L shaped to see the quadrupole of GW (characteristic signature!)

Waves and associated particles

- Particle/wave duality : particles are associated to a field
- Spin → defined by the transformation properties of the field under spatial rotations

- Electro-Magnetic wave → Photons
  - wave propagates at speed of light => massless particle
  - 2 polarization states : vector in x-y plane => elicity 1 = spin 1



each polarization mode is invariant under a 2π rotation

↑ "in 1 round it spins once"

=> photon = massless spin-1 particle

- Neutrino : massive spin $-\frac{1}{2}$ particle , invariance under 4π rotation

↑ in 1 round it "spins" half a way backward

- Gravitational wave → Graviton
  - wave propagates at speed of light => massless particle
  - 2 polarization states : Tensor in x-y plane => elicity 2 = spin 2

each polarization mode is invariant under a π rotation

↑ "in 1 round it spins twice"

=> Graviton = massless spin-2 particle

Quantum gravity theory

- theory of spin-2 gravitons
- linear theory => Linearized Einstein tensor => Lagrangian
- Lagrangian $\mathcal{L} = \frac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\gamma})(\partial_\rho h^{\mu}{}_{\gamma}) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\gamma})(\partial_\nu h_{\rho\gamma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right]$
- interpret it as a real physical field propagating in a Minkowski space
  not a perturbation to a dynamical metric (!)
- assume field to couple to its own energy-momentum tensor and the one of matter
- couplings → rise of higher order terms → repetition of procedure → infinite series
  → recover non-linearity of the theory

Generation of GW: summary

- Link metric perturbation ⇔ source

$\Box \gamma^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}$ the source is $T^{\mu\nu}$ ⇒ use inhomogeneous solution

consider localized far away non relativistic source

Hilbert gauge only

$$\gamma^{\mu\nu}(x^\alpha) = \frac{4G}{c^4}\int \frac{T^{\mu\nu}(x^0 - |\bar{x}-\bar{x}'|, \bar{x}')}{|\bar{x}-\bar{x}'|} d^3x'$$

$|\bar{x}-\bar{x}'| \approx |\bar{x}| = r$

$ct_r = x^0 - |\bar{x}-\bar{x}'| \simeq c\left(t - \frac{r}{c}\right)$

$$\simeq \frac{4G}{c^4 r}\int T^{\mu\nu}(t - r/c, \bar{x}') d^3x'$$

$\int T^{\ell k} d^3x = \frac{1}{2c^2}\partial_t^2 \int T^{00} x^\ell x^k d^3x$

energy-mom. cons.

$$\begin{cases} \partial_\nu T^{\mu\nu} = 0 \\ \int \partial_j (T^{j0} x^\ell x^k) d^3x = 0 \end{cases}$$

Gauss theorem

$$= \frac{2G}{c^6 r}\partial_t^2 \underbrace{\int T^{00} x^\ell x^k d^3x}_{\equiv I^{\ell k}(t_r)}$$

$T^{00} \simeq \rho c^2$ (slow velocities)

⇒ GW emission if $\partial_t^2 I^{\ell k} \neq 0$! "change in shape of asymmetric source"

- Energy carried by a GW

$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$ need at least 2° order in h (at 1° order $\partial_\nu T^{\mu\nu} = 0$ ⇒ no energy transfer to the field)

$G^{(1)}_{\mu\nu}(\eta + h^{(2)}) + \underbrace{G^{(2)}_{\mu\nu}(\eta + h^{(1)})}_{} = 0$ $G^{(1)}_{\mu\nu}(\eta + h^{(2)}) = \frac{8\pi G}{c^4} t_{\mu\nu}$

$t_{\mu\nu} \equiv -\frac{c^2}{8\pi G} G^{(2)}_{\mu\nu}(\eta + h^{(1)})$

$E = \int_\Sigma t_{00} d^3x$ total energy on a surface of constant time Σ

$\Delta E = \int_S t_{0\mu} n^\mu d^3x\, dt$ total energy radiated through to infinity

Generation of GW

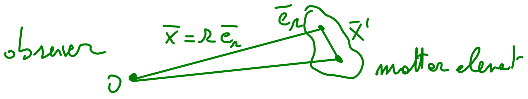
- There is a vacuum solution, GW:

$$\Box \gamma^{\mu\nu} = 0, \qquad \gamma_{\mu\nu} = \mathrm{Re}\left(\varepsilon_{\mu\nu}\, e^{ik_\alpha x^\alpha}\right), \qquad (\varepsilon_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{TT gauge}$$

- To look at the source of the perturbation, we need the inhomogeneous (linearized) eq:

$$\Box \gamma^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \qquad \gamma^{\mu\nu}(x^\alpha) = \frac{4G}{c^4}\int \frac{T^{\mu\nu}(x^0 - |\bar{x}-\bar{x}'|, \bar{x}')}{|\bar{x}-\bar{x}'|}\, d^3x' \qquad \text{Hilbert gauge only}$$

- Approximations:



- changes with velocities $\ll c$
- sources are far away (i.e. source small compared to its distance from observer) $|\bar{x} - \underset{\sim}{\bar{x}'}| \approx |\bar{x}| = r$

Approximate retarded time, t_r, as:

$$c t_r = x^0 - |\bar{x}-\bar{x}'| = x^0 - \sqrt{(\bar{x}-\bar{x}')^2} = x^0 - \sqrt{\bar{x}^2 + \bar{x}'^2 - 2\bar{x}\bar{x}'} = x^0 - r\sqrt{1 - 2\frac{\bar{x}\bar{x}'}{r^2}} \simeq x^0 - r\left(1 - \frac{\overset{\bar{x}=r\bar{e}}{\bar{x}}\bar{x}'}{r^2}\right)$$

$$= x^0 - r + \bar{x}'\bar{e}_r$$

$$= c\left(t - \frac{r + \bar{x}'\bar{e}_r}{c}\right)$$

neglect directional dependence of retarded time

$$\simeq c\left(t - \frac{r}{c}\right)$$

$$\gamma^{\mu\nu}(x^\alpha) \simeq \frac{4G}{c^4 r}\int T^{\mu\nu}(\underbrace{t - r/c}_{t_r}, \bar{x}')\, d^3x'$$

work out integral

- Express $\int T^{\mu\nu} d^3x'$ as an integral of density ρ $\qquad (\mu, \nu, .. = 0,1,2,3 \quad i,j,k,.. = 1,2,3)$

1) Exploit energy-momentum conservation $\qquad \partial_\nu T^{\mu\nu} = 0$

$$0 = \int_V x^k \underbrace{\partial_\nu T^{\mu\nu}}_{=0} d^3x' = \int_V x^k \partial_0 T^{0\mu} d^3x' + \underbrace{\int_V x^k \partial_i T^{i\mu} d^3x'}_{\text{by part}} = \frac{1}{c}\partial_t \int_V x^k T^{0\mu} d^3x' + \underbrace{x^k T^{i\mu}\Big|}_{=0} - \int_V \underbrace{\partial_i x^k}_{\delta^k_i} T^{i\mu} d^3x'$$

source completely within the volume

$$\Rightarrow \boxed{\int_V T^{k\mu} d^3x' = \frac{1}{c}\partial_t \int_V x^k T^{0\mu} d^3x'} \quad (*)$$

2) Exploit Gauss theorem $\qquad \int_V \partial_j (T^{j0} x^\ell x^k)\, d^3x = 0 \qquad$ (=0 if field disappears on the surface)

$$\int_V \partial_\nu (T^{\nu 0} x^\ell x^k) - \partial_0 (T^{00} x^\ell x^k)\, d^3x = 0$$

$$\frac{1}{c}\partial_t \int T^{00} x^\ell x^k d^3x = \int \partial_\nu (T^{\nu 0} x^\ell x^k)\, d^3x = \int \Big[\underbrace{\partial_\nu T^{\nu 0}}_{\substack{\text{conservation} \\ =0}} (x^\ell x^k) + T^{\nu 0} \underbrace{\partial_\nu (x^\ell x^k)}_{\delta^\ell_\nu x^k + x^\ell \delta^k_\nu}\Big] d^3x = \int (T^{\ell 0} x^k + T^{k0} x^\ell)\, d^3x$$

$$\partial_t \downarrow$$

$$\boxed{\frac{1}{c}\partial_t^2 \int T^{00} x^\ell x^k d^3x} = \partial_t \int (\underbrace{T^{\ell 0} x^k}_{(*)} + \underbrace{T^{k0} x^\ell}_{(*)})\, d^3x = c\int (T^{\ell k} + T^{k\ell})\, d^3x = \boxed{2c\int T^{\ell k} d^3x} \quad (**)$$

$$\boxed{\gamma^{\ell k}(x^\alpha)} \simeq \frac{4G}{c^4 r} \overbrace{\int T^{\ell k}(t - r/c, \bar{x}')\, d^3x'}^{(**)} = \frac{\overset{2}{\cancel{4}}G}{c^4 r} \frac{1}{\cancel{2}c^2} \partial_t^2 \int T^{00} x'^{\ell} x'^{k}\, d^3x' = \boxed{\frac{2G}{c^6 r} \partial_t^2 I^{\ell k}} \quad \propto c^{-4} \text{ small!!}$$

quadruple momentum tensor: $I^{\ell k}(t_r) \equiv \int T^{00}(t - r/c, \bar{x}')\, x^\ell x^k\, d^3x \simeq c^2 \int \rho\, x^\ell x^k\, d^3x$

(characterizes the source)

$T^{00} \simeq \rho c^2$ (slow velocities)

$\Rightarrow$ GW emission if $\partial_t^2 I^{\ell k} \neq 0$! i.e. $I^{\ell m} \neq 0$ (spherical sources do not emit GW)

i.e. static sources do not emit GW

- Exploit Hilbert gauge condition to derive the remaining components : $\partial_\nu \gamma^{\mu\nu} = 0$ $\quad \partial_0 \gamma^{\mu 0} + \partial_i \gamma^{\mu i} = 0$

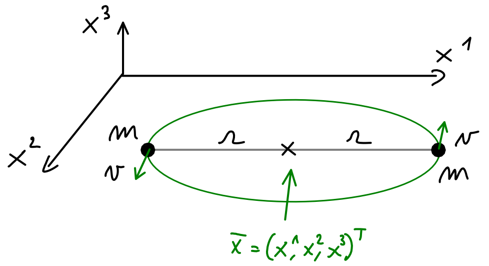
– <u>Why emission $\leftrightarrow$ quadruple momentum?</u>

- <u>monopole</u>: $I = \int T^{00}\, d^3x$ "mass"
- <u>dipole</u>: $I^k = \int T^{00} x^k\, d^3x$ barycenter of energy density
  ↳ for an isolated system:
  - electromagnetic radiation $\Rightarrow$ dipole momentum of charge density can change
  - gravitational " $\Rightarrow$ " " can not vary because of momentum conservation

    $\Rightarrow$ no GW signal
- <u>quadrupole</u>: "change of shape" can happen $\Rightarrow$ GW signal

Example: binary system

2 stars ("standard", neutron stars, ..), 1 star + 1 Black hole, 2 black holes



for simplicity: $m_1 = m = m_2$

circular orbit

Newtonian gravity for their motion

Energy loss via GW negligible

→ This approx is fine when r is "large"

- Tangential velocity v: $\frac{Gm^2}{(2r)^2} = \frac{mv^2}{r}$ (potential = centrifugal) $\Rightarrow v = \left(\frac{Gm}{4r}\right)^{1/2}$
- Orbital period: $T = \frac{2\pi r}{v}$ → angular frequency of orbit: $\Omega \equiv \frac{2\pi}{T} = \left(\frac{Gm}{4r^3}\right)^{1/2}$
- Positions: object A: $x^1_A = r\cos(\Omega t)$ $x^2_A = r\sin(\Omega t)$ $x^3_A = 0$

  " B: $x^1_B = -r\cos(\Omega t)$ $x^2_B = -r\sin(\Omega t)$ $x^3_B = 0$

- Quadrupole momentum: $I^{\ell k}(t_r) \simeq \int T^{00} x^\ell x^k d^3x$

$m = \int \rho_i \, d^3x$ $\rho_i = m\,\delta_D(\bar{x} - \bar{x}_i) = m\,\delta_D(x^1 - x^1_i)\,\delta_D(x^2 - x^2_i)\,\delta_D(x^3)$ $i = A, B$ $\rho = \rho_A + \rho_B$

$$T^{00} \simeq c^2\rho = mc^2\delta_D(x^3)\left[\delta_D(x^1 - r\cos\Omega t)\,\delta_D(x^2 - r\sin\Omega t) + \delta_D(x^1 + r\cos\Omega t)\,\delta_D(x^2 + r\sin\Omega t)\right]$$

$I^{\ell k}(t_r) = \int T^{00} x^\ell x^k d^3x$ ↪ the δ_D "select" ↴

$$I^{11} = \int T^{00}(x^1)^2 d^3x \simeq mc^2 \int \delta_D(x^3)[\cdots\cdots](x^1)^2 d^3x$$

$x^1 = r\cos(\Omega t)$ (A)

$x^1 = -r\cos(\Omega t)$ (B)

$$= mc^2 2 r^2 \cos^2(\Omega t) = mc^2 r^2 [1 + \cos(2\Omega t)]$$

$$I^{12} = I^{21} \simeq mc^2 2 \left(r\cos(\Omega t)\, r\sin(\Omega t)\right) = mc^2 r^2 \sin(2\Omega t)$$

$$I^{22} = mc^2 r^2 [1 - \cos(2\Omega t)]$$

$$I^{i3} = I^{3i} = 0$$

↓ 2Ω frequency!

$$\Rightarrow \boxed{\gamma^{\ell m}(x^\alpha)} = \frac{2G}{c^6 r}\partial_t^2 I^{\ell k} \simeq \boxed{\frac{2G}{c^{\underset{4}{\cancel{6}}} r} m c^{\overset{1}{\cancel{2}}} r^2 4\Omega^2 \begin{pmatrix} -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ -\sin(2\Omega t) & \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

→ other components of $\gamma^{\mu\nu}$ derived by imposing the harmonic gauge

$(\Box x^\mu = 0 \Rightarrow \partial_\mu \gamma^{\nu}{}_{\mu} = 0)$

Energy carried by gravitational waves

- T_{0i} expresses the energy density flux (like Pointing vector in electrodynamics)
- We want to identify the one of the gravitational field
- We linearized the equations (1° order in h) but to do that we need at least 2° order terms
- in fact, in the linear theory we have $\partial_\mu T^{\mu\nu}=0$ not $\nabla_\mu T^{\mu\nu}=0$

  1) free particles move along straight trajectories like in flat space (inconsistent!)

  2) we can not estimate the amount of energy that goes in the grav. field

recall $\nabla_\nu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\,T^{\mu\nu}) + \Gamma^\mu_{\nu\gamma}T^{\gamma\nu} = 0 \Rightarrow \partial_\nu T^{\mu\nu} = \underbrace{-\sqrt{-g}\,\Gamma^\mu_{\nu\gamma}T^{\gamma\nu} - T^{\mu\nu}\partial_\nu\sqrt{-g}}_{\text{this "got lost" in the linearization}}$

- **Keep terms up to 2° order**

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$$

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

$R^{(0)}_{\mu\nu} = 0$ flat background

$R^{(1)}_{\mu\nu}$ are of order $h^{(1)}_{\mu\nu}$

$h^{(2)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are of order $(h^{(1)}_{\mu\nu})^2$

- $R_{\mu\nu} = 0$ Einstein field eq. in vacuum
- $R^{(1)}_{\mu\nu}[h^{(1)}] = 0$ (1) 1° order vacuum equation $\Rightarrow$ defines $h^{(1)}$ up to gauge transformations
- $R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}] = 0$ 2° order vacuum equation

  $R^{(2)}_{\mu\nu}$: quadratic part of the Ricci tensor up to 2° order (wald 7.153)

  $R^{(1)}_{\mu\nu}$: linear in the perturbations (the one we derived) applied to second order perturbations $h^{(2)}_{\mu\nu}$

No cross terms (eg. $R^{(2)}[h^{(2)}]$) because they would be of higher order, neglect

- $R_{\mu\nu} = 0$ is equivalent to $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 0$

$$\Rightarrow \underbrace{R^{(1)}_{\mu\nu}[h^{(2)}] - \frac{1}{2}\overset{*}{\eta}^{\alpha\beta}R^{(1)}_{\alpha\beta}[h^{(2)}]\,\eta_{\mu\nu}}_{\equiv G^{(1)}_{\mu\nu}[h^{(2)}]} + \underbrace{R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\overset{*}{\eta}^{\alpha\beta}R^{(2)}_{\alpha\beta}[h^{(1)}]\,\eta_{\mu\nu}}_{\equiv G^{(2)}_{\mu\nu}[h^{(1)}]} = 0$$

$$\boxed{G^{(1)}_{\mu\nu}[h^{(2)}] = \frac{8\pi G}{c^4}t_{\mu\nu} \qquad t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}G^{(2)}_{\mu\nu}[h^{(1)}]}$$

interpret $t_{\mu\nu}$ as "energy-momentum tensor" of the "vacuum": GW

* Note: these terms $h^{(1)\alpha\beta}R^{(1)}_{\alpha\beta}[h^{(1)}] = 0$ because of (1)

- Careful!

$G^{(1)}_{\mu\nu}[h^{(2)}]$: not the full 2° order Einstein tensor: we moved part of it on the right side to "create" $t_{\mu\nu}$

$t_{\mu\nu}$:
- symmetric
- quadratic in $h_{\mu\nu}$
- conserved $\partial_\mu t^{\mu\nu} = 0$ thanks to Bianchi identity $\partial_\mu G^{\mu\nu} = 0$

} good! :)

- not a tensor
- not invariant under gauge transf. => average over wavelengths

} bad! :(

Careful here:

- Not possible to measure gravitational energy-momentum purely local

⇒ Average over several wavelength to capture the physical curvature in a small region $\overset{\text{average}}{\overset{\downarrow}{\langle\cdots\rangle}}$

$\langle f\rangle \equiv \frac{1}{L}\int_0^L f(\ell)\,d\ell$ average of f

- Get a gauge invariant measure
- Note: all terms with derivatives vanish $\langle\partial_\mu(x)\rangle = 0$

integrating by part the averaging in general one has: $\langle A(\partial_\mu B)\rangle = -\langle(\partial_\mu A)B\rangle$ ⊛

in fact: $\langle A(\partial_\mu B)\rangle \equiv \frac{1}{L}\int_0^L A(\partial_\mu B)\,d\lambda = \frac{1}{L}\left[\overset{=0}{A\cdot B\Big|_0^L} - \int_0^L(\partial_\mu A)B\,d\lambda\right]$

- Use expression $t_{\mu\nu}(\overset{\downarrow}{R}{}^{(2)}_{\mu\nu}[\overset{\downarrow}{h}{}^{(1)}])$ averaged ⟩

here in TT gauge for simplicity (not necessary)

$\langle R^{(2)TT}_{\mu\nu}[h^{(1)}]\rangle = -\frac{1}{4}\langle(\partial_\mu h^{TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{TT}) + \cancel{2\eta^{\rho\lambda}(\underbrace{\Box h^{TT}_{\rho\nu}}_{=0})}\,h^{TT}_{\lambda\mu}\rangle$ 1° order vacuum eq. of motion $\Box h^{TT}_{\rho\nu} =$

integrate by part and use trick ⊛ you get a term that drops: $\langle\eta^{\mu\nu}R^{(2)TT}_{\mu\nu}\rangle = 0$

$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\overset{\downarrow}{G}{}^{(2)}_{\mu\nu}[h^{(1)}]$

|=> $\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\langle(\partial_\mu \underset{\underset{ij\,*}{\sim}}{h^{TT}_{\rho\sigma}})(\partial_\nu \overset{\overset{ij\,*}{\sim}}{h^{\rho\sigma}_{TT}})\rangle}$ remember that in TT gauge $h^{TT}_{0\nu} = 0$ => $\rho\sigma \to ij$ $i,j = 1,2,3$

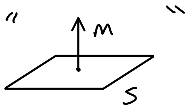
- but.... : there are quantities that are invariant for some gauge transf.

$E = \int_\Sigma t_{00} \, d^3x$ total energy on a surface at constant time Σ

$\Delta E = \int_S t_{0\mu} \, n^\mu \, d^3x \, dt$ total energy radiated through to infinity

S = time-like surface

n^μ = space-like vector orthogonal to S



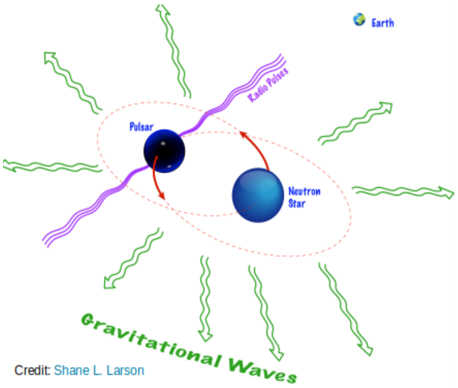
(r at infinity between 2 time interval Δt)

$\Delta E = \int P \, dt$ $\quad P = \frac{2^7}{5} G m^2 r^4 \Omega^6$ $\quad P = \frac{2}{5} \frac{G^4 m^5}{r^5}$

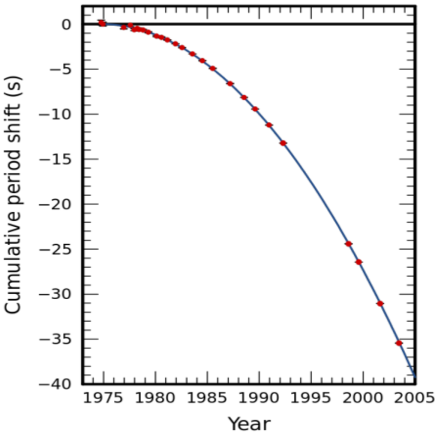
- 1st evidence of energy loss via gravitational wave emission

PSR1913+16 (Star + pulsar) Hulse & Taylor (1974) (Nobel 1993)

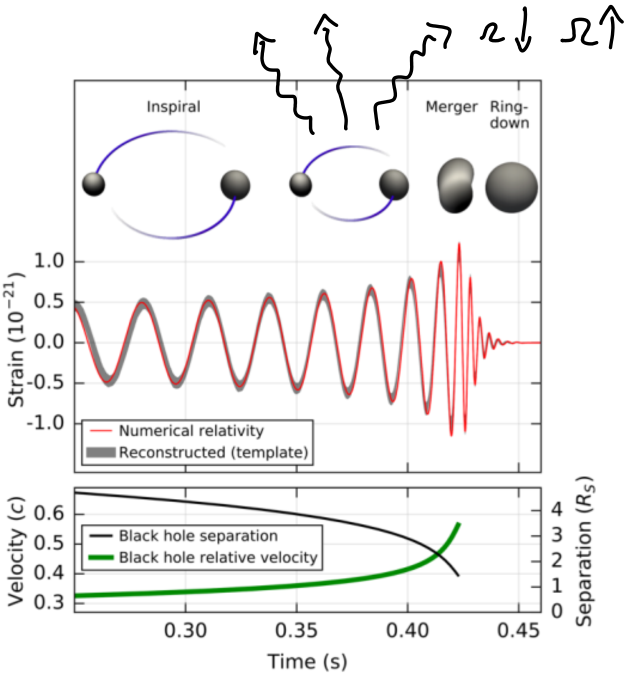
↑ accurate clock

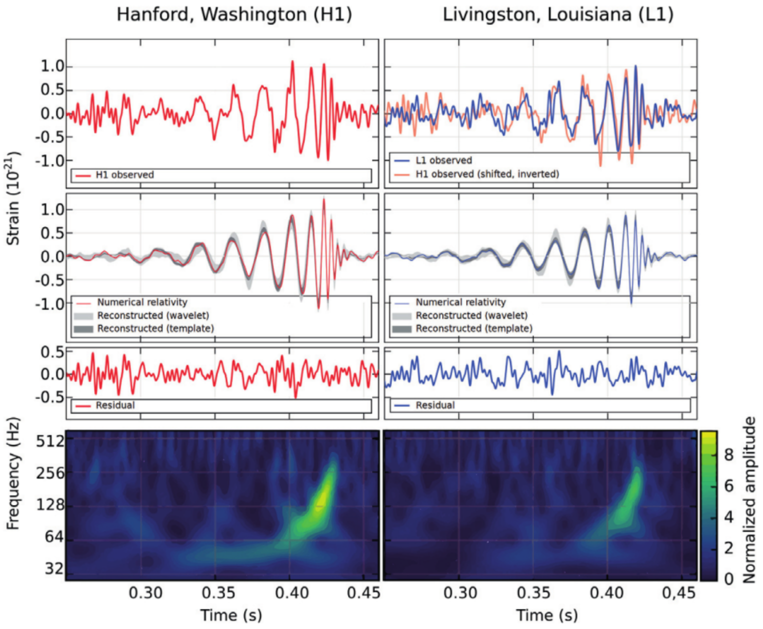


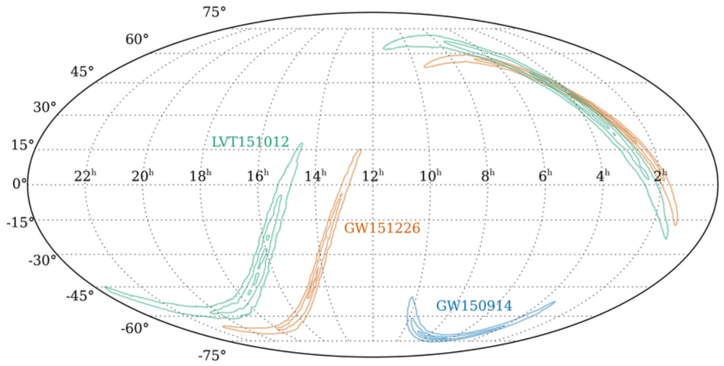
Credit: Shane L. Larson



- **1st detection: GW150914** BH-BH merger







Distance: 410^{+160}_{-180} Mpc (luminosity distance)

$m_1 = 35^{+5}_{-3}\, M_\odot$ $m_2 = 30^{+3}_{-4}\, M_\odot$ $\longrightarrow$ $62^{+4}_{-3}\, M_\odot$

Energy release: $3 \pm 0{,}5\, M_\odot c^2 = 1{,}8 \cdot 10^{54}$ erg

- Lensed gravitational wave sources ....

Spherically symmetric systems

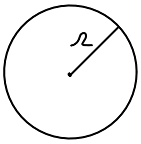
- Spherically symmetric metric

– spherical symmetry: $\bar{x} = (r, \theta, \phi)^T$ 3D polar coordinates

– flat space, observer at rest in the center of symmetry:

$dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)$ 3D part of interval

$2\pi r$ circonference



$4\pi r^2$ surface of sphere

$dA = r^2 \sin\theta\, d\theta\, d\phi$ surface element of sphere

– Curved space time is different! distinct "radiouses" : R, r

$$dR^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)$$

↑ radious for a fixed R (r = in "angular" coordinates)

↑ " " " θ, ϕ ($dR = f(r)dr$ $f(r)$ some function)

sphere has radious R but $\begin{cases} 2\pi r = \text{circonference} \\ 4\pi r^2 = \text{surface} \end{cases}$

– 4-interval:

$$\boxed{ds^2} = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$= g_{00}c^2dt^2 + g_{rr}dr^2 + \overbrace{g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2}^{r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2) \equiv r^2 d\Omega^2} \rightarrow g_{\theta\theta} = r^2 \quad g_{\phi\phi} = r^2\sin(\theta)$$

$$+ 2\underset{(1)}{\cancel{g_{r\theta}\, dr\, d\theta}} + 2\underset{(1)}{\cancel{g_{r\phi}\, dr\, d\phi}} + 2\underset{(1)}{\cancel{g_{\theta\phi}\, d\theta\, d\phi}} + \underset{(3)}{\cancel{g_{0r}\, c\, dt\, dr}} + \underset{(2)}{\cancel{g_{0\theta}\, c\, dt\, d\theta}} + \underset{(2)}{\cancel{g_{0\phi}\, c\, dt\, d\phi}}$$

$$= \boxed{g_{00}c^2dt^2 + g_{rr}dr^2 + r^2d\Omega^2}$$

(1) r, θ, ϕ ortogonal to each other i.e. if I move along θ or ϕ: $r = \text{const}$ and so on ... $\Rightarrow \begin{cases} g_{r\theta} = \bar{e}_r\bar{e}_\theta = 0 \\ g_{r\phi} = \bar{e}_r\bar{e}_\phi = 0 \\ g_{\theta\phi} = \bar{e}_\theta\bar{e}_\phi = 0 \end{cases}$

(2) $c\,dt(g_{0\theta}\, d\theta + g_{0\phi}\, d\phi) = c\,dt(V_i\, dx^i)$ $i = \theta, \phi$ $\bar{V} = (g_{0\theta}, g_{0\phi})^T$ $d\bar{x} = (d\theta, d\phi)^T$

$\bar{V}$ would define a priviledge direction (braking isotropy) $\Rightarrow$ $\underline{\bar{V} \overset{!}{=} 0}$

(3) symmetry under time inversion ($t \to -t$) $\Rightarrow$ $\boxed{g_{0r} = 0}$ otherwise $ds^2(-t) \neq ds^2(t)$

- Assume static metric : $\boxed{\partial_0 g = 0}$ => (time like Killing vector) energy conservation

- For convenience use: $g_{00} \equiv -e^{2A(r)}$ $g_{rr} \equiv e^{2B(r)}$ $e^{2A(r,\overset{!}{\cancel{t}})}$, $e^{2B(r,\overset{!}{\cancel{t}})}$

$$\boxed{ds^2} = g_{00}c^2dt^2 + g_{rr}dr^2 + r^2 d\Omega^2 = \boxed{-e^{2A(r)}c^2dt^2 + e^{2B(r)}dr^2 + r^2 d\Omega^2}$$

- "exp" to have a positive value
- e^2 for convenience (square)
- Far from the source => Minkowski metric $\lim_{r\to\infty} g_{00} = -1$, $\lim_{r\to\infty} g_{rr} = 1$ for $\lim_{r\to\infty} A(r) = 0$, $\lim_{r\to\infty} B(r) = 0$

↳ most general static spherical metric (rest frame at center of system)

- Use it for isolated spherically symmetric systems such as planets, stars, black-holes...

- Non rotating! Rotation gives you a priviledge direction
  => axial symmetry : Kerr metric!

- Proper radial distance $(dt = d\theta = d\varphi = 0)$: $l = \int_{r_1}^{r_2} e^{B(r)}\, dr$

  ↳ completely not related to surface of sphere $4\pi r^2$

Coupling metric with source: gravitatinal field equations

- We already have $g_{\theta\theta}=r^2$ $g_{\phi\phi}=r^2\sin(\theta)$ just because of symmetry arguments
- To find g_{rr}, g_{00} (i.e. $A(r), B(r)$) we need to solve Einstein equations

$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^3} T_{\mu\nu}}$ compute $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}R g_{\mu\nu}$ and $T_{\mu\nu}$ for this metric and a fluid

- Einstein tensor (non zero components)

$$\left[\begin{array}{l} G_{00} = \frac{1}{r^2} e^{2A} \frac{d}{dr}\left[r\left(1-e^{-2B}\right)\right] \\ G_{rr} = -\frac{1}{r^2} e^{2B}\left(1-e^{-2B}\right) + \frac{2}{r}\frac{dA}{dr} \\ G_{\theta\theta} = r^2 e^{-2B}\left(A'' + A'^2 + \frac{A'}{r} - A'B' - \frac{B'}{r}\right) \\ G_{\phi\phi} = G_{\theta\theta}\sin^2\theta \end{array}\right.$$

$\frac{d}{dr} =$ "$'$"

other components $= 0$

- Energy-Momentum tensor ("star" = self gravitating fluid)

1) observer at rest with the star $\Rightarrow$ $\bar{u} = (u^0, \overset{r}{0}, \overset{\theta}{0}, \overset{\phi}{0})^T$ $u^0 = \frac{du^0}{d\tau}$ $u^1 = \frac{dr}{d\tau}$ $u^2 = \frac{d\theta}{d\tau}$ $u^3 = \frac{d\phi}{d\tau}$

(matter) → $d\tau$

↳ $\underline{u^i} = 0$

↳ $\underline{u^0} = ce^{-A}$ because: $\begin{cases} u_\mu u^\mu = -c^2 \\ u_\mu u^\mu = g_{\mu\nu}u^\nu u^\mu = g_{00}u^0u^0 + 0 = -e^{2A}(u^0)^2 = -c^2 \end{cases}$

↳ $\underline{u_0} = -ce^A$ because $u_\mu u^\mu = u_0 c e^{-A} = -c$

2) star = self gravitating system (fluid): $T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u_\mu u_\nu + P g_{\mu\nu}$

$$\left[\begin{array}{l} T_{00} = \left(\rho+\frac{P}{c^2}\right)\overset{\downarrow}{u_0}\overset{\downarrow}{u_0} + P\overset{\downarrow}{g_{00}} = \left(\rho + \cancel{\frac{P}{c^2}}\right)c^2e^{2A} - \cancel{Pe^{2A}} = \rho c^2 e^{2A(r)} \quad (\to \rho c^2 \text{ for } r\to\infty) \\ T_{rr} = \left(\rho+\frac{P}{c^2}\right)\overset{=0}{u_r}u_r + P g_{rr} = p e^{2B(r)} \\ T_{\theta\theta} = p r^2 \quad \hookleftarrow g_{\theta\theta} = r^2 \\ T_{\phi\phi} = p r^2 \sin^2\theta = \sin^2\theta \, T_{\theta\theta} \quad \hookleftarrow g_{\phi\phi} = r^2\sin(\theta) \end{array}\right.$$

[all other components are zero: e.g. $T_{0r} = P \underset{=0}{g_{0r}} = 0$ $\Longrightarrow$ T is diagonal

- **Solve** $G_{00} = \frac{8\pi G}{c^4} T_{00}$ get g_{rr}

$$\frac{1}{\cancel{r^2}_{1}} \cancel{e^{2A(r)}} \frac{d}{dr}\left[\underline{r\left(1-g_{rr}^{-1}\right)}\right] = \frac{8\pi G}{c^{\cancel{4}\,2}} \rho \cancel{c^2}^{\,1} \cancel{e^{2A(r)}}\, r^2$$

integrate

$$r\left(1-g_{rr}^{-1}\right) = \frac{8\pi G}{c^2}\int \rho r^2 dr = \frac{2G}{c^2}\cdot \overbrace{\underbrace{4\pi \int_0^r \rho r'^2 dr'}_{\text{mass within } r}}^{\text{mass } m(r)}$$

[density]·[volume]

mass within r

$$m(r) \equiv \frac{1}{2}\frac{c^2}{G} r\left(1-\underline{\underline{g_{rr}^{-1}}}\right)$$

gravitational mass

$$\boxed{g_{rr} = \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}}$$

- **Solve** $G_{rr} = \frac{8\pi G}{c^4} T_{rr}$ get A i.e. $g_{00} = -e^{2A}$

$$-\frac{1}{r^2} g_{rr}\left(1-g_{rr}^{-1}\right) + \frac{2}{r}\frac{dA}{dr} = \frac{8\pi G}{c^4} p\, g_{rr}$$

$$\left(1-g_{rr}^{-1}\right) = m(r)\frac{2G}{c^2 r}$$

$$\frac{\cancel{2}^{\,1}}{\cancel{r}}\frac{dA}{dr} = \frac{1}{r^2} g_{rr}\, m(r) \frac{\cancel{2}^{\,1} G}{c^2 \cancel{r}} + \frac{\cancel{8}^{\,4}\pi G}{c^4} p\, g_{rr}\; r$$

$$\boxed{\frac{dA}{dr}} = \frac{G}{c^2} g_{rr}\left(\frac{m(r)}{r^2} + 4\pi p r\right) = \boxed{\frac{G}{c^2}\frac{\left(m(r) + 4\pi p r^3\right)}{r\left(r - \frac{2Gm(r)}{c^2}\right)}}$$

=> A depends on $m(r)$, i.e. on how matter is distributed

=> need to solve simultaneously for matter eq. of motion!

solve for A $\boxed{g_{00} = -e^{2A}}$

- **Because of the symmetries** $\boxed{g_{\theta\theta} = r^2}$ $\boxed{g_{\phi\phi} = r^2 \sin(\theta)}$, all other $g_{\mu\nu} = 0$

$$\Rightarrow \boxed{ds^2 = -e^{2A} c^2 dt^2 + \left(1-\frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2 \underbrace{\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right)}_{\equiv d\Omega^2}}$$

- Specify P and ρ of object to get A explicitely

eg. "star" = self gravitating system (isolated fluid in grav. field) => Hydrostatic equilib.

	Variables		Equations		
Gravity	g_{00}, g_{rr} $g_{\theta\theta}, g_{\phi\phi}, \dots$	$\longrightarrow$	2 eq. Einstein field eq. symmetries of system		
Hydrodynamics	$\rho, P, \bar{v}, \varepsilon$	$\longrightarrow$	1 eq $\nabla_\mu j^\mu = 0$	matter conservation	
			4 eq. $\nabla_\mu T^{\mu\nu} = 0$	energy "	(1)
			1 eq. $P = P(\rho, \cancel{T})$	eq. of state	(2)

(1) Energy conservation $\nabla_\mu T^{\mu\nu} = 0$ (∇_μ not ∂_μ, it contains gravity, no external force!)

e.g. $\nu = r$: $\overset{\downarrow \text{diagonal}}{\nabla_\mu T^{\mu r}} = 0$ => Relativistic Euler eq. ; using this metric: $(\rho + \frac{P}{c})\frac{dA(r)}{dr} = -\frac{dP}{dr}$ *

combine * with $\overset{\searrow}{\left(\frac{dA}{dr}\right)} = \frac{G}{c^2}\frac{(m(r) + 4\pi p r^3)}{r\left(r - \frac{2Gm(r)}{c^2}\right)}$ => $\boxed{\frac{dP}{dr} = \frac{G}{c^2}\frac{(m(r) + 4\pi p r^3)}{r\left(\frac{2Gm(r)}{c^2} - r\right)}\left(\rho + \frac{P}{c}\right)}$

Tolman-Oppenheimer-Volkoff equation

i.e. the Hydrostatic equilibrium eq. for a self gravitating system

(2) Equation of state:

- relates pressure to density
- depends on properties of matter
- eg. ideal gas $PV = NRT$
- often parameterized as $P(\rho) = P_0\left(\frac{\rho_0}{\rho}\right)^\gamma$ γ = politropic index
- valid for any fluids under adiabatic conditions

$\gamma = \frac{c_p}{c_v}$ $c_{p/v}$ = heat capacity at const. pressure / volume

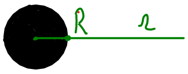
$\gamma = 5/3$ with dwarfs

$\gamma = 4/3$ other stars

Schwarzshild metric: exterior soution

- This was $g_{\mu\nu}$ in presence of matter $\rho \neq 0$ $p \neq 0$ (object interior)
- Now outside the object: exterior solution $r > R =$ size of object $\rho = 0 = p$

$$\Rightarrow \quad m(r) \overset{r>R}{=} 4\pi \int_0^R \rho r^2 dr \overset{\text{const}}{=} m \quad \text{object mass}$$



$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{from previous result}}}{\frac{dA}{dr}} = \frac{Gm}{r\left(r - \frac{2Gm}{c^2}\right)c^2}$$

$$A = \frac{Gm}{c^2} \int_0^r \left[r'\left(r' - \frac{2Gm}{c^2}\right)\right]^{-1} dr' = \frac{1}{2} \log\left(1 - 2\frac{Gm}{rc^2}\right)$$

$A = 0$ $r \to \infty$ to have the Minkowski metric

$$g_{00} = -e^{2A} = -\left(1 - \frac{2Gm}{rc^2}\right) \qquad r_s \equiv \frac{2Gm}{c^2} \quad \text{Schwarzschild radius}$$

$$\Rightarrow \quad \boxed{ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2} \qquad \underline{\text{Schwarzschild metric}}$$

$$\simeq -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 + \frac{r_s}{r}\right) dr^2 + r^2 d\Omega^2 \qquad (\text{for large } r)$$

- It agrees with the weak field limit we already derived, in fact $\frac{r_s}{r} = \frac{2Gm}{c^2 r} = \frac{2\phi}{c^2}$
- Killing vectors: $\partial_\alpha g_{\mu\nu} = 0$ for $\alpha = t, \phi$ $\Rightarrow$ energy and angular momentum conservation
  $P_0 = \text{const.}$ $P_\phi = \text{const}$

- Wierdnesses...

1) Swap between time- and space-like classification

g_{00} and g_{rr} change sign when crossing r_s

$\Rightarrow$ t, r change time- and space-like nature

2) Singularities for metric coefficients g_{00}, g_{rr}: $(\underset{\substack{\uparrow \\ \text{real}}}{r = 0})$ and $(\overset{\substack{\text{curvature is finite} \\ \downarrow}}{\underset{\substack{\uparrow \\ \text{just a coordinate singularity}}}{r = r_s}})$

- Sun: $r_s = \frac{2GM_\odot}{c^2} = 1.5\,\text{Km} \ll R_\odot$ $\Rightarrow$ r_s is in the interior where this solution is not valid
- Black holes: objects for which $r_s > R$ $\Rightarrow$ r_s is exposed, more later

- **Birkhoff's theorem**

By dropping assumption on static metric $\cancel{\partial_0 g_{\mu\nu}} = 0$

$$ds^2 = -e^{2A(r,\overset{\downarrow}{t})}c^2dt^2 + e^{2B(r,\overset{\downarrow}{t})}dr^2 + r^2 d\Omega^2$$

Birkhoff's theorem : "the metric outside a general spherically symmetric matter distribution is the Schwarzschild metric"

(the opposite is not guaranteed)

i.e. is the unique solution in vacuum given spherical symmetry
i.e. outside a pulsating spherical star you have the Schwarzschild metric
i.e. no production of gravitational waves (because this is the metric)

- **Interpreting the spatial interval**

– set $\theta = \frac{\pi}{2}$, $t=0$ (on the equator, simultaneous)

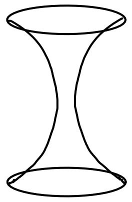
$$dl^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad \text{a plane} \qquad d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2 = d\theta^2 + d\phi^2$$

– looking at 3D euclidean cilindrical coord. (r, ϕ, z)

$$dl^2 = dz^2 + dr^2 + r^2 d\varphi^2 = \left(\frac{dz}{dr}\right)^2 dr^2 + dr^2 + r^2 d\varphi^2 = (1 + z'^2)\, dr^2 + r^2 d\varphi^2$$

– interpret by imposing

$$\left(1 - \frac{r_s}{r}\right)^{-1} \overset{!}{=} (1 + z'^2) \qquad z'^2 = -1 + \frac{r}{r - r_s} = \frac{r_s - \cancel{r} + \cancel{r}}{r - r_s} \qquad z' = \frac{dz}{dr} = \left(\frac{r_s}{r - r_s}\right)^{1/2} \qquad z = \pm\sqrt{8r_s(r - 2r_s)}$$



geometry on the equatorial plane (spatial slice) = rotational paraboloid

This is so that: $2r\pi =$ circonference circle
$4\pi r^2 =$ surface of sphere
so it is not just a physical distance from the center

Remember: $dR^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)$

$$\underbrace{\left(1 - \frac{r_s}{r}\right)^{-1} dr^2}_{dR^2} + r^2 d\Omega^2$$

Particles in a Schwarzshild spacetime: summary

- $S = -mc\int\sqrt{-\langle\bar{u},\bar{u}\rangle}\,d\tau$ $\rightarrow$ $\delta\hat{S} = \delta\int\langle\bar{u},\bar{u}\rangle\,d\tau = 0$ $\quad \hat{\mathcal{L}} = \langle\bar{u},\bar{u}\rangle = -c^2\varepsilon \begin{cases}\varepsilon = 1\\ \varepsilon = 0\end{cases}$ $\quad u^\nu = \frac{dx^\nu}{d\lambda}$

action $\qquad$ same eq. of motion $\qquad$ effective Lagrangian

- $\boxed{t, \phi = \text{cyclic coordinates}}$ <=> $\boxed{\bar{\partial}_0, \bar{\partial}_\phi = \text{Killing vectors}}$ <=> $\boxed{\text{time symmetry}, \phi \text{ symmetry}}$

$\frac{\partial\hat{\mathcal{L}}}{\partial\dot{x}^\mu} = 0 \quad \mu = t, \phi$

(2)

$\Updownarrow$

for $\mu = 0, \phi$ $\quad \boxed{\partial_\mu g_{\alpha\beta} = 0}$ (1) $\quad$ because the "good" coord system

$\boxed{\mathcal{L}_{V_{(\mu)}} g_{\alpha\beta} = 0}$ $\quad$ with $V_{(\mu)}(x^\nu) \equiv \bar{\partial}_\mu$ Killing vector field

$P_\phi = \text{const} \qquad P_0 = \text{const.}$

$\underline{\dot{\phi}} = \frac{L}{r^2} \qquad c\underline{\dot{t}} = -E\left(1 - \frac{r_s}{r}\right)^{-1}$ $\qquad$ eg. $\dot{\phi} = \frac{\partial\phi}{\partial\tau} = p^\phi$ per unit mass

- Effective potential $\qquad \hat{\mathcal{L}} = -c^2\varepsilon = -\left(1 - \frac{r_s}{r}\right)c^2\overset{\downarrow}{\dot{t}}{}^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\underset{\approx}{\dot{r}^2} + r^2\overset{\downarrow}{\dot{\phi}}{}^2$

- eq. of motion, radial component $\qquad \dot{r}^2 - E^2 + \underbrace{\left(1 - \frac{r_s}{r}\right)\left(\frac{L^2}{r^2} + c^2\varepsilon\right)}_{V(r)} = 0 \qquad \dot{r}^2 + V(r) = E^2$

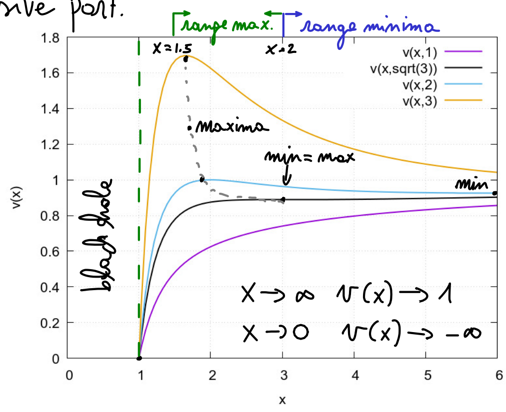
$\downarrow$

Effective potential energy

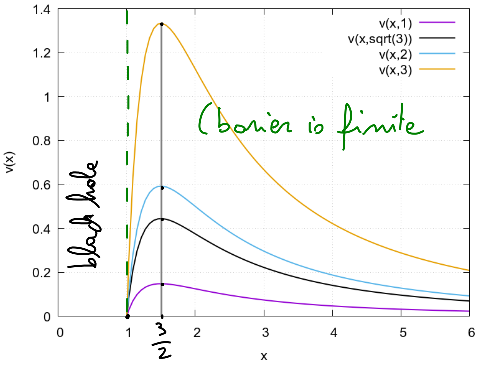
Kinetic energy

- Dimensionless effective potential $\qquad \boxed{v(x) \equiv \frac{V(x)}{c^2} = \left(1 - \frac{1}{x}\right)\left(\frac{\lambda^2}{x^2} + \varepsilon\right)} \qquad x \equiv \frac{r}{r_s} \qquad \lambda^2 = \frac{r_s L^2}{c^2}$

massive part.



massless part.



- Effective gravitational potential $\qquad \varphi(r) = -Gm\left(\frac{1}{r} + \underline{\frac{L^2}{c^2 r^3}}\right)$

Particles in a Schwarzshild spacetime

- Geodesic lines: use (a) geodesic eq. or (b) Euler-Lagrange eq.

  $\overset{||}{\text{(eq. of motion free particle)}}$

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda^2}\frac{dx^\beta}{d\lambda^2} \qquad \frac{d}{d\lambda}\frac{\delta\mathcal{L}}{\delta\dot{x}^\mu}-\frac{\delta\mathcal{L}}{\delta x^\mu}=0$$

- Action of a free particle → effective lagrangian

- $S=-mc\int\sqrt{-g_{\mu\nu}u^\mu u^\nu}\,d\tau=-mc\int\sqrt{-\langle\bar{u},\bar{u}\rangle}\,d\tau$ ← action

- $\partial S=+mc\int\frac{1}{2}\frac{\partial\langle\bar{u},\bar{u}\rangle}{\sqrt{-\langle\bar{u},\bar{u}\rangle}}d\tau\overset{!}{=}0 \rightarrow \partial\hat{S}=\partial\int\langle\bar{u},\bar{u}\rangle d\tau=0$ ← least action principle

  const ⇒ rescale affine parameter

  same eq. of motion

- $\hat{\mathcal{L}}=\langle\bar{u},\bar{u}\rangle$ ← effective lagrangian

  $=g_{\mu\nu}u^\mu u^\nu$

  $=-\left(1-\frac{r_S}{r}\right)c^2\dot{t}^2+\left(1-\frac{r_S}{r}\right)^{-1}\dot{r}^2+r^2\left(\dot{\theta}^2+\sin^2(\theta)\dot{\phi}^2\right)$ $\qquad u^\mu=\frac{dx^\mu}{d\tau}\equiv\dot{x}^\mu$ ← $d\lambda$ for photons!

  integral and differentiations with respect to τ, not t !

- $\hat{\mathcal{L}}=g_{\mu\nu}u^\mu u^\nu=-c^2\varepsilon$ $\begin{cases}\varepsilon=1 & \text{for mass particles} \\ \varepsilon=0 & \text{" massless particles}\end{cases}$ (because $\langle\bar{u},\bar{u}\rangle=-c^2$) (because $\langle\bar{u},\bar{u}\rangle=0$, $u^\mu=\frac{dx^\mu}{d\lambda}$ not τ)

- Set $\theta\overset{!}{=}\frac{\pi}{2}$, $\dot{\theta}\overset{!}{=}0$ for initial motion (just rotate the axis)

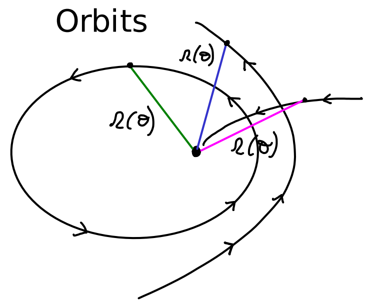
$$\frac{d}{d\tau}\frac{\hat{\mathcal{L}}}{\delta\dot{\theta}}-\frac{\delta\hat{\mathcal{L}}}{\delta\theta}=\frac{d}{d\tau}(2r^2\dot{\theta})-r^2 2\sin(\theta)\underbrace{\cancel{\cos(\theta)}}_{=0}\dot{\phi}^2=0 \Rightarrow r^2\dot{\theta}=\text{const}=0$$

i.e. $\dot{\theta}$ remains 0, θ remains $\pi/2$

$$\Rightarrow \boxed{\hat{\mathcal{L}}=-\left(1-\frac{r_S}{r}\right)c^2\dot{t}^2+\left(1-\frac{r_S}{r}\right)^{-1}\dot{r}^2+r^2\dot{\phi}^2}=-c^2\varepsilon$$

effective lagrangian

- Immediate results:
  1) identify cyclic coordinates → conserved quantities
  2) identify effective radial potential
  3) derive the eq. of motion (radial motion only)
  4) define an effective gravitational potential



1) Cyclic coordinates, conserved quantities and effective potential

- identify the cyclic coordinates, i.e. those not explicity present in $\mathcal{L}$
- to each of them we have an associated conserved quantity $\left(\dot{x}^\nu \equiv \frac{dx^\nu}{d\tau}\right)$

$$\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\phi}} = r^2\dot{\phi} \equiv \textcircled{L} = \text{const} \qquad \text{(Angular momentum)} \quad \Rightarrow \quad \dot{\phi} = \frac{L}{r^2}$$

$$\frac{\partial\hat{\mathcal{L}}}{c\,\partial\dot{t}} = -\frac{1}{\cancel{c}}\left(1-\frac{r_s}{r}\right)\overset{1}{\cancel{c^2}}\,2\dot{t} \equiv \textcircled{2\tilde{E}} = \text{const} \quad \text{(energy per unit mass)} \quad \Rightarrow \quad c\dot{t} = -\tilde{E}\left(1-\frac{r_s}{r}\right)^{-1}$$

- $\boxed{t, \vartheta = \text{cyclic coordinates}} \iff \boxed{\bar{\partial}_0, \bar{\partial}_\phi = \text{Killing vectors}} \iff \boxed{\text{time symmetry}, \vartheta \text{ symmetry}}$

(2)

$$\updownarrow$$

for $\mu = 0, \phi$ $\boxed{\partial_\mu g_{\alpha\beta} = 0}$ (1) because the "good" coord system

$\boxed{\mathcal{L}_{V_{(\mu)}} g_{\alpha\beta} = 0}$ with $V_{(\mu)}(x^\nu) \equiv \bar{\partial}_\mu$ Killing vector field

$P_\phi = \text{const} \qquad P_0 = \text{const.}$

(a) $\partial_0 g_{\alpha\beta} = 0$: e.g. $P_0 = g_{0\underset{\smile}{\mu}} P^{\bar{\mu}} = g_{00}P^0 = -\underset{\substack{\uparrow \\ \text{in fact no time dependency (!)}}}{\left(1-\frac{r_s}{r}\right)} m\overset{*}{\gamma} c^2 = E$ as above $\left(*\ \dot{t} \equiv \frac{dt}{d\tau} = \gamma\right)$

(b) Killing vector e.g. $\bar{\partial}_0 = (1,0,0,0)^T$ $\quad \partial_0^\nu P_\nu = g_{\underset{\smile}{\nu}\alpha}\partial_0^{\hat{\nu}} P^\alpha = g_{0\underset{\smile}{\alpha}} P^{\bar{\alpha}} = g_{00}P^0 = E$ as above

Recall: this comes from Euler-Lagrange eq.:

$$\frac{d}{d\tau}\frac{\partial\hat{\mathcal{L}}}{\partial\dot{q}} - \underbrace{\cancel{\frac{\partial\hat{\mathcal{L}}}{\partial q}}}_{=0} = 0 \quad \Rightarrow \quad \frac{\partial\hat{\mathcal{L}}}{\partial\dot{q}} = 0$$

$q \equiv$ cyclical coord. $P_q = \text{const.}$ conserved quantity

here $\dot{q} \equiv \frac{dq}{d\tau}$ because $S = \int \hat{\mathcal{L}}\, d\tau$

Conserved quantities and Killing vectors

Recall: $\bar{\xi}$ killing vector if $\nabla_{\dot{\gamma}}(\dot{\gamma}_\mu \xi^\mu) = 0$ i.e. $\langle \bar{\dot{\gamma}}, \bar{\xi} \rangle = \text{const.}$ along $\bar{\dot{\gamma}}$

$\bar{\dot{\gamma}}(\lambda) =$ vector tangent to geodesic line $\bar{\dot{\gamma}} = (c\dot{t}, \dot{r}, \dot{\vartheta}, \dot{\phi}) = (\dot{\gamma}^0, \dot{\gamma}^r, \dot{\gamma}^\vartheta, \dot{\gamma}^\phi)$ basis $\bar{\dot{\gamma}} = \dot{\gamma}^\mu \bar{\partial}_\mu$

$\uparrow$ eg. 4-momentum, 4-frequency

Here: $\bar{\partial}_0, \bar{\partial}_\phi =$ Killing vectors:

1) $\langle \bar{\dot{\gamma}}, \bar{\partial}_0 \rangle = \langle \dot{\gamma}^\mu \bar{\partial}_\mu, \bar{\partial}_0 \rangle = \dot{\gamma}^\mu \langle \bar{\partial}_\mu, \bar{\partial}_0 \rangle = \dot{\gamma}^\mu g_{\mu 0} = \dot{\gamma}^0 g_{00} = -\left(1-\frac{r_s}{r}\right)c\dot{t} = \text{const.} \equiv E$

2) $\langle \bar{\dot{\gamma}}, \bar{\partial}_\phi \rangle = \dot{\gamma}^\phi g_{\phi\phi} = r^2\dot{\phi} = \text{const.} \equiv L$

$\hookrightarrow$ in fact: eg. $\mu = t$ $\quad \partial_t(r^2\dot{\phi}) = 2r\dot{r}\underset{L/r^2}{\dot{\phi}} + r^2\underset{L/r^2}{\ddot{\phi}} = 2\cancel{r}\dot{r}\frac{L}{r^{\cancel{2}}} - \cancel{r^2}\frac{2L}{r^{\cancel{3}}}\dot{r} = 0$

2) Effective radial potential $V(r)$

$\dot{\phi} = \frac{L}{r^2} \quad c\underline{\dot{t}} = -E\left(1-\frac{R_s}{r}\right)^{-1}$

- Use the conserved quantities to get rid of $\dot{\phi}$ and $\dot{t}$

$$\hat{\mathcal{L}} = -c^2\varepsilon = -\left(1-\frac{R_s}{r}\right)c^2\overset{\downarrow}{\dot{t}}^2 + \left(1-\frac{R_s}{r}\right)^{-1}\dot{r}^2 + r^2\overset{\downarrow}{\dot{\phi}}^2 = -\cancel{\left(1-\frac{R_s}{r}\right)}E^2\left(1-\frac{R_s}{r}\right)^{\cancel{-2}\,-1} + \left(1-\frac{R_s}{r}\right)^{-1}\underline{\underline{\dot{r}^2}} + \cancel{r^2}\frac{L^2}{r^{\cancel{4}\,2}}$$

$$\dot{r}^2 - E^2 + \underbrace{\left(1-\frac{R_s}{r}\right)\left(\frac{L^2}{r^2}+c^2\varepsilon\right)}_{V(r)} = 0 \qquad \dot{r}^2 + V(r) = E^2$$

eq. of motion, radial component

Effective potential energy

Kinetic energy

$\downarrow$

$$\boxed{v(x) \equiv \frac{V(x)}{c^2} = \left(1-\frac{1}{x}\right)\left(\frac{\lambda^2}{x^2}+\varepsilon\right)} \qquad x \equiv \frac{r}{R_s} \qquad \lambda^2 = \frac{R_s^2 L^2}{c^2}$$

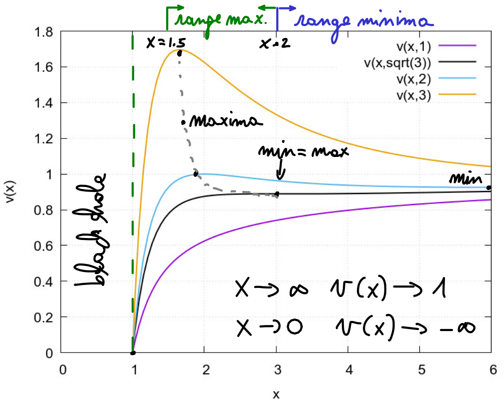
Dimensionless effective potential

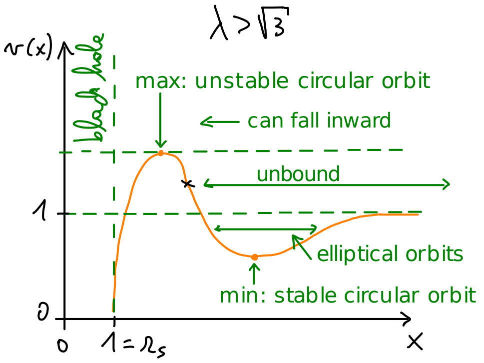
- extrema :

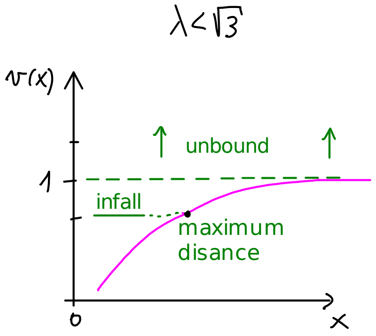
purely GR term

$$\frac{dv}{dx} = \frac{1}{x^2}\left(\varepsilon+\frac{\lambda^2}{x^2}\right) - \left(1-\frac{1}{x}\right)\frac{2\lambda^2}{x^3} = 0 \quad \xrightarrow{(\cdot x^4)} \quad \left(\varepsilon+\frac{\lambda^2}{x^2}\right)x^2 - \left(1-\frac{1}{x}\right)2\lambda^2 x = \boxed{\varepsilon x^2 - 2\lambda^2 x + 3\lambda^2 = 0}$$

- For massive particles: $\varepsilon = 1$ $\Rightarrow$ $x_\pm = \lambda^2 \pm \lambda\sqrt{\lambda^2-3}$ min / max radious







$\underline{\lambda > \sqrt{3}}$ $\Rightarrow$ $x_\pm \in \mathbb{R}$: non circular closed orbits : maxima at x_-, minima at x_+
2 possible circular orbit (stable / unstable)
particles with $E > \max(V)$ can fall to $r = 0$

$\underline{\lambda = \sqrt{3}}$ $\Rightarrow$ innermost circular stable orbit $x = 3$ $r = 3R_s$

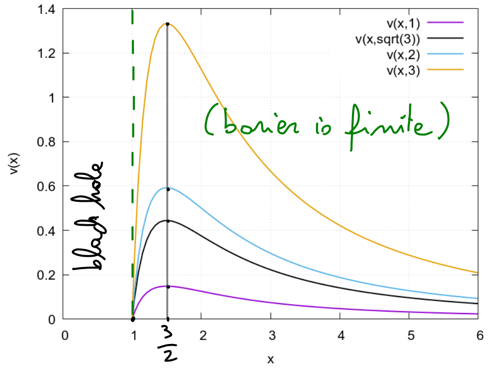
$\underline{\lambda < \sqrt{3}}$ $\Rightarrow$ $x_\pm$ complex : no maxima , particles with $E^2 < 1$ fall towards $r = R_s$

- Summary, bound particles:

unstable circular orbits:	$\frac{3}{2}R_s < r < 3R_s$	possible maxima
stable " " :	$r > 3R_s$	possible minima

- accelerated particle (non geodesic path) can enter $r = 3R_s$ and escape again if it does not cross $r < R_s$

- **For massless particles :** $\varepsilon = 0$ $\Rightarrow$ $-2\lambda^2 x + 3\lambda^2 = 0$ $x = 3/2$ always 1 maxima



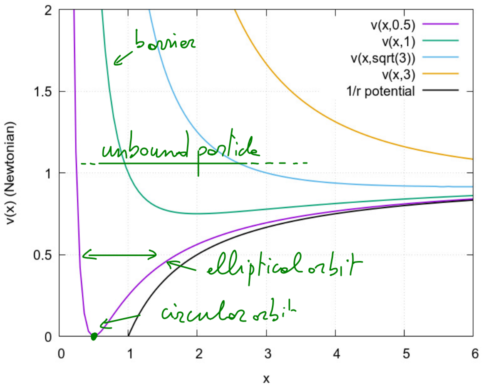
- 1 unstable circular orbit $x = \frac{3}{2}$ (small perturbation and $r \begin{smallmatrix}\nearrow 0\\ \searrow \infty\end{smallmatrix}$)
- No other bound orbits
- particle can fall to $r=0$

finite barrier $\Rightarrow$ photons can go to $r=0$

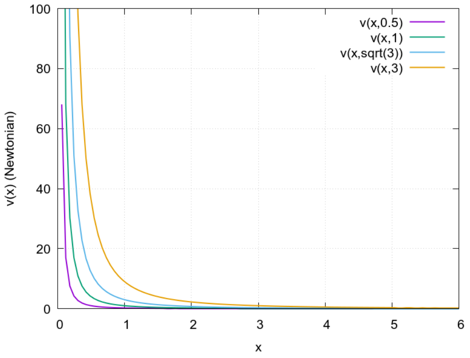
(sufficient for a given L) regardless their v,
it is all about the impact parameter

* * *

- **Newtonian gravity** : no GR term $\Rightarrow$ $\varepsilon x^2 - 2\lambda^2 x + \cancel{3\lambda^2} = 0$ $x = \frac{2\lambda^2}{\varepsilon}$ 1 minimal always



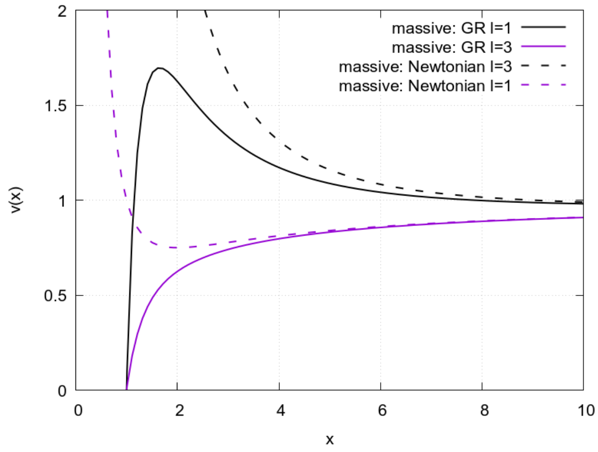
- Massive : $\varepsilon = 1$ $x = 2\lambda^2$
  - always angular momentum barrier $\Rightarrow$ can not reach $r=0$
  - stable circular orbit is always possible
  - Elliptical orbits possible $E<1$
  - Unbound particles for $E>1$

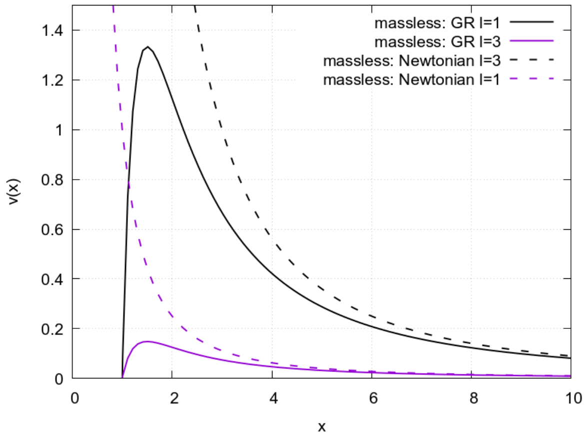


Massless $\varepsilon = 0$

- $x = \infty$ no bound orbits (light always escapes!)
- infinite barrere

- **Comparison GR <--> Netown**





3) Equation of motion (radial)

Use the energy conservation law

$\dot{r}^2 + V(r) = E^2$ look for $r(\vartheta)$ $\Rightarrow$ $\frac{\dot{r}}{\dot{\vartheta}} = \frac{dr}{d\tau}\frac{d\tau}{d\vartheta} = \frac{dr}{d\vartheta} \equiv r'$ $\dot{r} = \dot{\vartheta} r' = \frac{L}{r^2} r'$

$\frac{L^2}{r^4} r'^2 + V(r) = E^2$ $u \equiv 1/r$ $u' = -r'/r^2 = -u^2 r'$

$$L^2 \cancel{u^4} \frac{u'^2}{\cancel{u^4}} + V(1/u) = -L^2 u'^2 + (1 - r_s u)(L^2 u^2 + c^2 \mathcal{E}) = E^2$$

$\Big\downarrow \neq L^2$

$$u'^2 + u^2 + \frac{c^2\mathcal{E}}{L^2} - r_s u^3 - \frac{r_s c^2 \mathcal{E}}{L^2} u = E^2/L^2$$

$\Big\downarrow$ "1" get rid of constants

$$2u'u'' + 2uu' - 3r_s u^2 u' - \frac{r_s c^2 \mathcal{E}}{L^2} u' = 0$$

$u' = 0$: trivial solution with $u' = -\frac{r'}{r^2} = 0$, i.e. $\boxed{r = \text{const}}$ $\Rightarrow$ circular orbit

$u' \neq 0$: $2\cancel{u'}u'' + 2u\cancel{u'} - 3r_s u^2 \cancel{u'} - \frac{r_s c^2 \mathcal{E}}{L^2}\cancel{u'} = 0$

$$\boxed{u'' + u = \underbrace{\frac{3}{2} r_s u^2}_{(1)} + \frac{r_s c^2}{2L^2}\mathcal{E}}$$

(1) pure GR Term

driven harmonic oscillator

- Newtonian case

Same procedure as before (but now gravity is an external force with a potential)

$\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2) - \varphi(r)$ $\varphi(r) = -\frac{Gm}{r}$

E.-L. eq: $r^2\dot{\vartheta} \equiv L = \text{const}$, $\ddot{r} = \frac{L}{r^3} - \frac{Gm}{r^2} = \frac{L}{r^3} + \frac{\varphi}{r}$

$\Big\downarrow$ $r_s \equiv \frac{2Gm}{c^2}$ $u \equiv r^{-1}$

$$\boxed{u'' + u = \frac{1}{L^2 u^2}\frac{d\varphi}{dr}} = \frac{c^2 r_s}{2L^2}$$

like before but missing GR $\frac{3}{2} r_s u^2$ term!

4) Effective gravitational potential

(to interpret gravity as a conservative force)

Impose: $3r_s u^2 + \frac{r_s c^2}{L^2} \overset{!}{=} \frac{2}{L^2 u^2}\frac{d\varphi}{dr}$ as in Newtonian case $\Rightarrow$ effective force given by φ

$$\frac{d\varphi(r)}{dr} = \frac{3}{2} r_s L^2 u^4 + \frac{r_s c^2}{2} u^2 = \frac{3 r_s L^2}{2r^4} + \frac{r_s c^2}{2r^2}$$

$u \equiv 1/r$

$$\varphi(r) = -\frac{1}{\cancel{3}}\frac{\cancel{3} r_s L^2}{2r^3} - \frac{r_s c^2}{2r} + A$$

$r \to \infty$ $\varphi = 0$ $\Rightarrow$ $A = 0$, $r_s = \frac{2Gm}{c^2}$

$$\boxed{\varphi(r) = -Gm\left(\underbrace{\frac{L^2}{c^2 r^3}}_{(1)} + \frac{1}{r}\right)}$$

(1) the "famous" perturbation causing the perihelion shift!

Predictions for the Schwarzshild metric

Perihelion shift

- Newtonian case   Closed Keplerian orbits

$$2u''+2u=\frac{r_s c^2}{L^2} \qquad u_0=\frac{1}{p}(1+e\cos\theta) \qquad p=a(1-e^2)=\frac{L^2}{m}$$

a = semi major axis
e = eccentricity

- GR

$$2u''+2u=\overset{(1)}{3r_s \overset{}{\underset{\uparrow\, u_0^2}{u^2}}}+\overset{(2)}{\frac{r_s c^2}{L^2}} \qquad \text{assume } (1)<<(2) \qquad \frac{(1)}{(2)}=\frac{3u^2 L^2}{c^2}\approx 7{,}7\cdot 10^{-8} \text{ (Mercury)}$$

↳ plug u^0 in $3r_s u^2$ for an approximate result

$$\simeq 3r_s\frac{1}{p}(1+e\cos\theta)+\frac{r_s c^2}{L^2}$$

complete solution: $u=\frac{1}{p}(1+e\cos\theta)+\frac{3m}{p^2}\left[1+e\theta\sin\theta+\frac{e^2}{2}\left(1-\frac{1}{3}\cos 2\theta\right)\right]$

$\frac{du}{d\theta}=0$ has maxima (i.e $r=\frac{1}{u}$ minima = perihelion) at $\theta=0$ $\frac{du(0)}{d\theta}=0$

consider a full revolution + deviation $\theta=2\pi+\delta\theta$

look for the $\delta\phi$ for which $\frac{du(2\pi+\delta\theta)}{d\theta}=0$ $\Rightarrow$ $\delta\theta\simeq\frac{3\pi r_s}{a(1-e^2)}$ (= 43'' Mercury)

Gravitational lensing

We already found $u(\theta)$ $u=r^{-1}$ $2u''+2u=3r_s u^2+\overset{=0}{\cancel{\frac{r_s c^2\varepsilon}{L^2}}}$ $\varepsilon=0$! photons

$\uparrow$ very small $\left(\frac{3r_s u^2}{2u}=\frac{3}{2}\frac{r_s}{r}\ll\frac{r_s}{R_\odot}\sim 10^{-6}\right)$ at surface of the Sun

=> almost homogeneous solution

$u''+u=0$ $\qquad u_0=A\sin\theta+B\cos\theta$ $\qquad$ impose closest impact $u_0=1/b$ at $\theta=\pi/2$

$\Rightarrow B=0\ A=1\ u_0=\sin\theta/b$ $\qquad u=r^{-1}$

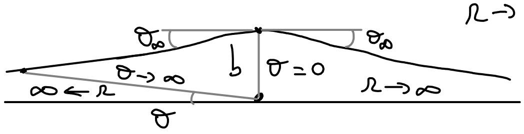
0-th order

plug u_0

$$2u''+2u=3r_s u^2=\frac{3r_s}{b^2}\sin^2\theta=\frac{3r_s}{b^2}(1-\cos^2\theta) \Rightarrow u=\frac{\sin\theta}{b}+\frac{3r_s}{2b^2}-\frac{3r_s}{4b^2}\left(1-\frac{1}{3}\cos 2\theta\right)$$

$\theta/2$ closest approach $\Rightarrow$ $\theta=0$ from $r=\infty$ $\Rightarrow$

$r\to\infty = \theta\to 0$ $\qquad u\simeq\frac{\theta_\infty}{b}+\frac{3r_s}{2b^2}-\frac{\cancel{3}r_s}{\underset{2}{\cancel{4}}b^2}\frac{\overset{1}{\cancel{2}}}{\cancel{3}}=\frac{\theta_\infty}{b}+\frac{r_s}{b^2}\simeq\frac{1}{r}=0$



$\Rightarrow \theta_\infty=-\frac{r_s}{b}$

Total deflection angle $\alpha=2|\theta_\infty|\simeq\frac{2r_s}{b}$

Gravitational redshift

$$\cancel{-c^2}d\tau_i^2 = \cancel{-}\left(1-\frac{r_s}{r_i}\right)\cancel{c^2}dt_i$$

$$\boxed{\frac{\nu_2}{\nu_1}} = \frac{d\tau_1}{d\tau_2} = \boxed{\left(1-\frac{r_s}{r_1}\right)^{1/2}\left(1-\frac{r_s}{r_2}\right)^{-1/2}} \quad \text{(exact solution)}$$

$$\frac{r_s}{r} = \frac{2Gm}{c^2 r} = \frac{2\varphi}{c^2}$$

$$\simeq \left(1-\frac{r_s}{2r_1}\right)\left(1+\frac{r_s}{2r_2}\right) = 1+\frac{r_s}{2r_2}-\frac{r_s}{2r_1}-\frac{r_s^2}{4r_1r_2} \simeq 1+\frac{r_s}{2r_2}-\frac{r_s}{2r_1} = \boxed{1+\frac{\varphi_2}{c^2}-\frac{\varphi_1}{c^2}} \quad (\text{for } r \gg r_s)$$

For an observer at $r_2=\infty$: $\frac{\nu_\infty}{\nu_1} = \left(1-\frac{r_s}{r_1}\right)^{1/2}$ (exact solution)

or if you which $r_2 \gg r_s$

for $r_1 = r_s$: $\nu_\infty = 0$ i.e. you can not have a photon coming out of r_s!

Schwarzshild black holes

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \qquad \text{Schwarzschild metric}$$

When r_s is exposed the sourcing object is called Black hole r_s = event horizon

Sun:	$m_\odot = 1.9\cdot10^{30}$ Kg	$R_\odot = 4{,}3\cdot10^{6}$ Km	$r_s = 1{,}5$ Km
white dwarf:	$m \sim m_\odot$	$R \sim 6\cdot10^{3}$ Km	"
Neutron star:	$m \sim 1.4\, m_\odot$	$R \sim 10$ Km	$r_s = 2.1$ Km
Black hole:	object for which	$R < r_s$	r_s is "exposed"!

$r \gg r_s$
NOT
Black holes

1) Singularities for the metric coefficients g_{00}, g_{rr}: $\underset{\text{real}}{\underset{\uparrow}{(r=0)}}$ and $\underset{\text{just a coordinate sing}}{\underset{\uparrow}{(r=r_s)}}$

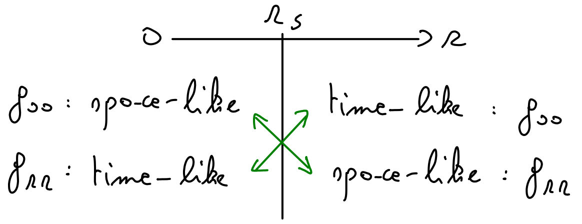
- Recall: coefficients are coordinate dependent $g_{\mu\nu} = g(\vec{e}_\mu, \vec{e}_\nu)$ $\vec{e}_\mu$ basis
- They might just be "coordinate singularities" not intrinsic problems of the manifold
- As a test evaluate curvature related quantities at the suspected singularities
  eg. Ricci scalar R (curvature singularities are real...) (scalar ⇒ invariant in all frames)
  - here $R^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma\mu} = \frac{12G^2m^2}{c^2r^6} \in \mathbb{R}$ $r \neq 0$ ⇒ only $r=0$ is a real singularity $\left(\underset{R_{\mu\nu}=0}{\underset{\Downarrow}{\text{vacuum}}}\right)$
  - at r_s: $R \in \mathbb{R}$ there is always a scale L for wich $g = \eta$ at 1° order
- Singularities are a real problem only if something can "reach" them (along a world-line)

2) Swap between time- and space-like classification

$$g_{00} = -\left(1-\frac{r_s}{r}\right) \begin{cases} <0 & r>r_s \\ >0 & r<r_s \end{cases}$$

$$g_{rr} = \left(1-\frac{r_s}{r}\right)^{-1} \begin{cases} >0 & r>r_s \\ <0 & r<r_s \end{cases}$$

⇒



- Light cone while approaching the Schwarzshild radious r_s

- light cone given by world line of photons
- Take a radial trajectory => $d\theta = 0 \quad d\phi = 0$

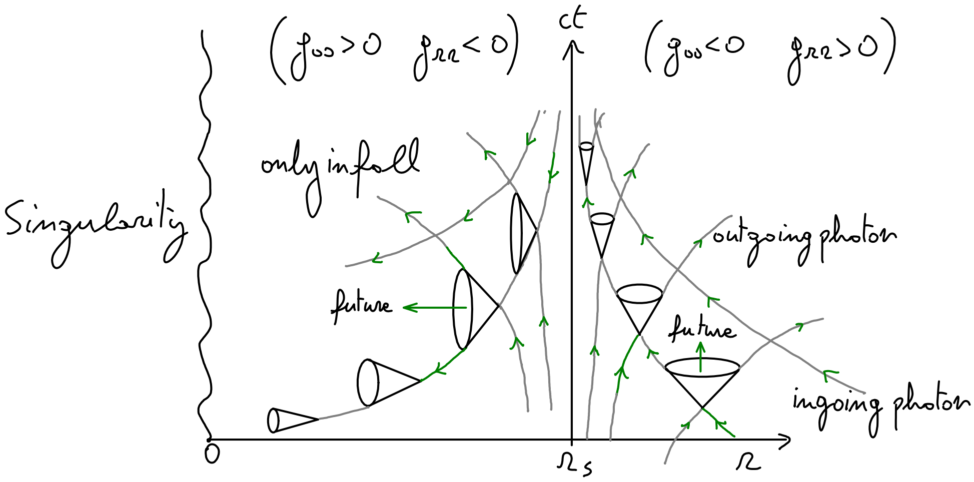
$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 = 0 \qquad (\text{photon}) \qquad \Rightarrow \boxed{\frac{dr}{cdt} = \pm\left(1-\frac{r_s}{r}\right)}$$

it gives you the inclination of the light cone

at $r \gg r_s \qquad \frac{dr}{cdt} = \pm 1$ like in Minkowski

for $r \to r_s \qquad \frac{dr}{cdt} \to \mp 0$ i.e. vertical

- Space-Time diagram in Schwarzshild space-time



Note: the future delimited by the light-cones inside $r<r_s$ points inward toward $r=0$

=> Once you cross $r=r_s$ there is no return → Blak-Hole!

Similarly for massive particles, they world-line is always within the light cone centered on their location

It seems:

from discussion on grav. redshift

they approach r_s at $t=\infty$: $\quad \frac{dt_1}{d\tau} = \left(1-\frac{r_s}{r_1}\right)^{1/2}\left(1-\frac{r_s}{r}\right)^{-1/2} \to \infty$ observer $r_1 = \infty$, $r \to r$

but....

...But in their own rest frame they arrive in r_s in a finite τ:

Investigate massive particle ($\varepsilon=1$) alon radial trajectory ($L=0$)

$$\dot{r}^2-E^2+\left(1-\frac{r_s}{r}\right)\left(\overset{=0}{\cancel{\frac{L^2}{r^2}}}+c^2\overset{\stackrel{!}{=}1}{\cancel{\varepsilon}}\right)=0 \quad \leftarrow \text{Radial eq. of motion}$$

Energ. per unit mass

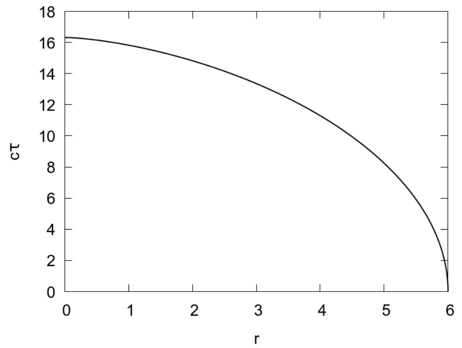
$$\dot{r}^2-E^2+\left(1-\frac{r_s}{r}\right)c^2=0 \quad \leftarrow \text{init. conditions: (at rest in } R) \Rightarrow E^2=\left(1-\frac{r_s}{R}\right)c^2,\ E^2<c^2$$

$$\frac{\dot{r}^2}{c^2}=\left(1-\frac{r_s}{R}\right)-\left(1-\frac{r_s}{r}\right)=r_s\left(\frac{1}{r}-\frac{1}{R}\right) \quad \leftarrow \dot{r}=\frac{dr}{d\tau}$$

parameter

$$c\,d\tau=\left[r_s\left(\frac{1}{r}-\frac{1}{R}\right)\right]^{-1/2}dr \qquad \text{parametric solution for: } r=\frac{R}{2}(1+\cos\xi) \quad dr=-\frac{R}{2}\sin\xi\, d\xi$$

$$\Rightarrow\ c\tau=\sqrt{\frac{R^3}{4r_s}}\,(\xi+\sin\xi) \qquad r\stackrel{!}{=}0 \xrightarrow{\text{i.e.}} \xi=\pi \quad \Rightarrow \quad \boxed{\tau=\frac{\pi}{c}\sqrt{\frac{R^3}{4r_s}}}\ \text{free fall time} \in \mathbb{R}$$



→ we have seen that $t\to\infty$ for $r\to r_s$, but:

1) a free falling object can reach $r=0$ in a finite τ

2) we have seen that $R^\mu{}_\nu \in \mathbb{R}$ (finite curvature at r_s)

⇒ objects can cross r_s!

the problem is in the coordinate (t) which explodes in r_s

- Same argument holds for photons: just use $\varepsilon=0$, $d\tau\to d\lambda$

Try to solve these issues

Study photon moving along a radial trajectory (i.e. $\dot{\theta}=0$)

- We had: $\hat{\mathcal{L}}=-\left(1-\frac{r_s}{r}\right)c^2\dot{t}^2+\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2+r^2\dot{\theta}^2$ $\qquad \frac{d}{d\lambda}\frac{\delta\mathcal{L}}{\delta\dot{x}^\mu}-\frac{\delta\mathcal{L}}{\delta x^\mu}=0 \qquad \dot{x}^\mu=\frac{dx^\mu}{d\lambda}$

Photons

$$\mu=r:\ \frac{d}{d\lambda}\left[\left(1-\frac{r_s}{r}\right)^{-1}2\dot{r}\right]-\left[-\frac{r_s}{r^2}c^2\dot{t}^2-\left(1-\frac{r_s}{r}\right)^{-2}\frac{r_s}{r^2}\dot{r}^2\right]=0$$

$$-\left(1-\frac{r_s}{r}\right)^{-2}\frac{r_s}{r^2}\dot{r}\cancel{2}\dot{r}+\frac{r_s}{r^2}c^2\dot{t}^2+\cancel{\left(1-\frac{r_s}{r}\right)^{-2}\frac{r_s}{r^2}\dot{r}^2}=0$$

$$-\left(1-\frac{r_s}{r}\right)^{-2}\cancel{\frac{r_s}{r^2}}\underset{\left(\frac{dr}{d\lambda}\right)^2}{\dot{r}^2}+\cancel{\frac{r_s}{r^2}}c^2\underset{\left(\frac{dt}{d\lambda}\right)^2}{\dot{t}^2}=0 \quad \Rightarrow \quad \boxed{\frac{dr}{c\,dt}=\pm\left(1-\frac{r_s}{r}\right)}$$

This is just an exercise, you already got this earlier frome the 4-interval

- Use it to define a new radious: $d\bar{r}\equiv\left(1-\frac{r_s}{r}\right)^{-1}dr \Rightarrow \boxed{\frac{d\bar{r}}{c\,dt}=\pm1}$

with $\bar{r}$ light cones have always the same shape! :)

"tortois coordinate"

$$d\bar{r}_{(+)}\equiv\frac{dr}{\left(1-\frac{r_s}{r}\right)}=\frac{dr}{\frac{r_s}{r}\left(\frac{r}{r_s}-1\right)}=\frac{r/r_s+1-1}{\left(\frac{r}{r_s}-1\right)}dr=dr+\frac{dr}{\left(\frac{r}{r_s}-1\right)} \Rightarrow \boxed{\bar{r}_{(+)}=r+r_s\ln\left(\frac{r}{r_s}-1\right)}\ r>r_s$$

$$d\bar{r}_{(-)}\equiv\frac{-dr}{\left(1-\frac{r_s}{r}\right)}=\frac{dr}{\left(\frac{r_s}{r}-1\right)} \Rightarrow \boxed{\bar{r}_{(-)}=r+r_s\ln\left(1-\frac{r}{r_s}\right)}\ r<r_s$$

Kruskal coordinates

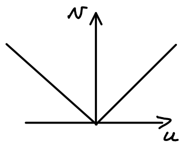
- Schwarzshild $(ct, r, \theta, \phi) \rightarrow (v, u, \theta, \phi)$ new coordinates u, v

- Define 4-interval as: $ds^2 = -f^2(u,v)(dv^2 - du^2) + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)$

$$g' = \mathrm{diag}(-f^2, f^2, r^2, r^2 \sin\theta)$$

- Look at the light-cone structure: $ds^2 \overset{!}{=} 0 \quad d\theta \overset{!}{=} 0 \overset{!}{=} d\phi$

$$0 = -\cancel{f^2(u,v)}^{1}(dv^2 - du^2) \qquad \boxed{\frac{dv}{du} = \pm 1}$$



i.e. L-cone is like in Minkowski space everywhere by construction

- Identify the $f(u,v)$ representing the Scwarzshild metric

$$g \overset{!}{=} J g' J^T = \begin{pmatrix} -f^2(v_{,t}^2 - u_{,t}^2) & -f^2(v_{,t}v_{,r} - u_{,t}u_{,r}) & 0 & 0 \\ -f^2(v_{,t}v_{,r} - u_{,t}u_{,r}) & -f^2(v_{,r}^2 - u_{,r}^2) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin\theta \end{pmatrix} \qquad J = \begin{pmatrix} \partial_t v & \partial_t u & 0 & 0 \\ \partial_r v & \partial_r u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Jacobian of the transformation

impose: $-\left(1 - \frac{r_s}{r}\right) \overset{!}{=} -f^2(v_{,t}^2 - u_{,t}^2)$ **(1)** $\qquad \left(1 - \frac{r_s}{r}\right)^{-1} \overset{!}{=} -f^2(v_{,r}^2 - u_{,r}^2)$ **(2)** $\qquad 0 \overset{!}{=} -\cancel{f^2}^{1}(v_{,t}v_{,r} - u_{,t}u_{,r})$ **(3)**

- Change of variables: $f(u,v) \rightarrow f(r,t) = f(r)$ because we the metric is static

- Change of variable, use $\bar{r}_{(r)} = r + r_s \ln\left(\frac{r}{r_s} - 1\right)$ from eq. of motion

$$\Downarrow$$

$$\frac{dr}{d\bar{r}} = \left(1 - \frac{r_s}{r}\right)$$

$$v_{,\bar{r}} = v_{,r}\frac{dr}{d\bar{r}} = v_{,r}\left(1 - \frac{r_s}{r}\right)$$

$$u_{,\bar{r}} = u_{,r}\left(1 - \frac{r_s}{r}\right)$$

(2): $\left(1 - \frac{r_s}{r}\right)^{-1} \overset{!}{=} -f^2(v_{,r}^2 - u_{,r}^2) \qquad \cancel{\left(1 - \frac{r_s}{r}\right)^{-1}} = -f^2\left(1 - \frac{r_s}{r}\right)^{\cancel{-2}\,-1}(v_{,\bar{r}}^2 - u_{,\bar{r}}^2) \qquad \underbrace{f^{-2}\left(1 - \frac{r_s}{r}\right)}_{\equiv F(r)} = (v_{,\bar{r}}^2 - u_{,\bar{r}}^2)$

(1): $\cancel{-}\left(1 - \frac{r_s}{r}\right) \overset{!}{=} \cancel{-}f^2(v_{,t}^2 - u_{,t}^2) \qquad F(r) = (v_{,t}^2 - u_{,t}^2)$

(3): $0 = v_{,t}v_{,r} - u_{,t}u_{,r} \qquad v_{,t}\cancel{\left(1 - \frac{r_s}{r}\right)^{-1}} v_{,\bar{r}} - u_{,t}\cancel{\left(1 - \frac{r_s}{r}\right)^{-1}} u_{,\bar{r}} = 0 \qquad v_{,t}v_{,\bar{r}} - u_{,t}u_{,\bar{r}} = 0$

- Solve for $F(r)$ + express v, u as a func. of r, t see Bartelmann 10.2.2

$$\boxed{f^2 = \frac{4 r_s^3}{r} e^{-r/r_s} \qquad \begin{aligned} u &= \left[\left(\frac{r}{r_s}-1\right)^{1/2} e^{r/2r_s}\right] \cosh\left(\frac{ct}{2r_s}\right) \\ v &= \left[\left(\frac{r}{r_s}-1\right)^{1/2} e^{r/2r_s}\right] \sinh\left(\frac{ct}{2r_s}\right) \end{aligned}} \quad r > r_s$$

(v, u, θ, ϕ) Kruskal-Szekeres (early 60s)

using $\bar{r}_{(-)} = r + r_s \ln\left(1-\frac{r}{r_s}\right)$

$$\boxed{\begin{aligned} u &= \left[\left(1-\frac{r}{r_s}\right)^{1/2} e^{r/2r_s}\right] \sinh\left(\frac{ct}{2r_s}\right) \\ v &= \left[\left(1-\frac{r}{r_s}\right)^{1/2} e^{r/2r_s}\right] \cosh\left(\frac{ct}{2r_s}\right) \end{aligned}} \quad r_s < r$$

- Link $u, v \to r, t$

Get $r(u,v)$ by inverting $u^2 - v^2 = \left[\left(\frac{r}{r_s}-1\right) e^{r/2r_s}\right] \left(\overset{=1}{\cancel{\cosh^2(\dots) - \sinh^2(\dots)}}\right)$

Get $t(u,v)$ by inverting $\frac{v}{u} = \tanh\left(\frac{ct}{2r_s}\right)$ $\quad t = \frac{2r_s}{c} \tanh^{-1}\left(\frac{v}{u}\right)$

P.S. The transformation distinguish the case $r_s > r$, $r_s < r$ .... :(

this is not an "issue" coming from the u,v coordinates, it comes from the misbehavejour of ct, r! v, u keep their nice behavejours regardless r_s

- 4-interval / metric of Schwarzshild solution in Kruskal coord.

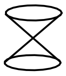
$$\boxed{ds^2 = -\frac{4r_s^3}{r} e^{-r/r_s} (dv^2 - du^2) + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2)}$$

func. of u, v

no coordinate singularity in r_s!

- Kruskal diagram: (u,v) plane

- $g_{00} < 0 \Rightarrow v$ has time-like structure

- radial, null lines: $ds^2 = 0 = -\cancel{\frac{4r_s^3}{r}}(dv^2 - du^2) \Rightarrow \frac{dv}{du} = \pm 1$  (0)

  light cones are always "diagonal" everywhere

- Each point (u,v) identifies a two-sphere (θ,ϕ)

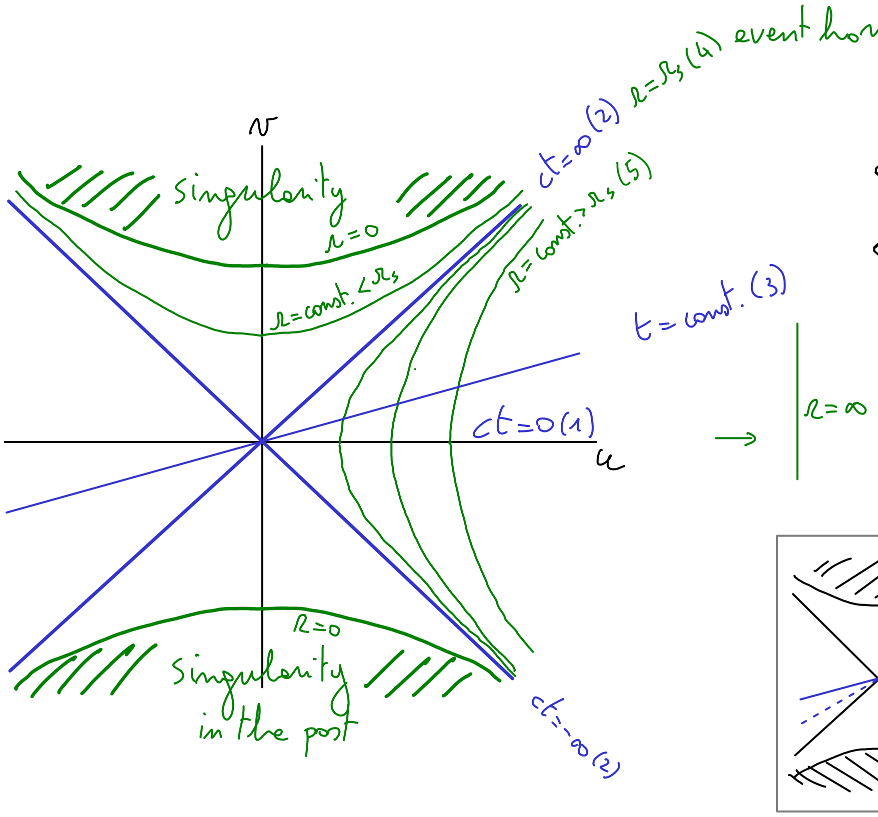
- From $\frac{v}{u} = \tanh\left(\frac{ct}{2r_s}\right)$ $t = \frac{2r_s}{c}\tanh^{-1}\left(\frac{v}{u}\right)$:

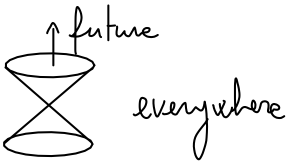
  1) $v=0 \Rightarrow t=0$
  2) $\frac{v}{u}=1$ $t=+\infty$ $\frac{v}{u}=-1$ $t=-\infty$ $\Rightarrow$ diagonals set time limits
  3) $\frac{v}{u}=$ const. lines with $t=$ const

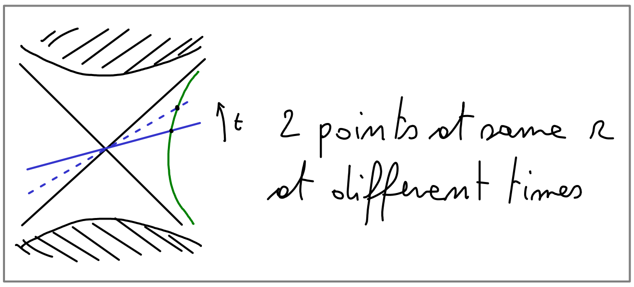
- From $u^2-v^2 = \left(\frac{r}{r_s}-1\right)e^{r/2r_s}$:

  4) $r=r_s$: $u=\pm v$ diagonals (ok), in agreement with $t=\pm\infty$
  5) $r=$ const. : $u^2-v^2=$ const. iperbolae in u,v plane
  6) $r=0$: $u^2-v^2=-1$ curvature singularity (not related to coordinates)

- Allowed (u,v) regions $-\infty < u < +\infty$ $v^2-u^2<1$

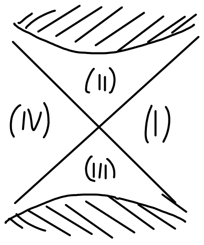






2 points at same r at different times

- 4 Regions



Regions defining the manifold

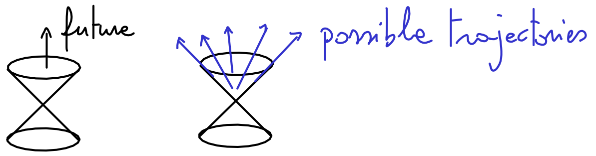
(I) outside the event horizon $r > r_s \ \forall t$

(II) following future direction $r < r_s$ Black Hole!

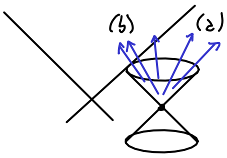
(III) following past direction

(IV) asymptotically flat space-time like (I)

- Allowed trajectories limited by the lightcone structure



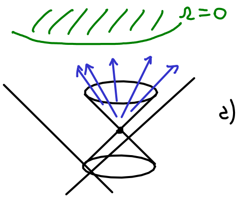
(I)



a) can escape BH,

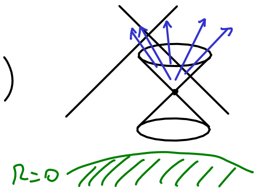
b) can enter r_s

(II)



a) can not escape BH must fall on r=0 (Black hole)

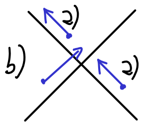
(III)



must go away from r=0 : White Hole!

it can not be reached it exists in the past

(IV) asymptotically flat space-time like (I)

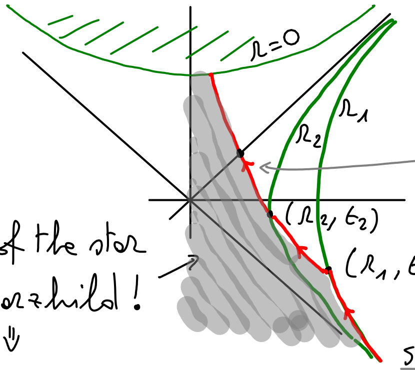


a) unreachable by (I) and (II) nor from past nor from future

b) nothing can reach (I) from (IV)

Connected to region (I) only through origin : "Wormhole"

- Formation of a stellar Black Hole



$r_1 > r_2$ $t_2 > t_1$

step 3 : formation of the Black Hole

step 2 : t_1 = time of collapse

step 1 : r_1 = const surface of a stable star

interior of the star not Schwarzschild!

⇓

no white holes / no wormholes

Appendix: more about horizons

Carroll 6.2

Black holes : whatever enters r_S (event horizon) cannot come out
this is their distinctive feature (not the central singularity)

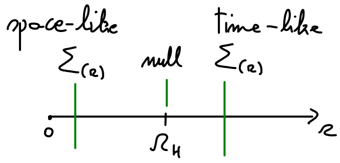
event horizon r_H

- hypersurface separating space-time points that are connected to infinity by a time-like path from those that are not
- surface beyond which time-like curves cannot escape to infinity
- they are null hypersurfaces, i.e. $n_\mu n^\mu = 0$ ($\bar{n}$ vectors orthogonal to it)

being $n_\mu n^\mu = 0$: $\bar{n}$ also tangent to surface
can be defined as a collection of null geodesics $x^\mu(\lambda)$
-> generators of the hypersurface ↖ (tangent lines)

Find Possible event horizons

- when a clever frame is specified, consider hypersurfaces $\Sigma_{(r)}$ ($r=\text{const}$) running along r from $\infty \to 0$ $\Sigma_{(r)}$ is time-like until $r = r_H$ here $\Sigma_{(r)}$ is null or space-like for all θ, ϕ
↑
(if frame is not so clever, only for some θ, ϕ)



- Find possible null $\Sigma_{(r)}$ hypersurfaces :
  - $\partial_\mu f(r)$ is a 1-form orthogonal to a surface (recall: gradients are prototypes of 1-forms)
  - consider $\partial_\mu r$ ($f(r)=r$) 1-form normal to $r=\text{const}$ surface ($\frac{\partial r}{\partial x^\mu}$ is also basis for r)

  => search when $\partial_\mu r$ norm is null : $g^{\mu\nu}(r_H)\,\partial_\mu r\,\partial_\nu r = \boxed{g^{rr}(r_H) \overset{!}{=} 0}$

  eg. Schwarzschild, Reissner-Nordström

Killing horizon :

ξ^μ Killing vector, if $\xi^\mu \xi_\mu = 0$ (i.e. null vector) => hypersurface Σ is a Killing horizon of ξ^μ

event horizons are not necessarily Killing horizons

eg. Schwarzschild metric : killing vector $\xi = \partial_t$ goes from time-like to space-like at $r = r_s$

Part V

Applications

Electrically charged bodies: the Reissner-Nordström solution

☐ Spherically symmetric static metric with $A(r)$, $B(r)$ as for neutral bodies

$$ds^2 = -e^{2A(r)} c^2 dt^2 + e^{2B(r)} dr^2 + r^2 d\Omega^2$$

☐ Object with an electric charge => electro-mag. field in the surroundings i.e. $T_{\mu\nu} \neq 0$

$$T^{(EM)}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$$ energy-momentum tensor of electro-magnetic field

$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$ e.-m. field tensor (antisymmetric)

$A^{\mu\nu} = (\phi, \vec{A})$ 4-potential

$T^\mu{}_\mu \equiv T = 0$ because of antisymmetry of $F_{\mu\nu}$

☐ Impose symmetry to $A^{\mu\nu}$ as well : $\vec{A}$: points radially $\vec{A} = (A,0,0)$ ϕ: spherical iso-levels $\phi(r)$

- All components $F_{\mu\nu} = 0$ exept for
- $F_{tr} = -F_{rt} = f(r)$ ← radial electric field
- $F_{\vartheta\phi} = -F_{\phi\vartheta} = g(r)\sin\vartheta$ ← " magnetic field ← only considering magnetic monopole contribution

coming from r component of $B^\ell = \epsilon^{0\ell\mu\nu} F^{(EM)}_{\mu\nu}$ $\epsilon^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\alpha\beta\mu\nu} \propto (\sin\vartheta)^{-1}$ spherical symmetry

☐ A and B set by the field equations : Einstein + Maxwell

Outside the body, we are not in vacuum, the E.M. field also surrounds the source but $j^\nu = 0$, because all charges are in the body

$$\left.\begin{aligned} R_{\mu\nu} &= \frac{8\pi G}{c^4}\left(T^{(EM)}_{\mu\nu} - \frac{1}{2}\overset{T=0}{\cancel{T^{(EM)}}} g_{\mu\nu}\right) \\ D_{[\mu}F_{\nu\gamma]} &= 0 \\ g^{\mu\nu} D_\mu F_{\nu\gamma} &= 0 \end{aligned}\right\} \text{coupled together:}$$

$T_{\mu\nu}$ enters in gravity

$g_{\mu\nu}$ " " electrodynamics

$$\Rightarrow \boxed{F_{tr} = -q/r^2 \qquad F_{\vartheta\phi} = p\sin\vartheta \qquad \text{all other } F_{\mu\nu} = 0}$$

$$\boxed{ds^2 = -\Delta c^2 dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2 \qquad \Delta = 1 - \frac{2Gm}{rc^2} + \underbrace{\frac{G(p^2+q^2)}{c^4 r^2}}}$$ Reissner-Nordstrøm metric

$p = 0 = q$ => Schwarzschild; c^{-4} contribution!

m = mass

q = total electric charge

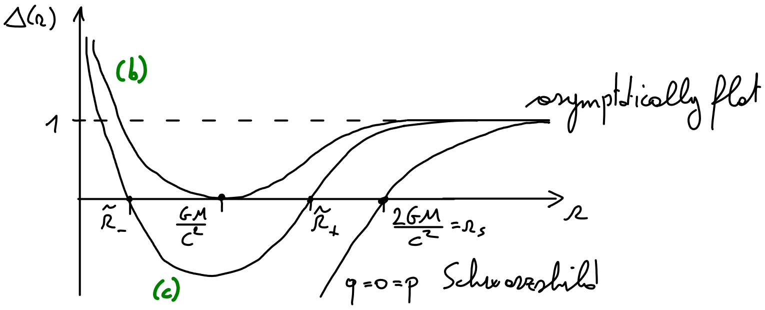
p = total magnetic charge : isolated magnetic charge (monopole) theoretically predicted but never observed (very rare) set p=0

Reissner-Nordström black holes

- Singularities : (1) $r=0$ true curvature singularity (eg. $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$)

  (2) $\Delta = 1 - \frac{2Gm}{rc^2} + \frac{G(p^2+q^2)}{r^2c^4} = 0$ $\qquad \tilde{r}_{\pm} = \frac{Gm \pm \sqrt{G^2m^2 - G(p^2+q^2)}}{c^2}$

  $\hookrightarrow$ 0, 1 or 2 solutions according to $Gm^2 - (p^2+q^2)$



3) $Gm^2 < (p^2+q^2)$

- $r_\pm$ No real solution $\Rightarrow$ metric is regular in (t, r, ϑ, ϕ) coordinates everywhere (except for $r=0$)

- $r=0$: curvature singularity (real singularity)
  is a time-like line, i.e. can be avoided
  naked singularity i.e. not shield by an horizon, you can reach $r=0$ and go away

- $\Delta > 0$ $\Rightarrow$ t is always time-like
  r " " space-like
  No event horizon

- looking at geodesics:
  $r=0$ "is repulsive" : time-like geodesics never meet $r=0$,
  they approach and then move away without reaching it
  null-geodesics and non-geodesic world lines can reach it
  $\llcorner$(with external force)

- cosmic censorship conjecture = the collapse of an object will never produce a naked singularity
  i.e. gravitational collapse never form black-holes with $Gm^2 < (p^2+q^2)$
  $\Rightarrow$ This shenario is considered unphysical i.e. such BH are expected not to exist

R
th
e
th
to

b) $\underline{Gm^2 = (p^2+q^2)}$

- is called extreme Reissner-Nordstrom solution
- Very unlikely to occours and very unstable: if it accretes some matter and you get case (c)
- $\Delta = 0$ for $\tilde{r} = Gm$ One solution (1 event horizon)
- $\Delta \geqslant 0 \ \forall r \neq \tilde{r}$ ⇒ r always space-like

  ⇒ $r = 0$ is time-like ⇒ can be avoided

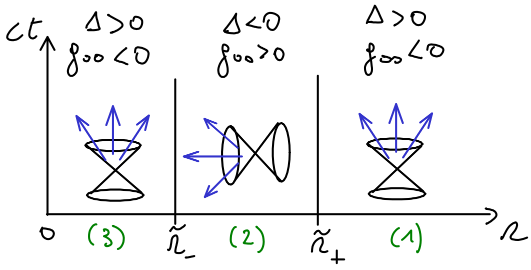
c) $\underline{Gm^2 > (p^2+q^2)}$

- expected in a real gravitational collapse of matter (energy in electro-mag. field < total energy)
- 2 coordinate singularities: $\tilde{r}_+, \tilde{r}_-$ (can be removed)
- $\underline{\Delta(\tilde{r}_\pm) = 0}$ Surfaces at $r = \tilde{r}_\pm$ are null-like ⇒ 2 horizons

  $\underline{\Delta(r) > 0}$ for $r < \tilde{r}_-$ and $r > \tilde{r}_+$, $r = 0$ time-like line ⇒ can be avoided

  $\underline{\Delta(r) < 0}$ for $\tilde{r}_- < r < \tilde{r}_+$: r, t swap nature, i.e. r = time t = space

- Ingoing particle ←



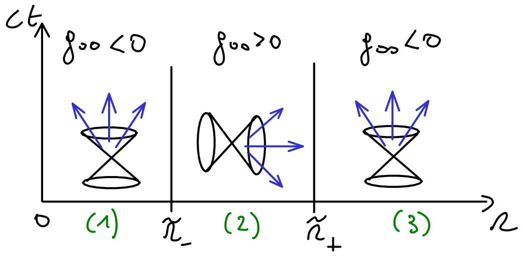
(1) can enter the horizon

(2) must go toward $r=0$ (Black hole)

$\tilde{r}_+$ behaves as r_s in Schwarzschild

(3) can do whatever: avoid/reach $r=0$, remain, exit $\tilde{r}_-$

- Outgoing particle →



(1) can exit $\tilde{r}_-$

(2) must go towards $\tilde{r}_+$ and cross it (like a white hole)

particle looks back in time ....

(3) can do whatever: re-enter $\tilde{r}_+$ (a different hole)

- but! if you entered the hole, your presence perturbs the metric

  ⇒ no more Reissner-Nordstrom metric (it is very "sensitive" to perturbers)

  ⇒ all of this should not happen

Explicit derivation (no magnetic monopole)

1) Energy momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$ energy-momentum tensor of electro-magnetic field

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$$ electro-magnetic field tensor (anti-symmetric)

$$(A^\mu) = (\phi(r), a(r), 0, 0)^T$$ 4-potential (here we imposed the symmetries) static + spherically symmetric

$$\left.\begin{aligned} F_{\mu\mu} &= A_{\mu,\mu} - A_{\mu,\mu} = 0 \\ F_{\mu\nu} &= -F_{\nu\mu} \end{aligned}\right\}$$ because F anti symmetric

i=θ,φ,0 ↓

$$F_{0\mu} = A_{0,\mu} - A_{\mu,0} = A_{0,\mu} \Rightarrow \quad F_{0r} = \phi(r)_{,r} \quad F_{0i} = \overset{=0}{\phi_{0,i}} - \overset{=0}{\phi_{i,0}} = 0$$ A_0 depends on r only + static

$$F_{r\mu} = A_{r,\mu} - A_{\mu,r} \qquad \Rightarrow \quad F_{ri} = a(r)_{,i} - 0 = 0$$ A_r depends on r only

↖ i=θ,φ,0

$$F_{\phi\mu} = 0 = F_{\mu\theta}$$ A_θ, A_ϕ null because of symmetry

$$\Rightarrow \text{All } F_{\mu\nu} = 0 \quad \text{except for: } \boxed{F_{0r} = \phi(r)_{,r} = -F_{r0}}$$ ← radial electric field

$$F = \phi_{,r}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- use local inertial frame ⇒ $g = \mathrm{diag}(-1, 1, 1, 1)$

$$F_{\alpha\beta}F^{\alpha\beta} = F_{0r}F^{0r} + F_{r0}F^{r0} = 2F_{0r}F^{0r} = 2\overset{1}{g^{00}}\overset{-1}{g^{rr}}F_{0r}F_{0r} = -2F_{0r}^2$$

$$F_{0\alpha}F_0{}^\alpha = g^{\alpha\beta}F_{0\alpha}F_{0\beta} = \overset{1}{g^{rr}}F_{0r}F_{0r} = F_{0r}^2 \longrightarrow$$

$$F_{r\alpha}F_r{}^\alpha = g^{\alpha\beta}F_{r\alpha}F_{r\beta} = \overset{-1}{g^{00}}F_{r0}F_{r0} = -F_{r0}^0 \longrightarrow$$

$$\text{All other } F_{\mu\alpha}F_\nu{}^\alpha = 0 \longrightarrow$$

$$\boxed{\begin{aligned} T_{00} &= \frac{1}{4\pi}\left(F_{0r}^2 + \frac{1}{\underset{2}{\cancel{4}}}\overset{-1}{g_{00}}\underset{1}{\cancel{2}}F_{0r}^2\right) = \frac{F_{0r}^2}{8\pi} \\ T_{rr} &= \frac{1}{4\pi}\left(-F_{0r}^2 + \frac{1}{\underset{2}{\cancel{4}}}\overset{1}{g_{rr}}\underset{1}{\cancel{2}}F_{0r}^2\right) = -T_{00} \\ T_{\theta\theta} &= T_{\phi\phi} = \frac{1}{4\pi}\frac{1}{\underset{2}{\cancel{4}}}\overset{1}{g_{\theta\theta}}\underset{1}{\cancel{2}}F_{0r} = \frac{F_{0r}}{8\pi} \end{aligned}}$$

2) Maxwell equations to constrain F_{0r}

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \overset{\text{symmetric}(\mu\beta)}{\Gamma^\mu_{\mu\beta}} F^{\beta\nu} + \underset{=0}{\cancel{\overset{\text{antisym.}}{\Gamma^\nu_{\mu\beta} F^{\mu\beta}}}} = \frac{1}{\cancel{\sqrt{-g}}} \partial_\mu \left(\sqrt{-g}\, F^{\mu\nu}\right) \overset{\text{vacuum}}{=} 0$$

$g_{\mu\nu}$ diagonal

$\Rightarrow \quad -g = g_{00}\, g_{rr}\, r^4 \sin^2\theta$

$\underline{\mu = 0, \theta, \phi}$ no information as identically null: $\partial_0(-) = 0$ static

$\partial_\theta(-) = 0 = \partial_\phi(-)$ no θ, ϕ dependency

$\underline{\mu = r}$
$$\partial_r\left(\sqrt{g_{00} g_{rr}}\, r^2 \sin\theta\, F^{0\nu}\right) = \cancel{\sin\theta}\, \partial_r\left(\sqrt{g_{00} g_{rr}}\, r^2 F^{0\nu}\right) = \partial_r\left(\frac{r^2 F_{0r}}{\sqrt{g_{00} g_{rr}}}\right) = 0$$

only $F^{0r} = g^{00} g^{rr} F_{0r} = \dfrac{\phi_{,r}}{g_{00} g_{rr}}$

integrate: $\dfrac{r^2 \phi_{,r}}{\sqrt{g_{00} g_{rr}}} = K$ $\quad \boxed{F_{0r} = K \dfrac{\sqrt{g_{00} g_{rr}}}{r^2}}$

$r \to \infty$ expect Coulomb $\Rightarrow$ $\left.\begin{array}{l} g_{00} \to 1 \\ g_{rr} \to 1 \end{array}\right\}$ at $r \to \infty$ Minkowski

identify $F_{0r} = \phi_{,r} = E(r) = q \dfrac{\sqrt{g_{00} g_{rr}}}{r^2}$ as radial electric field

$\Rightarrow$ $\underline{K = q}$ total charge of the object

3) Einstein Equations

$G_{\mu\nu}$ as in Schwarzschild (we are using the same symmetry assumptions)

$$\begin{cases} G_{00} = \dfrac{1}{r^2} e^{2A} \dfrac{d}{dr}\left[r\left(1 - e^{-2B}\right)\right] \\ G_{rr} = -\dfrac{1}{r^2} e^{2B}\left(1 - e^{-2B}\right) + \dfrac{2}{r} \dfrac{dA}{dr} \\ G_{\theta\theta} = r^2 e^{-2B}\left(A'' + A'^2 + \dfrac{A'}{r} - A'B' - \dfrac{B'}{r}\right) \\ G_{\phi\phi} = G_{\theta\theta} \sin^2\theta \end{cases}$$

$\dfrac{d}{dr} = {}'' \, ' \, ''$ $\qquad g_{00} = -e^{2A} \qquad g_{rr} = e^{2B}$

$$\begin{cases} G_{00} = \dfrac{8\pi G}{c^4} \overset{\downarrow}{T_{00}} = \dfrac{\cancel{8\pi} G}{c^4} \dfrac{\overset{\downarrow}{F_{0r}^2}}{\cancel{8\pi}} = \dfrac{G}{c^4} \dfrac{q^2 |g_{00} g_{rr}|}{r^4} \\ G_{rr} = \dfrac{8\pi G}{c^4} \overset{\downarrow}{T_{rr}} = -\dfrac{8\pi G}{c^4} T_{00} = -\dfrac{G}{c^4} \dfrac{q^2 |g_{00} g_{rr}|}{r^4} \\ \ldots \end{cases}$$

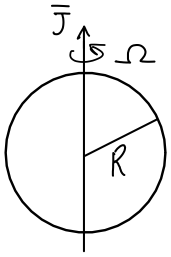
modulus because $F_{0r}^2 > 0$

$\leftarrow$ now only the $g_{\mu\nu}$ are the free functions to be evaluated

$\Rightarrow$ Solve for them $\Rightarrow$ $g_{00} = g_{rr}^{-1} = 1 - \dfrac{2Gm}{rc^2} + \dfrac{G(p^2 + q^2)}{c^4 r^2}$ ✓

Kerr and Kerr-Newman solution

- Spherical matter distribution
- Spinning around one axis
- Now: Axial symmetry around rotation axis, Not spherical!
- Consider $\partial_0 g_{\mu\nu} = 0$ stationary (not static!)



$R = \text{const.}$ $\quad$ $I =$ momentum of inertia

$\Omega = \text{const.}$ angular velocity $\quad$ $\bar{J} =$ angular momentum

e.g. $J = \Omega \overset{\swarrow}{I} = \Omega \int \rho r^2 \sin^2(\theta)\, r^2 dr\, d(\cos\theta)\, d\phi = \frac{8\pi}{5} \rho R^5 \Omega$ for $\rho = \text{const.}$

Solution: Kerr metric (1963!) 48 years Einstein field equations!

$$ds^2 = \left(-\frac{\Delta - a^2 \sin^2\theta}{\beta^2}\right) c^2 dt^2 + \left(\frac{\beta^2}{\Delta}\right) dr^2 + \left(\beta^2\right) d\theta^2 + \left(\frac{(r^2+a^2)^2 - a^2 \Delta \sin^2\theta}{\beta^2} \sin^2\theta\right) d\phi^2 \left(-a \frac{2Gmr\sin^2\theta}{c^2 \beta^2}\right) 2c\, dt\, d\phi$$

$$\Delta(r) = r^2 - \frac{2Gmr}{c^2} + a^2$$

$$\beta^2(r,\theta) = r^2 + a^2 \cos^2\theta$$

$a \equiv J/m \rightarrow$ black hole rotation
$m =$ mass
} 2 parameters only!

- $g_{t\phi} \neq 0$ implies rotation, there is no coord. system in which $g_{t\phi} = 0$
  $\Rightarrow$ frame dragging/gravitomagnetism (eg. Lens-Thirring effect)

- To include charge: $\Delta(r) \longrightarrow \Delta(r) - \frac{G(q^2+p^2)}{c^4}$ Kerr-Newman metric

- for $a = 0$: Schwarzschild metric

- for $r \to \infty$: Minkowski

- for $r \to \infty$ or $m \to 0$ or $a = 0$: $g_{t\phi} \to 0$, i.e. frame dragging disappears

- for $m = 0$, $a \neq 0$: $ds^2 = -c^2 dt^2 + \frac{(r^2 + a^2\cos^2\theta)}{(r^2+a^2)} dr^2 + (r^2 + a^2\cos^2\theta)^2 d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2$

  flat space-time but not standard polar coord.

  $\hookrightarrow$ elliptical coord.: $\begin{cases} x = (r^2+a^2)^{1/2} \sin\theta \cos\phi \\ y = (r^2+a^2)^{1/2} \sin\theta \sin\phi \\ z = r\cos\theta \end{cases}$

  (t, r, θ, ϕ) Boyer-Linquist coordinates

Kerr black-holes: singularities and horizons

(1) $\underline{\beta=0}$: · $g_{tt}=g_{\phi\phi}=g_{t\phi}=\infty$ $\beta^2=r^2+a^2\cos^2\theta=0$ only if $(r=0)$ <u>and</u> $(\theta=\frac{\pi}{2}$ or $a=0)$

· singularity is a point only if $a=0$ (Schwarzschild)

· singularity is a ring, not a point : $\{P \mid \beta^2=r^2+a^2\cos^2\theta=0\}$

· being a ring you can enter in the disk inside the singularity

· inside the ring there is anothe asyntotically flat space-time $\neq$ from ours.... Kerr metric with $r<0$

(2) $\underline{\Delta=0}$: $g_{rr}=\infty$

$$\Delta(r)=r^2-\frac{2Gmr}{c^2}+a^2=0 \quad \Rightarrow \quad \hat{r}_\pm=\frac{Gm}{c^2}\pm\sqrt{\frac{G^2m^2}{c^4}-a^2} \quad \Rightarrow \quad 3 \text{ scenarios}$$

a) $G^2m^2<a^2c^4$: 0 solutions × rare / impossible to realize
↳ rotational energy > total energy

b) $G^2m^2=a^2c^4$: 1 solution $\hat{r}$ unstable case: add some matter and you get (a)

c) $G^2m^2>a^2c^4$: 2 solutions $\hat{r}_+,\hat{r}_-$ possible ← become (c)

<u>Case (c)</u> · discussion similar to Ressner-Nordstrøm solution

· $\hat{r}_\pm$ booth null surfaces ⇒ horizons

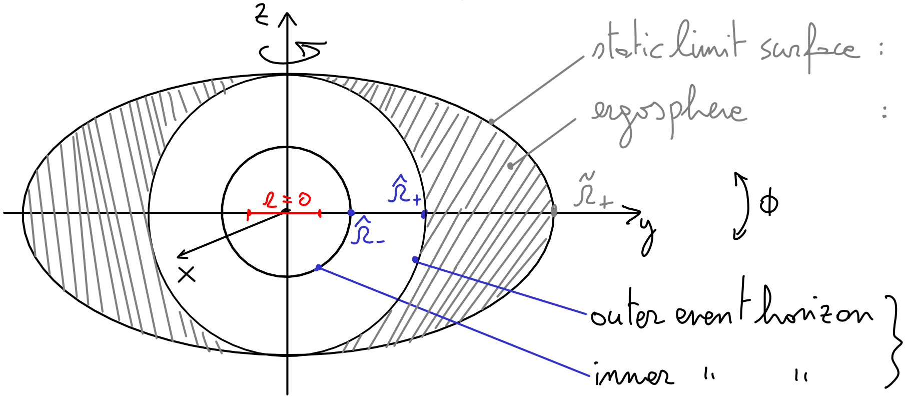
· you can get a coord system with no coordinates singularity

· $a=0 \Rightarrow \hat{r}_+=r_S \quad \hat{r}_-=0$ like Schwarzschild

· no dependency on θ ⇒ sphere

· $\hat{r}_+<\tilde{r}_+$ ⇒ $\hat{r}_+$ horizon contained in ergosphere

<u>Black hole structure</u> "part I"



static limit surface : time-like except at the poles

ergosphere : region in which it is impossible to have a stationary observer

outer event horizon, inner " " } null surfaces Constrain the possibility to cross them

- Killing vectors for Kerr metric

- Two Killing vectors: $\boxed{\bar{\xi} = \partial_t \quad \bar{\eta} = \partial_\phi}$

obvious because metric coefficients have no dependency on t, ϕ $\left(\partial_t g_{\mu\nu} = 0 ,\ \partial_\phi g_{\mu\nu} = 0 \right)$

$\bar{\eta}$ expresses the axial symmetry

$\bar{\xi}$ is not orthogonal to hypersurfaces with $t = \text{const}$, ... and to any other

→ the metric is stationary but not static!

i.e. it is not changing with time but it is spinning (frame dragging)

$$\Rightarrow \boxed{\begin{array}{l} P_\phi = \text{const.} \\ P_0 = \text{const.} \end{array}}$$

Now, investigte constraints on motion of particles

1) Free massive particle initially falling along the radious

- initial radial direction $P_\phi = 0$ => $P_\phi = 0$ always (conservation law $P_\phi = \text{const.}$)

$$p^\phi = g^{\phi\mu} p_\mu = g^{\phi t} P_t + \cancel{g^{\phi\phi} P_\phi}^{=0}$$

$$p^t = g^{t\mu} p_\mu = g^{tt} P_t + \cancel{g^{t\phi} P_\phi}^{=0}$$

$$\Rightarrow \frac{p^\phi}{p^t} = \frac{g^{\phi t}}{g^{tt}} = \frac{d\phi}{c\,dt} \equiv \frac{\omega}{c}$$ angular velocity

$$\Rightarrow \boxed{\omega = c \frac{g^{\phi t}}{g^{tt}}} = \cancel{c} \cdot a \frac{2Gmr\sin^2\theta}{c^{\cancel{2}} \cancel{\beta^2}} \Big/ \frac{\Delta - a^2 \sin^2\theta}{\cancel{\beta^2}}$$

ω of particle ≠ 0!

in the ratio β disappears, ω has same sign of a

i.e. particle is dragged by the rotation of the object

frame dragging/gravitomagnetism

2) Photon, initial tangential trajectory

along ϕ at some radius r $(dr=0)$ in equatorial plane $(\theta = \pi/2)$

$$ds^2 = 0 = g_{tt}\cancel{c^2dt^2} + 2g_{\phi t}\left(\frac{d\phi}{cdt}\right)\cancel{dt} + g_{\phi\phi}\left(\frac{d\phi^2}{c^2dt^2}\right) \Rightarrow \boxed{\frac{d\phi}{cdt} \equiv \frac{\omega_\pm}{c} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}}$$

- Surface of $g_{tt}=0$: 2 solutions

$$g_{tt} \propto \overset{\downarrow}{\Delta} - a^2\sin^2\theta = r^2 - \frac{2Gmr}{c^2} + a^2 - a^2\sin^2\theta = 0 \qquad \tilde{r}_\pm = \frac{Gm}{c^2} \pm \sqrt{\frac{G^2m^2}{c^4} - a^2\cos^2\theta}$$

here:

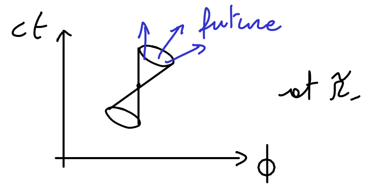
$$\frac{\omega_-}{c} = -2\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a}{2G^2M^2 + a^2}$$

co-rotating with central object: same sign of a
(interpret: photon goes along same direction of object rotation)

$$\frac{\omega_+}{c} = -\frac{g_{t\phi}}{g_{\phi\phi}} + \frac{g_{t\phi}}{g_{\phi\phi}} = 0$$

"at rest" (ϕ coordinate) when seen by an external observer
photon have no angular momentum p^ϕ

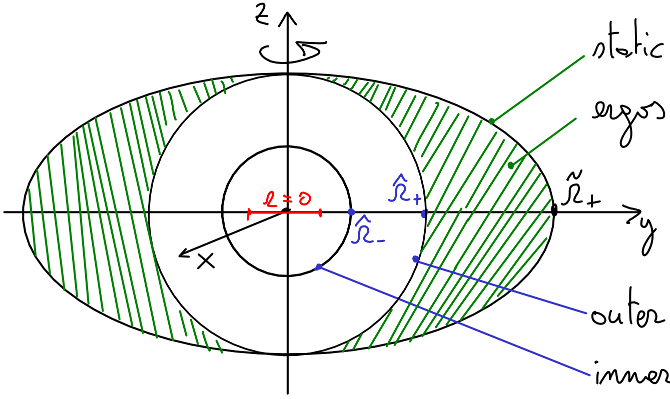
$\boxed{\frac{d\phi}{cdt} = \frac{\omega_\pm}{c}}$ give you the light cone in the $ct-\phi$ plane



$\Rightarrow$ dragging is so strong that all massive particles $(v<c)$ co-rotate with central object
if $r < \tilde{r}_+$ (i.e. $g_{tt}>0$) all particles inside are dragged (ergoregion)

"region", not sphere ... there is a $\cos^2\theta$ dependency

Black hole structure "part I"



static limit surface : time-like except at the poles

ergosphere : region in which it is impossible to have a stationary observer

outer event horizon
inner " "

null surfaces
Constrain the possibility to cross them

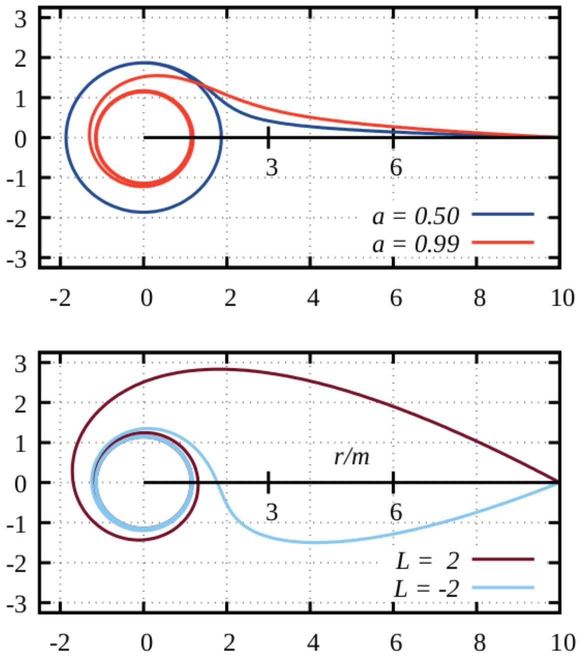


Figure 11.5 Trajectories of test particles in the equatorial plane of the Kerr metric. All orbits begin at $r = 10m$ and $\varphi = 0$. *Top*: Orbits with angular momentum $L = 0$ for $a = 0.5$ and $a = 0.9$. *Bottom*: orbits with angular momenta $L = \pm 2$ for $a = 0.99$.

from: General Relativity (Bartelmann)

Relativistic cosmological model

- Assumptions: homogeneity + isotropy on large scales = cosmological principle
  (observations)

- Non static universe (observations + stationary = fine tuning)

  Hubble law $\bar{v} = H_0 \bar{d}$ $H_0 = \underset{\sim}{70 \pm 4}$ km/s/Mpc $\rightarrow$ distance length $\frac{c}{H_0} = 3000$ Mpc/h

  (it depends to whom you ask...) $\rightarrow$ time length $\frac{1}{H_0} = 10$ Gyr/h

  because of uncertainty on H_0, convenient to use adimentional factor
  $h = H_0/(100 \text{Km/s/Mpc}) \approx 0{,}7$ $\longrightarrow$ scales times in units of h eg. Mpc/h

- Impose symmetries to metric

  (free falling observer)

  1) set time units such: $dt'^2 \equiv -g_{00}(t)\, dt^2$; $g_{00}(t) c^2 dt \rightarrow -c^2 dt$

  2) isotropy: $g_{0i} = 0$ $g_{ij} = 0$ $i \neq j$

     most general g_{ij} isotropic metric: $h_{ij} dx^i dx^j = e^{2B(r)} dr^2 + r^2 d\Omega^2$ } like in Scwarzschild

     $\Rightarrow$ same $G_{\mu\nu}$ of Schwarzschild with $A(r) = 0$

  3) same expansion everywhere: $dl^2 = a^2(t)(h_{ij}\, dx^i dx^j)$ $a(t)$ = scale factor
     (isotropy)
     rescaled by a time dependent factor only
     set distance units such that $a(\text{today}) \overset{!}{=} 1$
     expansion $\Rightarrow a(t) < 1$ for $t < t_{\text{today}}$

  4) homogeneity: $R^i_i = \text{const} \overset{*}{\Longleftrightarrow} G^i_i = 3K = \text{const}$ on $t = \text{const}$ hypersurfaces $i = 1,2,3$

     i.e. Same space curvature on the slice : foliation

     $* \left(G^i_i = R^i_i - \frac{1}{2} \underset{=R}{R}\, \underset{=4}{g^i_i} = -R^i_i \right)$

     $$G^i_i = g^{ij} G_{ij} = e^{-2B} \overset{\swarrow ✩}{G_{rr}} + r^{-2} \overset{\swarrow ✩}{G_{\theta\theta}} + r^{-2} \sin^{-2}\theta\, \overset{\swarrow ✩}{G_{\phi\phi}} = -\frac{1}{r^2}\left[r\left(1 - e^{-2B}\right)\right]' \overset{!}{=} 3K$$

     ✩ same in Schwarzschild but no r dependency on g_{00}

     $$\text{integrate: } \overset{1}{\cancel{r}}\left(1 - \underset{\sim}{e^{-2B}}\right) = -\cancel{3}\frac{K}{\cancel{3}} r^{\cancel{3}\,2} + \frac{C}{r} \qquad e^{2B} = g_{rr} = \left(1 + Kr^2\right)^{-1}$$

     (initial condition: flatness at $r = 0 \Rightarrow C = 0$)
     "on small scales you see no curvature"

⇒ Friedman-Lemaître-Robertson-Walker metric (FLRW)

$$\boxed{ds^2=-c^2dt^2+a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\Omega^2\right)}$$

r,θ,ϕ = comoving coordinates

- $k\in\mathbb{R}$ ($k<0$ $k=0$ $k>0$) but r can always be rescaled such that:

  $\left(k\to\frac{k}{|k|}\quad r\to\sqrt{|k|}\,r\quad a\to\frac{a}{\sqrt{|k|}}\right.$ leave the 4 interval the same)

$$\boxed{k=\begin{cases}0 & \text{flat}\\ 1 & \text{closed / spherical}\\ -1 & \text{open / hyperbolic}\end{cases}}$$

names are from geometry of space surfaces with $t=\text{const}$

- $\underline{k=1}$: define parameter χ: $\boxed{(d\chi)^2\equiv\frac{(dr)^2}{1-r^2}}$ ⇒ $\underline{r=\sin\chi}$ (spherical)

  $d\tilde{l}^2=d\chi^2+\sin^2\chi\,d\Omega^2$ surface element of 3-D sphere embedded in 4D space

  3D volume element: $V=\int\sqrt{\underset{\text{3D det.}}{g_{(3)}}}\,dx^3=\int\sin^2\chi\,d\Omega^2d\chi=4\pi\int_0^\pi\sin^2\chi\,d\chi=2\pi^2$ (finite)

  $(\chi=0\Leftrightarrow r=0)$ $(\chi=\frac{\pi}{2}\Leftrightarrow r=1)$ $(\chi=\pi\Leftrightarrow r=0)$ ⇒ $r\in[0,1]$ closed universe

- $\underline{k=-1}$: as above: $\boxed{(d\chi)^2\equiv\frac{(dr)^2}{1+r^2}}$ ⇒ $\underline{r=\sinh\chi}$ (hyperbolic)

  $V=4\pi\sinh^2\chi\to\infty$ for $\chi\to\infty$

  $(\chi=0\Leftrightarrow r=0)$ $(\chi\to\infty\Leftrightarrow r\to\infty)$ ⇒ $r\in[0,\infty)$ open universe

$$\Rightarrow\ \boxed{ds^2=-c^2dt^2+a^2(t)\left[d\chi^2+f_k^2(\chi)\left(d\theta^2+\sin\theta\,d\phi^2\right)\right]}\qquad f_k(\chi)=\begin{cases}\chi=r & k=0\\ \sin\chi & k=1\\ \sinh\chi & k=-1\end{cases}$$

- ## Cosmological redshift

Redshift: $z\equiv\frac{\lambda_0-\lambda}{\lambda}=\frac{\lambda_0}{\lambda}-1=\frac{\nu}{\nu_0}-1$ o = observed (us, t_0 = today $a_0=1$)

$$ds^2=0=-c^2dt^2+a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\Omega^2\right)=-c^2dt^2+a^2(t)R\qquad R=\text{const.}\ \text{(comoving coord.)}$$

$$\Rightarrow\ \frac{\nu}{\nu_0}=1+z=\frac{dt_0}{dt}=\frac{a(t_0)}{a(t)}=a^{-1}(t)\quad\Rightarrow\quad\boxed{a(z)=(1+z)^{-1}}$$

a = direct observable!

Dynamic of the universe: a(t)

- Find behavejour of $a(t)$ => Solve Einstein field eq.s
- Energy-Momentum tensor: Assume universe filled up by perfect fluid

homogeneous and isotropic => matter remains at rest in comoving coordinates

$T_{\mu\nu} = (\rho + \frac{P}{c^2}) u_\mu u_\nu + P g_{\mu\nu}$ i.e. fixed r, θ, ϕ => $\bar{u} = (c, 0, 0, 0)^T$

=> $T = \mathrm{diag}(\rho c^2, P, P, P)$ $T^\mu{}_\mu = T = -\rho c^2 + 3P$

- Geometry:

1) $\Gamma^0_{11} = \frac{a\dot{a}}{(1-kr^2)c}$ $\Gamma^0_{22} = \frac{a\dot{a}r^2}{c}$ $\Gamma^0_{33} = \Gamma^0_{22} \sin^2\theta$

$\Gamma^i_{0i} = \Gamma^i_{i0} = \frac{\dot{a}}{ac}$ $\Gamma^1_{11} = \frac{-kr}{(1-kr^2)}$ $\Gamma^1_{22} = -r(1-kr^2)$ $\Gamma^1_{33} = \Gamma^1_{22} \sin^2\theta$

$\Gamma^2_{12} = \Gamma^3_{13} = \Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r}$ $\Gamma^2_{33} = -\sin\theta\cos\theta$

$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$ factors $\frac{1}{c}$ come from $\partial_0 = \frac{\partial}{c\partial t}$ $\dot{a} = \frac{da}{dt}$

2) $R_{00} = -3\frac{\ddot{a}}{ac^2}$ $R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{(1-kr^2)c^2}$ $R_{22} = r^2(a\frac{\ddot{a}}{c^2} + 2\frac{\dot{a}^2}{c^2} + 2k)$ $R_{33} = r^2(a\frac{\ddot{a}}{c^2} + 2\frac{\dot{a}^2}{c^2} + 2k)\sin^2\theta$

3) $R = \frac{6}{a^2}(a\frac{\ddot{a}}{c^2} + \frac{\dot{a}^2}{c^2} + k)$

- Einstein eq.s:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

$g_{00} = -1$

(1) $\mu\nu = 00$ $-3\frac{\ddot{a}}{c^2 a} = \frac{8\pi G}{c^4}\left(\rho c^2 - \frac{\rho c^2}{2} + \frac{3}{2}P\right) = \frac{4\pi G}{c^2}\left(\rho + 3\frac{P}{c^2}\right)$ $\frac{\ddot{a}}{ac^2} = -\frac{4\pi G}{3c^2}\left(\rho + 3\frac{P}{c^2}\right)$

(2) $\mu\nu = ii$ all the same because of isotropy (eg. $\mu\nu = 22$) $\frac{\ddot{a}}{ac^2} + 2\left(\frac{\dot{a}}{ac}\right)^2 + \frac{2k}{a^2} = \frac{4\pi G}{c^2}\left(\rho - \frac{P}{c^2}\right)$

plug (1) in (2) =>

$$\boxed{\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) \quad (1) \\ H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (2) \end{aligned}}$$

Friedmann eq.s. ($H \equiv \frac{\dot{a}}{a}$ Hubble function)

- Adiabatic condition:

$E = \rho c^2 V$

$V = V_0 a^3$

$\nabla_\mu T^{\mu 0} = \underbrace{\boxed{\dot{\rho}c^2 + 3H(\rho c^2 + P) = 0}}_{(c)}$ <=> $\frac{d(c^2\rho a^3)}{dt} + p\frac{da^3}{dt} = 0$ i.e. $dE + pdV = 0$

1° law of thermodynamics

i.e. Adiabatic condition

Whit (c) you can obtain Friedmann eq. (1) from Friedmann eq. (2)

=> the 2 Friedmann eq.s are not independent!

- Important quantities:

- Friedmann (2), Set $K \overset{!}{=} 0$ $\Rightarrow$ $\boxed{\rho_c \equiv \frac{3H^2}{8\pi G}}$ critical density: density for which the universe is flat

- $\frac{\text{Friedmann (2)}}{H^2}$: $\boxed{1} = \frac{8\pi G}{3H^2}\rho - \frac{Kc^2}{a^2H^2} = \boxed{\Omega + \Omega_K}$ with $\boxed{\Omega \equiv \rho/\rho_c}$ density contrast

  must hold ∀ time

  $\boxed{\Omega_K \equiv -\frac{Kc^2}{a^2H^2}}$ density contrast of curvature

$\Rightarrow$ condition for flat universe: $K=0 \Rightarrow \Omega_K = 0 \Leftrightarrow \Omega = 1$

- Cosmological constant

Interpret Λ as an ideal fluid

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \Rightarrow T^{\Lambda}_{\mu\nu} = \frac{c^4\Lambda}{8\pi G}g_{\mu\nu}$$

↑

no dependency on observer motion in $T^{\Lambda}_{\mu\nu}$ (there is no $\bar{u}$)

↳ characteristics of empty space $\Rightarrow$ T^{Λ} associated to vacuum energy

$$T^{\Lambda}_{\mu\nu} = \frac{c^4\Lambda}{8\pi G}g_{\mu\nu} \overset{!}{=} \overbrace{\left(\rho + \frac{p}{c^2}\right)}^{\overset{!}{=}0} u_\mu u_\nu + p\, g_{\mu\nu} \Rightarrow \boxed{c^2\rho = -p}^{\,*} \text{ not to have } \bar{u} \text{ dependency}$$

$$= p\, g_{\mu\nu}$$

$$= -\rho c^2 g_{\mu\nu} \Rightarrow \boxed{\rho_\Lambda = \frac{c^2\Lambda}{8\pi G}}$$

* Eq. of state

Multi-component cosmological fluid

$$T_{\mu\nu} = \sum_i (T_{\mu\nu})_i \qquad \text{i-th component}$$

$$= \sum_i \left[\left(\rho_i + \frac{p_i}{c^2}\right) u_\mu u_\nu + p_i g_{\mu\nu}\right]$$

$$= u_\mu u_\nu \left(\sum_i \rho_i + \sum_i \frac{p_i}{c^2}\right) + g_{\mu\nu} \sum_i p_i$$

=> same as a single fluid with $\rho = \sum_i \rho_i$ $p = \sum_i p_i$

Assuming components are not interacting => $D_\mu (T^{\mu\nu})_i = 0$

(conservation holds for each of them independently)

=> independent evolution of densities

Parametric eq. of state

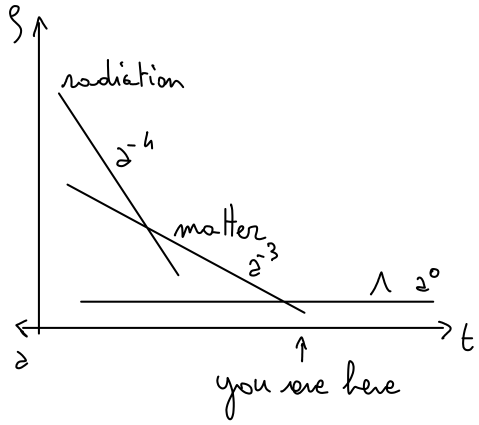
$p = w \rho c^2$ from adiabatic condition (d) => $\rho = \rho_0 a^{-3(1+w)}$

$w = 0$	dust	$\rho_m = \rho_{m0} a^{-3}$	matter
$w = 1/3$	relativistic matter	$\rho_r = \rho_{r0} a^{-4}$	photons, neutrinos
$w = -1$	cosmo. const.	$\rho_\Lambda = \text{const}$	vacuum

General Friedmann eq.

$$H^2 = \frac{8\pi G}{3c^2}\left(\rho_{m0} a^{-3} + \rho_{r0} e^{-4} + \rho_\Lambda - K a^{-2}\right) = H_0^2\left(\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_\Lambda - \Omega_{K0} a^{-2}\right) = H_0^2 E(a)$$

$H_0 \simeq 70$ $\Omega_{m0} \sim 0{,}3$ $\Omega_\Lambda \sim 0{,}7$ $\Omega_{K0} \sim 0$ flat



3 epocs : radiation domination
matter domination
Λ domination (future)

Cosmological distances

Distances depends on how they are performed (defined)

convenient to know: $H \equiv \frac{\dot a}{a}$ $\quad \dot a = a H(a)$

$$\dot a = \frac{da}{dt} \qquad dt = \frac{da}{\dot a} = \frac{da}{a H(a)}$$

- Proper distance: distance covered by a photon in dt

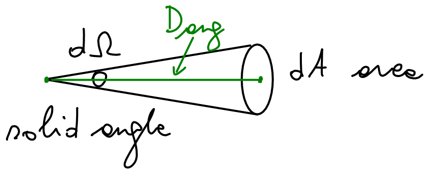
$$dD_p \equiv c\,dt = \frac{c\,da}{a\,H(a)} \qquad \boxed{D_p = \frac{c}{H_0}\int \frac{da}{a E(a)}}$$

- Comoving distance: distance between 2 hypersurfaces for t

$$ds^2 = -c^2dt^2 + a^2dr^2 = 0 \qquad dr = c\frac{dt}{a} = \frac{c\,da}{a^2 H(a)} \qquad \boxed{D_c = \frac{c}{H_0}\int\frac{da}{a^2 E(a)}}$$

- Angular diameter distance: distance obtained by measuring angles

$$dA \equiv d\Omega\, D_{ang}^2$$



$$D_{ang} = \left(\frac{dA}{d\Omega}\right)^{1/2}$$

solid angle →

area

$$\frac{d\Omega}{\cancel{4\pi}} = \frac{dA}{\cancel{4\pi}\, a^2 D_c^2} \qquad \frac{dA}{d\Omega} = a^2 D_c^2 \qquad \Rightarrow \quad \boxed{D_{ang} = a D_c}$$

solid angle of sphere ↗

↖ area of sphere

- Luminosity distance: distance obtained by measuring fluxes F

$$F \equiv \frac{L}{4\pi D_L^2} \quad \left[\frac{J}{s\,m^2}\right]$$

redshift: "stretching of λ" (a_1/a_2)

spatial dilution: $(a_1/a_2)^2$ $\Rightarrow \propto \left(\frac{a_1}{a_2}\right)^4$ on F

delayed arrival time: (a_1/a_2)

$$\boxed{D_L = \left(\frac{a_1}{a_2}\right)^2 D_{ang}(a_1, a_2)}$$

⇐ D_{ang} between a_1 and a_2

Explicit computations

- Constrain on spatial curvature (page 1)

G same as in Schwarzschild
but not r dependency in $g_{00}(t)$

$$G^i_{\ i} = g^{ij}G_{ij} = e^{-2B}G_{rr} + r^{-2}G_{\theta\theta} + r^{-2}\cancel{\sin^{-2}}\theta\, \overset{\swarrow \cancel{\sin^2\theta}\, G_{\theta\theta}}{G_{\phi\phi}} \qquad i=1,2,3$$

$$= e^{-2B}\overset{\swarrow}{G_{rr}} + 2r^{-2}\overset{\swarrow}{G_{\theta\theta}}$$

$$= -\cancel{e^{-2B}}\left(\frac{1}{r^2}\cancel{e^{2B}}(1-e^{-2B}) + \frac{2}{r}\overset{=0}{\cancel{\frac{dA}{dr}}}\right) + \cancel{r^{-2}}\,2\,\cancel{r^2}\,e^{-2B}\left(\overset{=0}{\cancel{A''}} + \overset{=0}{\cancel{A'^2}} + \overset{=0}{\cancel{\frac{A'}{r}}} - \overset{=0}{\cancel{A'B'}} - \frac{B'}{r}\right) \qquad \frac{d}{dr} = \text{“}\,'\,\text{”}$$

$$= -\frac{1}{r^2}(1-e^{-2B}) - 2e^{-2B}\frac{B'}{r} = -\frac{1}{r^2}\left[r(1-e^{-2B})\right]' \overset{!}{=} 3k \text{ const.}$$

Christoffel symbols: (page 3)

$$\Gamma^a_{\ bc} = \frac{1}{2}g^{a\gamma}(\partial_b g_{\gamma c} + \partial_c g_{b\gamma} - \partial_\gamma g_{cb}) \overset{\downarrow g \text{ diag.}}{=} \frac{1}{2}g^{aa}(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{cb}) \qquad i,j = 1,2,3 \qquad \mu,\nu = 0,1,2,3$$

$$\Gamma^0_{\ ij} = \frac{1}{2}g^{0\gamma}(\partial_i g_{\gamma j} + \partial_j g_{i\gamma} - \partial_\gamma g_{ij}) = \frac{1}{2}g^{00}(\partial_i \overset{=0}{\cancel{g_{0j}}} + \partial_j \overset{=0}{\cancel{g_{i0}}} - \partial_0 g_{ij}) = \cancel{+}\frac{1}{2}\overset{=-1}{\cancel{g^{00}}}\partial_0 g_{ij}$$

$$\Gamma^0_{\ ij} = 0 \quad i \neq j \qquad \Gamma^0_{\ 11} = \frac{1}{2}\frac{\partial}{c\partial t}\left(\frac{a^2}{(1-kr^2)}\right) = \frac{\cancel{2}a\dot{a}}{\cancel{2}c(1-kr^2)} \qquad \Gamma^0_{\ 22} = \frac{1}{2}\partial_0 a^2 = \frac{a\dot{a}}{c}r^2 \qquad \Gamma^0_{\ 33} = \Gamma^0_{\ 22}\sin^2\theta$$

$$\Gamma^1_{\ 11} = \frac{1}{2}g^{11}(\partial_1 g_{11} + \cancel{\partial_1 g_{11}} - \cancel{\partial_1 g_{11}}) = \frac{1}{2}\frac{(1-kr^2)}{a^2}\frac{\partial}{\partial r}\left[\frac{a^2}{(1-kr^2)}\right] = \frac{1}{\cancel{2}}\frac{\cancel{(1-kr^2)}}{\cancel{a^2}}\left[\cancel{a^2}\frac{-\cancel{2}kr}{(1-kr^2)^{\cancel{2}}}\right] = \frac{-kr}{(1-kr^2)}$$

.....

- Adiabatic condition (page 3)

$$D_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u^\mu u^\nu + P g^{\mu\nu} \qquad \swarrow \text{for metric compatibility } D_\gamma g^{\mu\nu} = 0$$

$$u^\mu u^\nu D_\mu\left(\rho + \frac{P}{c^2}\right) + \left(\rho + \frac{P}{c^2}\right)\left[u^\nu D_\mu u^\mu + u^\mu D_\mu u^\nu\right] + g^{\mu\nu}D_\mu P = 0 \qquad \rho(t) \quad D_\mu P = 0$$

$$\Big\downarrow \times u_\nu$$

$$-c^2u^\mu D_\mu\left(\rho + \frac{P}{c^2}\right) + \left(\rho + \frac{P}{c^2}\right)\left[-c^2 D_\mu u^\mu + \overset{\circledast 1}{0}\right] + u_\nu g^{\mu\nu}D_\mu P = 0$$

$$-u^\mu D_\mu(\rho c^2 + \cancel{P}) - (\rho c^2 + P)\overset{\swarrow}{D_\mu} u^\mu + \cancel{u^\mu D_\mu P} = 0$$

$$u^\mu \overset{\downarrow \rho = \text{scalar}}{\partial_\mu}(\rho c^2) + (\rho c^2 + P)(\partial_\mu \overset{\downarrow}{u^\mu} + \overset{\downarrow}{\Gamma^\mu_{\ \mu\gamma}} \overset{\downarrow}{u^\gamma}) = 0 \qquad u^\mu = \partial_0 x^\mu = \delta^\mu_0 \qquad \Gamma^\mu_{\ \mu\gamma}\delta^\gamma_0 = \Gamma^\mu_{\ \mu 0} = \frac{\dot{a}}{ac}$$

$$\partial_0(\rho c^2) + (\rho c^2 + P)\left(0 + 3\frac{\dot{a}}{ac}\right) = 0$$

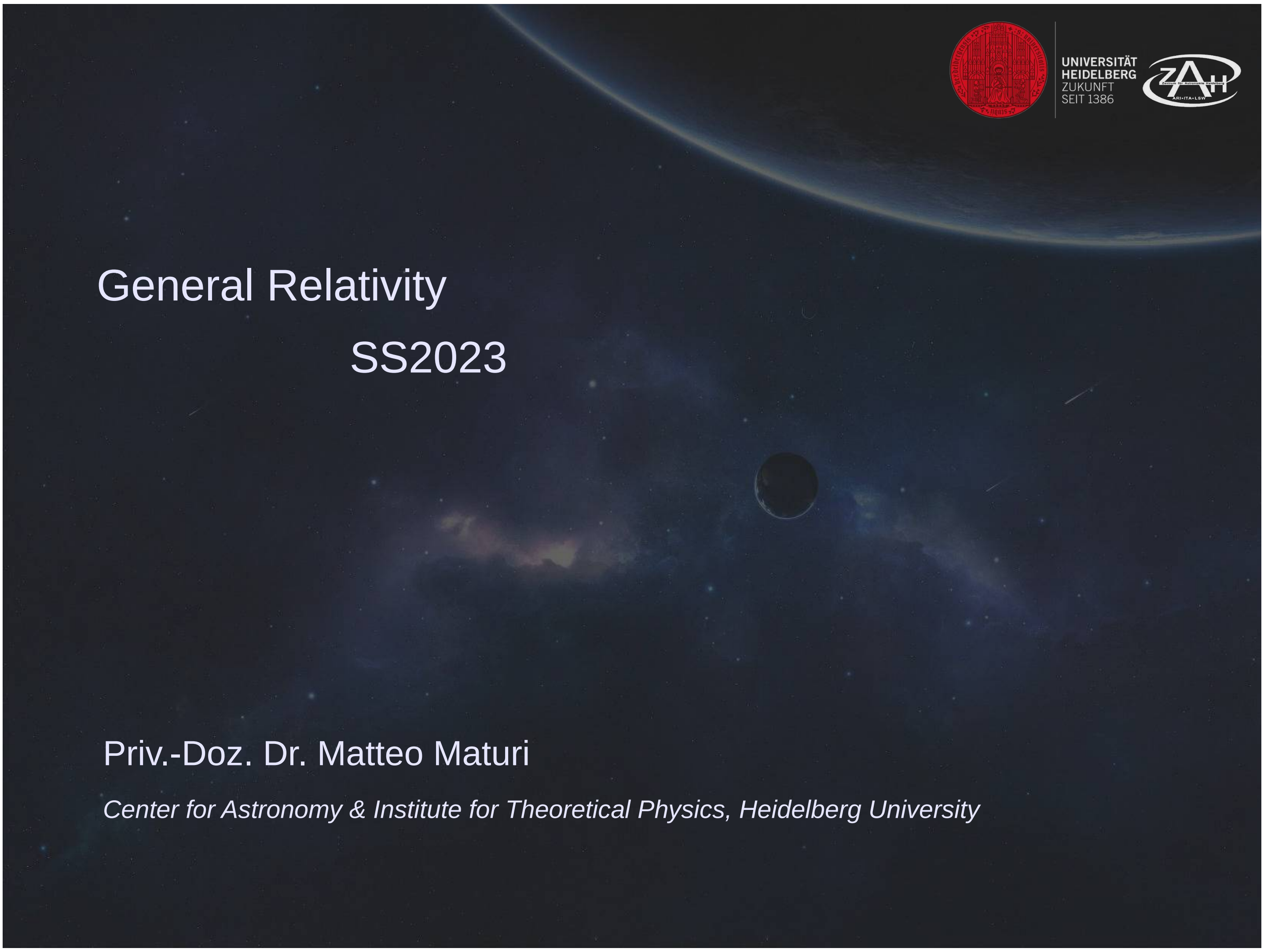
$$\dot{\rho}c^2 + 3\frac{\dot{a}}{a\cancel{c}}(\rho c^2 + P) = 0 \qquad \boxed{\dot{\rho}c^2 + 3H(\rho c^2 + P) = 0}$$

$$\circledast 1 \quad u_\nu D_\mu u^\nu + u^\nu D_\mu u_\nu = 2u_\nu D_\mu u^\nu = D_\mu(u^\nu u_\nu) = D_\mu(-c^2) = 0$$

- <u>Density of fluids</u> (p. 4) from adiabat. condition

$$\frac{d(c^2\rho a^3)}{dt} + \overset{\searrow}{p}\frac{da^3}{dt} = 0 \qquad d(\cancel{c^2}\rho a^3) + w\rho\cancel{c^2}\, da^3 = 0 \qquad \cancel{a^3}\, d\rho + 3\rho\cancel{a^2}\frac{da}{a} + w\rho\, 3\cancel{a^2}\frac{da}{a} = 0$$

$$\frac{d\rho}{\rho} = -3\cancel{\rho}(1+w)\frac{da}{a} \qquad \log\rho = -3(1+w)\log a + C \qquad \boxed{\rho = \rho_0\, a^{-3(1+w)}}$$



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General Relativity

SS2023

Priv.-Doz. Dr. Matteo Maturi

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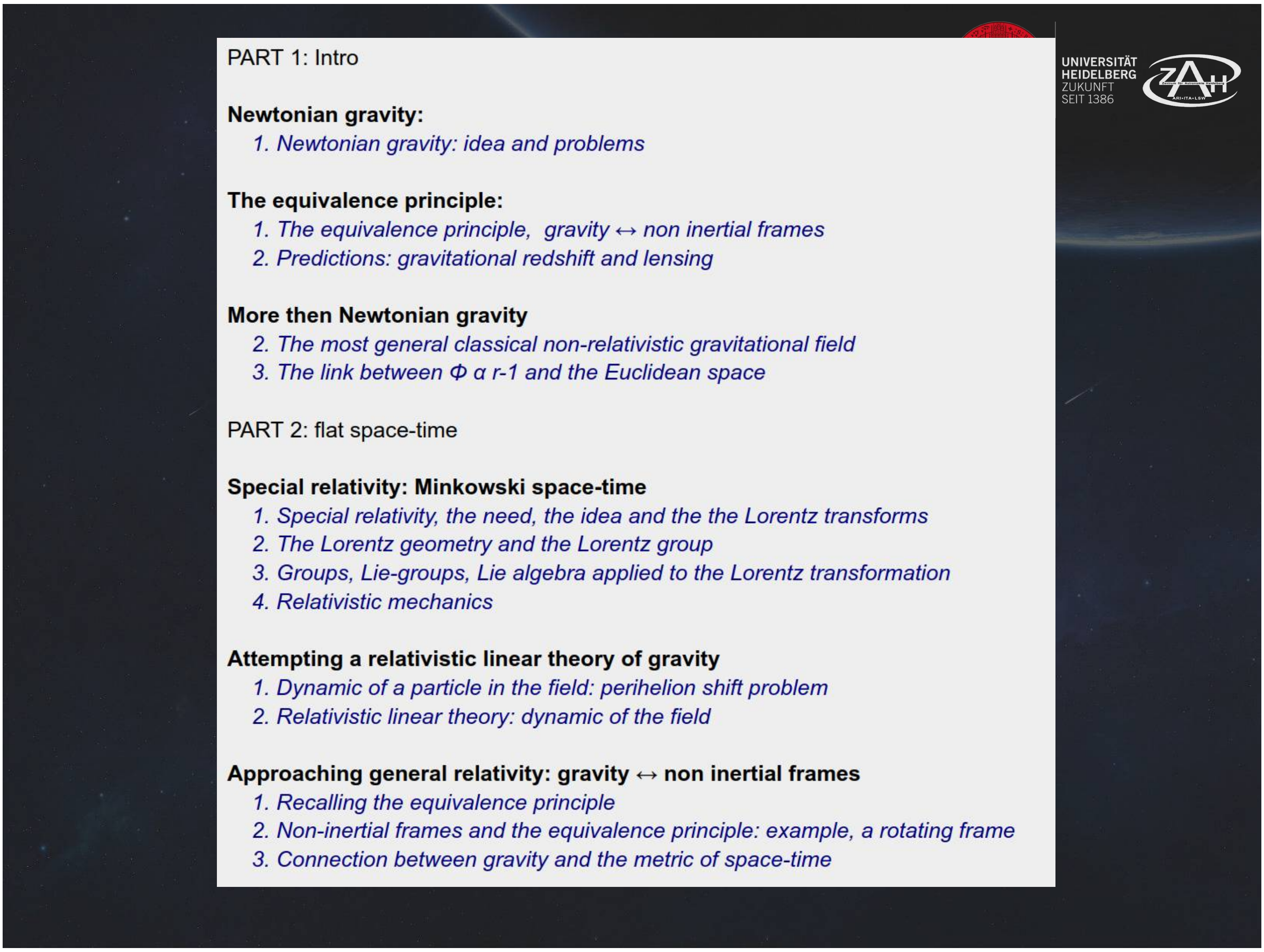


PART 1: Intro

Newtonian gravity:

1. *Newtonian gravity: idea and problems*

The equivalence principle:

1. *The equivalence principle, gravity ↔ non inertial frames*
2. *Predictions: gravitational redshift and lensing*

More then Newtonian gravity

2. *The most general classical non-relativistic gravitational field*
3. *The link between Φ α r-1 and the Euclidean space*

PART 2: flat space-time

Special relativity: Minkowski space-time

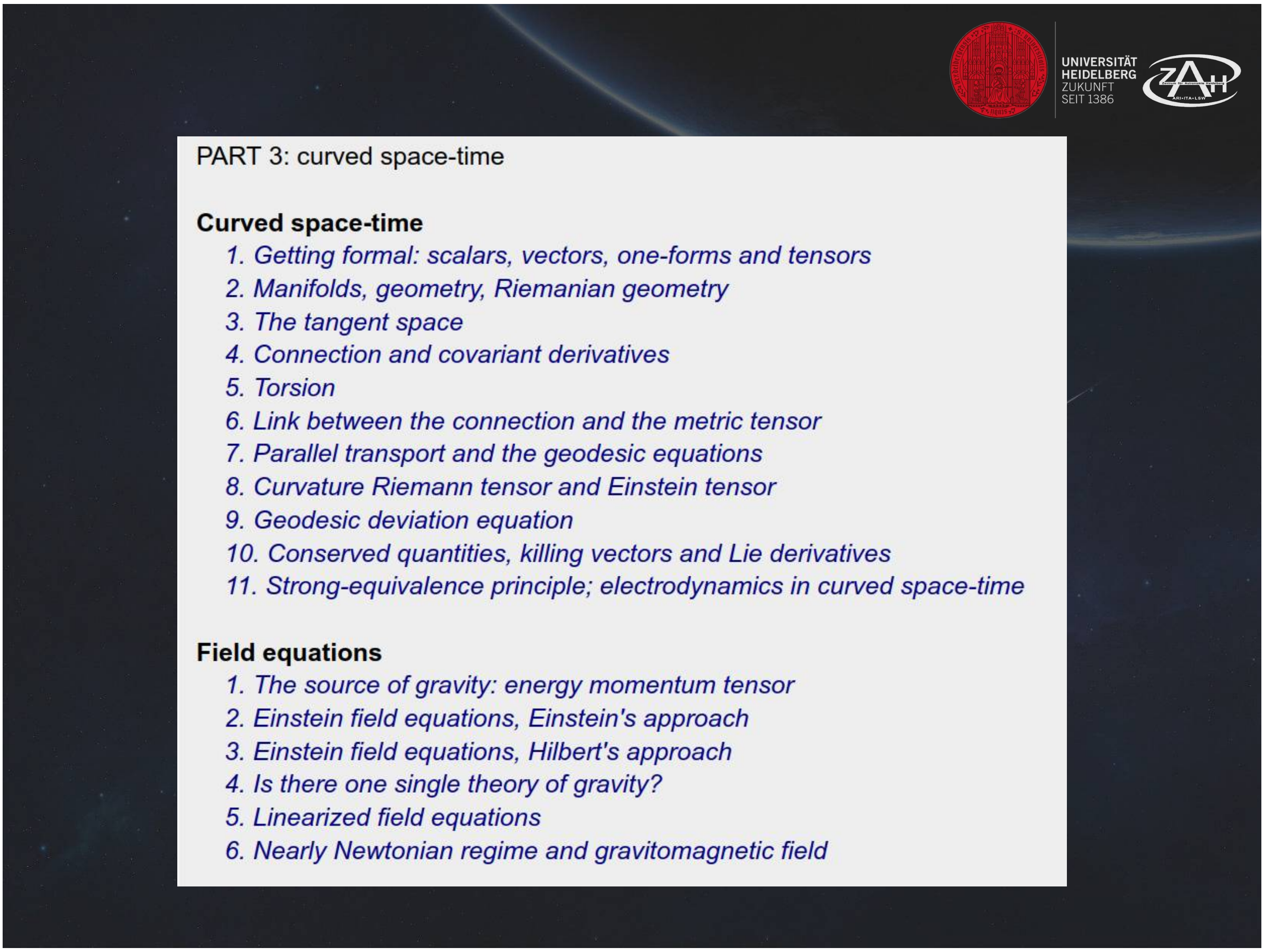
1. *Special relativity, the need, the idea and the the Lorentz transforms*
2. *The Lorentz geometry and the Lorentz group*
3. *Groups, Lie-groups, Lie algebra applied to the Lorentz transformation*
4. *Relativistic mechanics*

Attempting a relativistic linear theory of gravity

1. *Dynamic of a particle in the field: perihelion shift problem*
2. *Relativistic linear theory: dynamic of the field*

Approaching general relativity: gravity ↔ non inertial frames

1. *Recalling the equivalence principle*
2. *Non-inertial frames and the equivalence principle: example, a rotating frame*
3. *Connection between gravity and the metric of space-time*



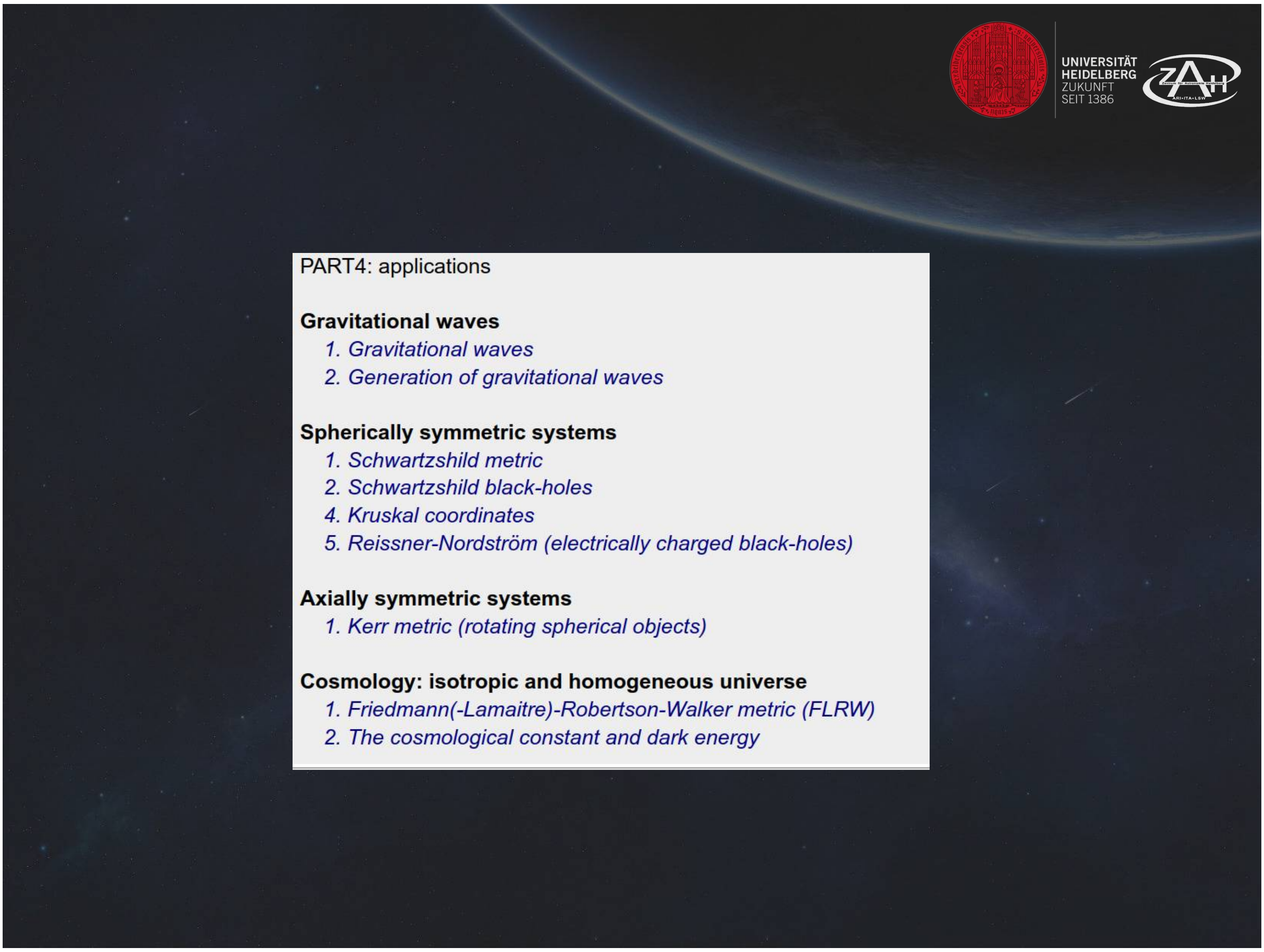
PART 3: curved space-time

Curved space-time

1. *Getting formal: scalars, vectors, one-forms and tensors*
2. *Manifolds, geometry, Riemanian geometry*
3. *The tangent space*
4. *Connection and covariant derivatives*
5. *Torsion*
6. *Link between the connection and the metric tensor*
7. *Parallel transport and the geodesic equations*
8. *Curvature Riemann tensor and Einstein tensor*
9. *Geodesic deviation equation*
10. *Conserved quantities, killing vectors and Lie derivatives*
11. *Strong-equivalence principle; electrodynamics in curved space-time*

Field equations

1. *The source of gravity: energy momentum tensor*
2. *Einstein field equations, Einstein's approach*
3. *Einstein field equations, Hilbert's approach*
4. *Is there one single theory of gravity?*
5. *Linearized field equations*
6. *Nearly Newtonian regime and gravitomagnetic field*



PART4: applications

Gravitational waves

1. *Gravitational waves*
2. *Generation of gravitational waves*

Spherically symmetric systems

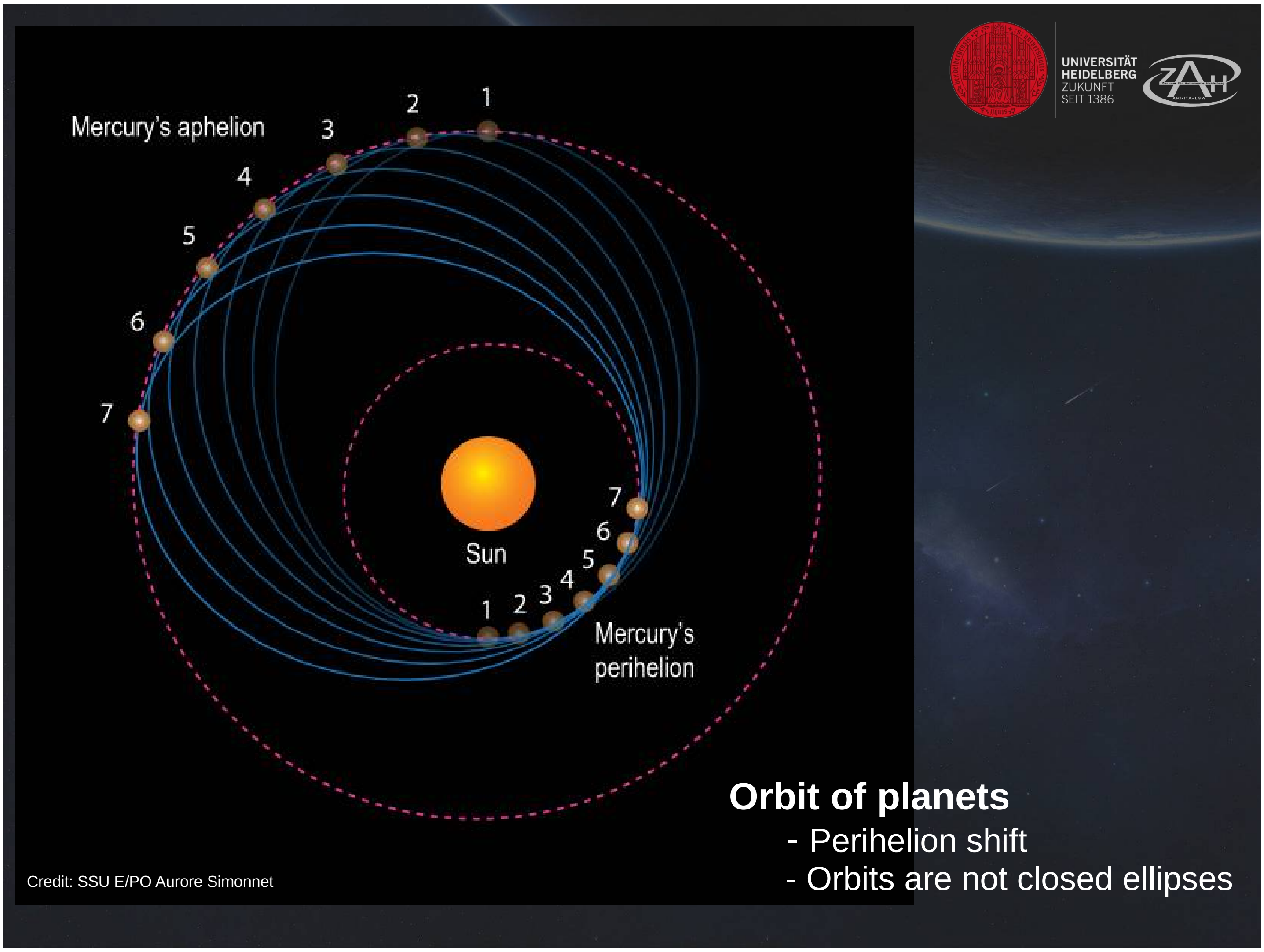
1. *Schwartzshild metric*
2. *Schwartzshild black-holes*
4. *Kruskal coordinates*
5. *Reissner-Nordström (electrically charged black-holes)*

Axially symmetric systems

1. *Kerr metric (rotating spherical objects)*

Cosmology: isotropic and homogeneous universe

1. *Friedmann(-Lamaitre)-Robertson-Walker metric (FLRW)*
2. *The cosmological constant and dark energy*



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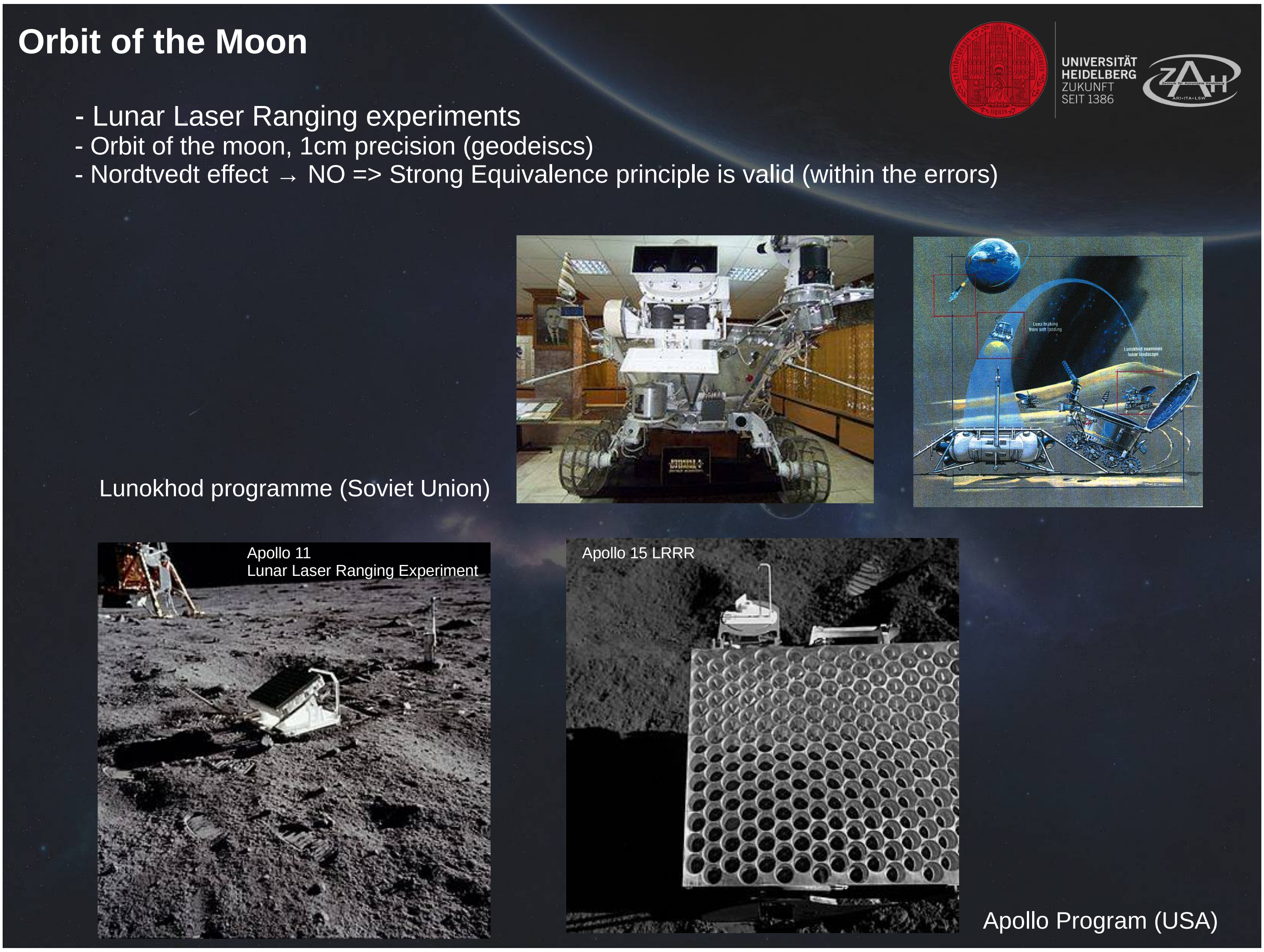
Credit: SSU E/PO Aurore Simonnet

Orbit of planets

- Perihelion shift
- Orbits are not closed ellipses



Orbit of the Moon

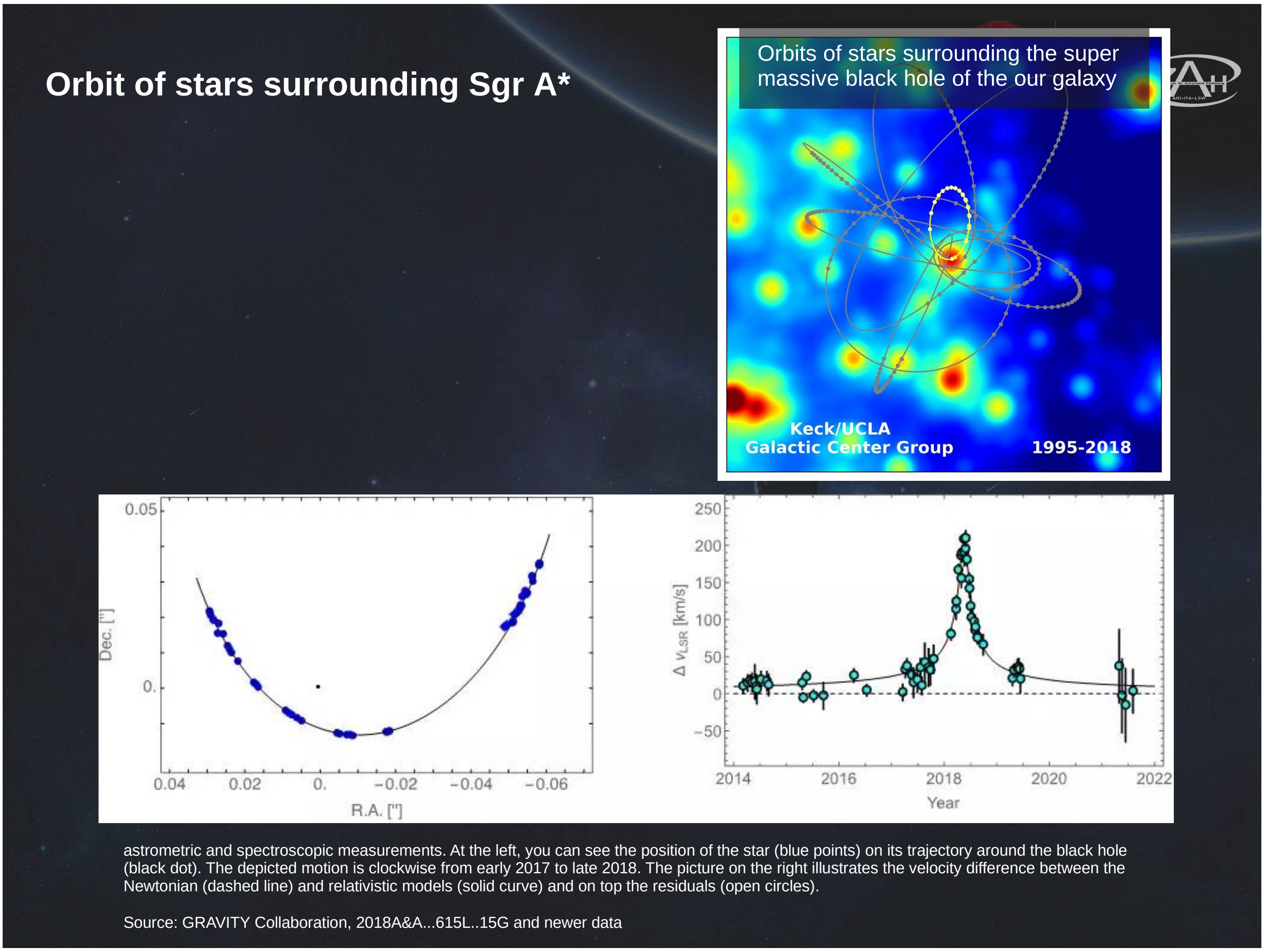
- Lunar Laser Ranging experiments
- Orbit of the moon, 1cm precision (geodeiscs)
- Nordtvedt effect → NO => Strong Equivalence principle is valid (within the errors)

Lunokhod programme (Soviet Union)

Apollo 11
Lunar Laser Ranging Experiment

Apollo 15 LRRR

Apollo Program (USA)



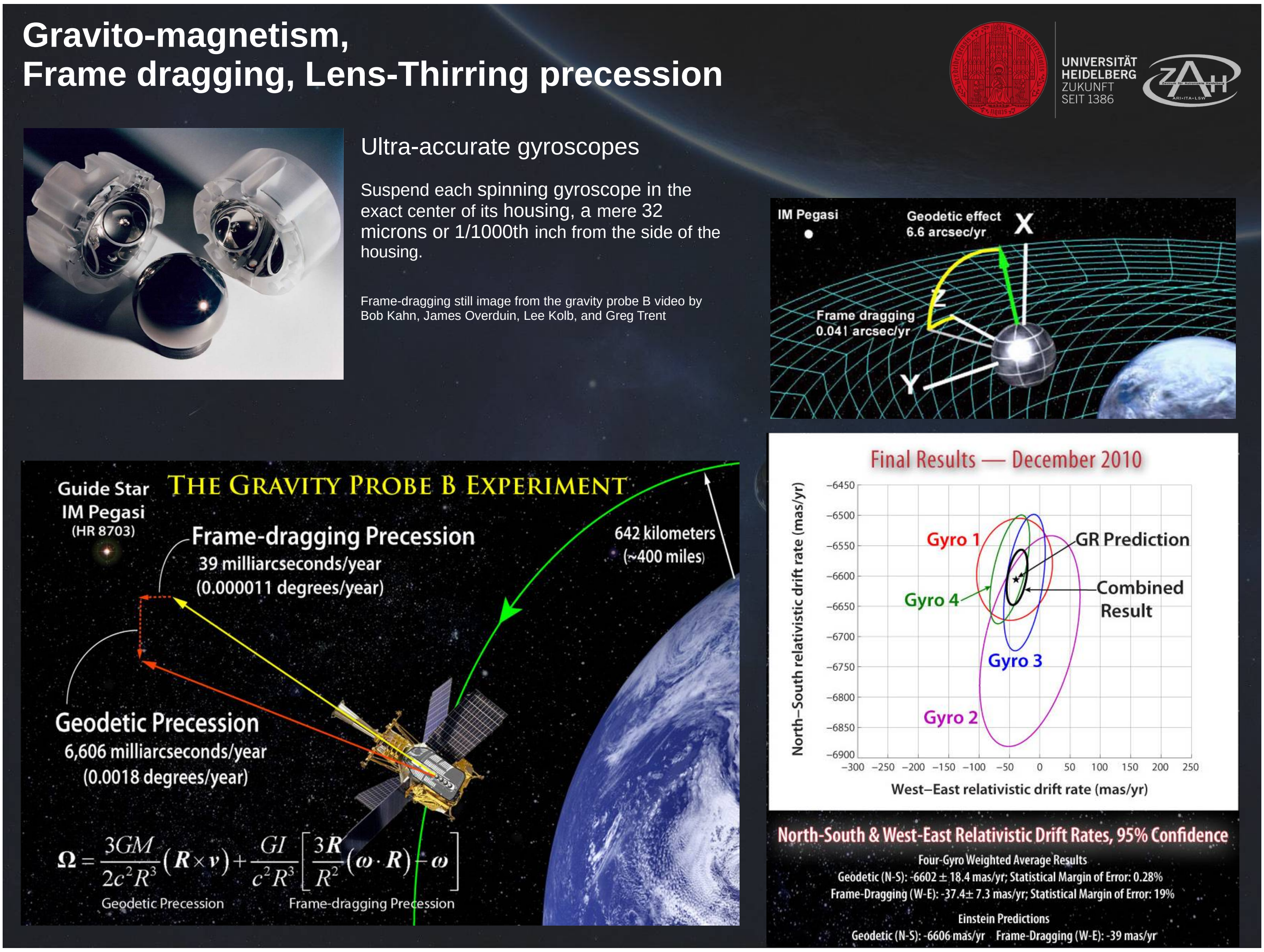
Orbit of stars surrounding Sgr A*

Orbits of stars surrounding the super massive black hole of the our galaxy

Keck/UCLA Galactic Center Group **1995-2018**

astrometric and spectroscopic measurements. At the left, you can see the position of the star (blue points) on its trajectory around the black hole (black dot). The depicted motion is clockwise from early 2017 to late 2018. The picture on the right illustrates the velocity difference between the Newtonian (dashed line) and relativistic models (solid curve) and on top the residuals (open circles).

Source: GRAVITY Collaboration, 2018A&A...615L..15G and newer data



Gravito-magnetism,
Frame dragging, Lens-Thirring precession

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Ultra-accurate gyroscopes

Suspend each spinning gyroscope in the exact center of its housing, a mere 32 microns or 1/1000th inch from the side of the housing.

Frame-dragging still image from the gravity probe B video by Bob Kahn, James Overduin, Lee Kolb, and Greg Trent

THE GRAVITY PROBE B EXPERIMENT

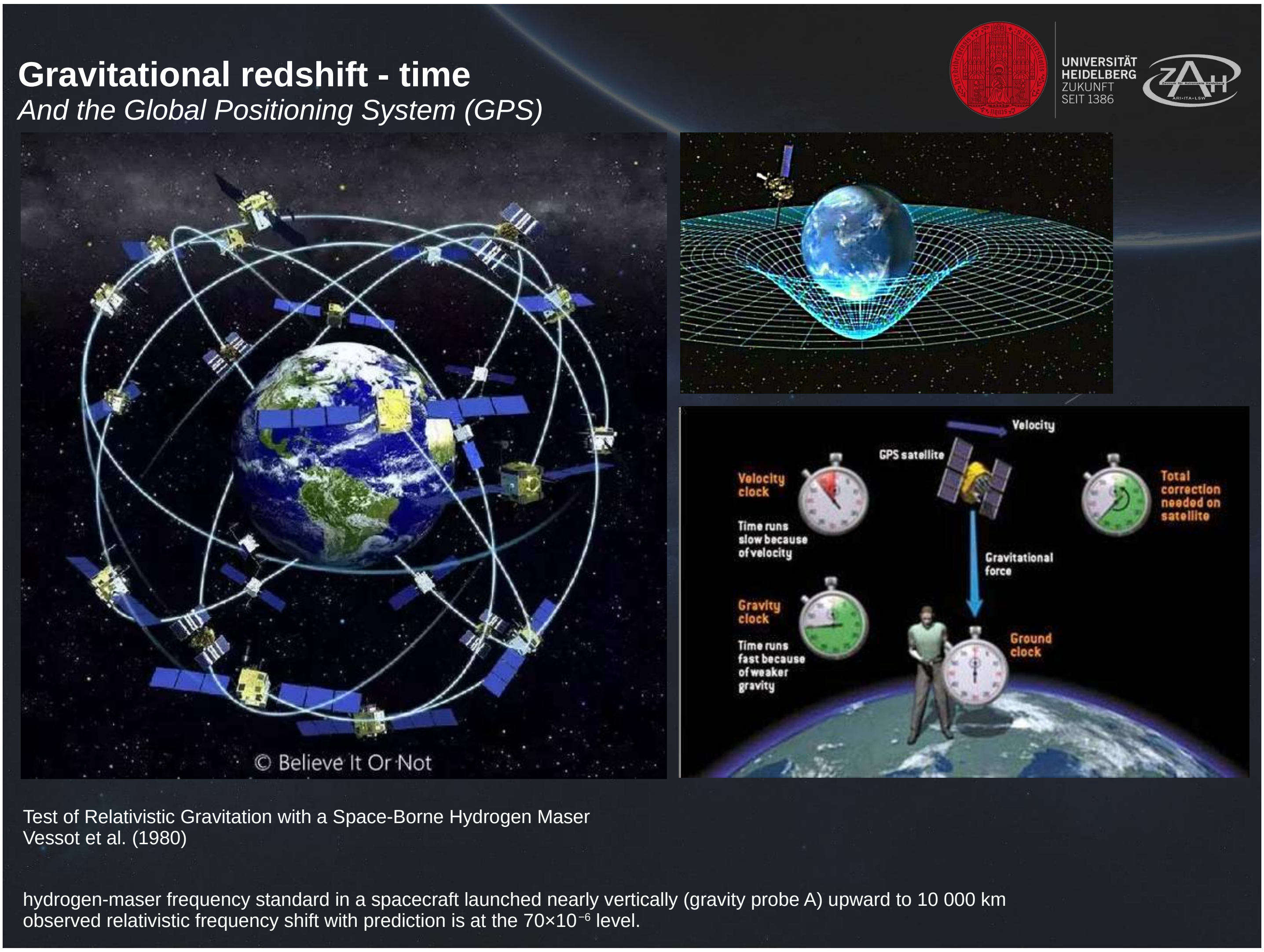
$$\mathbf{\Omega} = \underbrace{\frac{3GM}{2c^2R^3}(\boldsymbol{R}\times\boldsymbol{v})}_{\text{Geodetic Precession}} + \underbrace{\frac{GI}{c^2R^3}\left[\frac{3\boldsymbol{R}}{R^2}(\boldsymbol{\omega}\cdot\boldsymbol{R})-\boldsymbol{\omega}\right]}_{\text{Frame-dragging Precession}}$$

Final Results — December 2010

North-South & West-East Relativistic Drift Rates, 95% Confidence

Four-Gyro Weighted Average Results
Geodetic (N-S): -6602 ± 18.4 mas/yr; Statistical Margin of Error: 0.28%
Frame-Dragging (W-E): -37.4± 7.3 mas/yr; Statistical Margin of Error: 19%

Einstein Predictions
Geodetic (N-S): -6606 mas/yr Frame-Dragging (W-E): -39 mas/yr

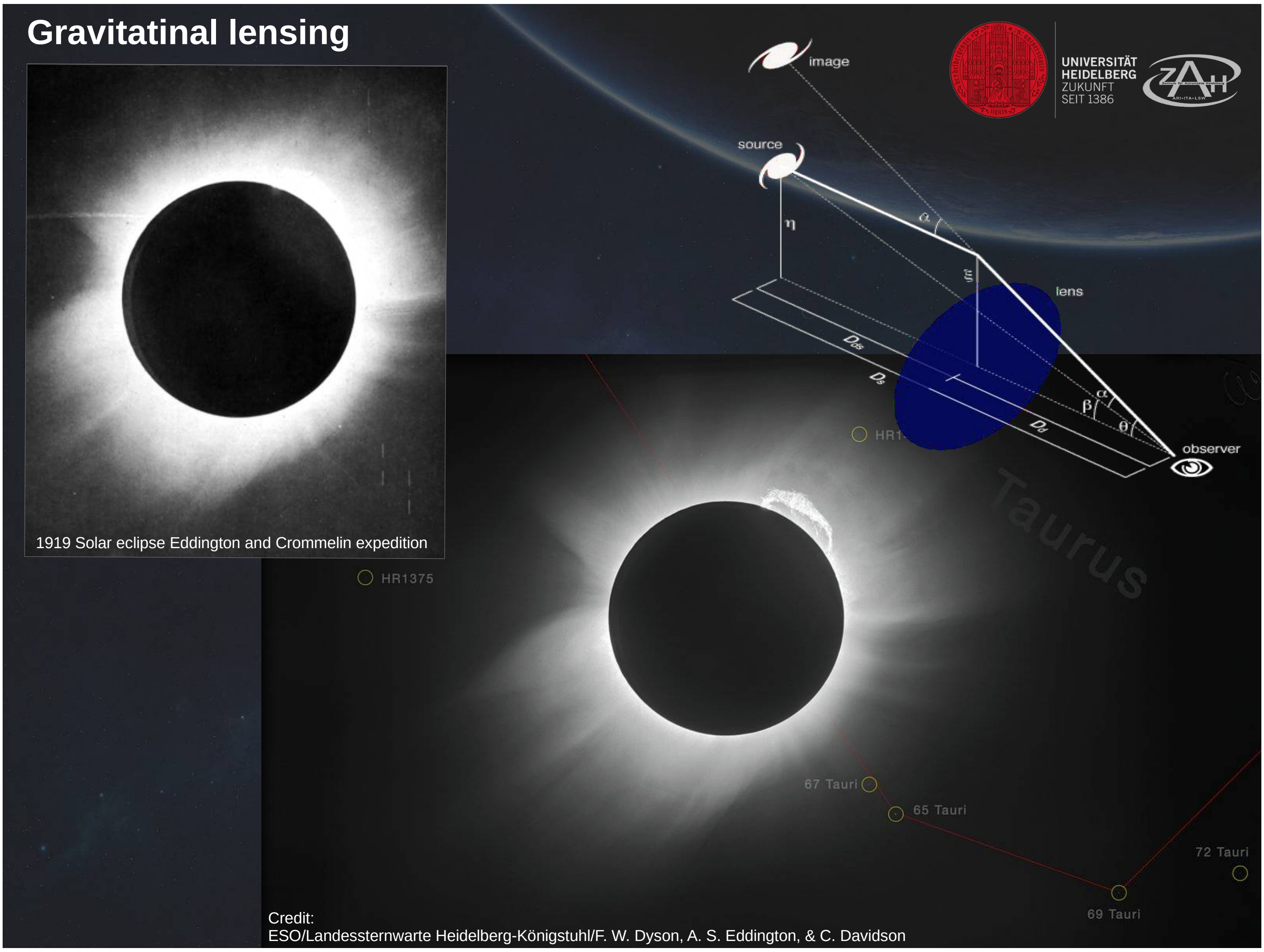


Gravitational redshift - time

And the Global Positioning System (GPS)

Test of Relativistic Gravitation with a Space-Borne Hydrogen Maser
Vessot et al. (1980)

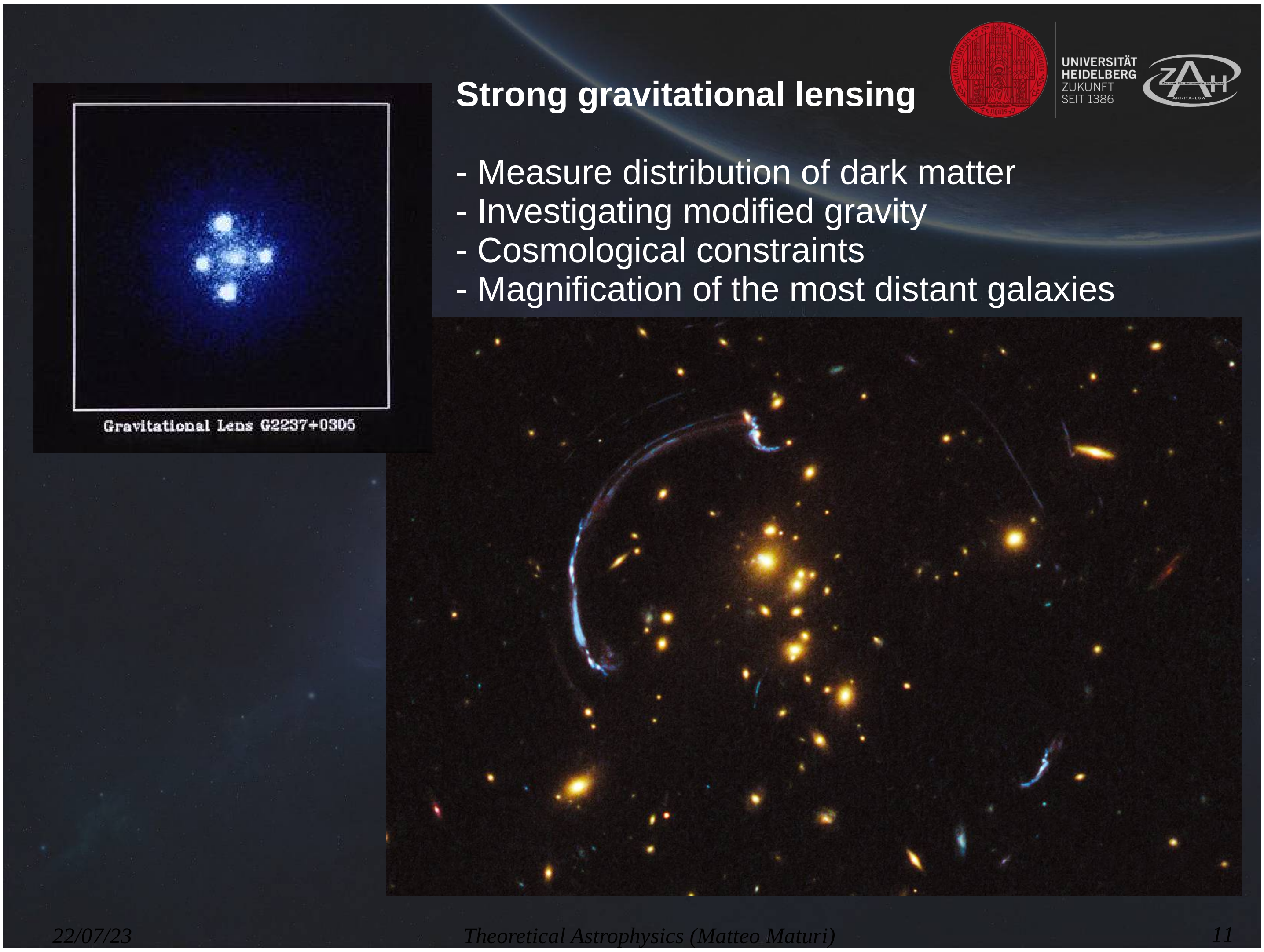
hydrogen-maser frequency standard in a spacecraft launched nearly vertically (gravity probe A) upward to 10 000 km
observed relativistic frequency shift with prediction is at the 70×10^{-6} level.



Gravitatinal lensing

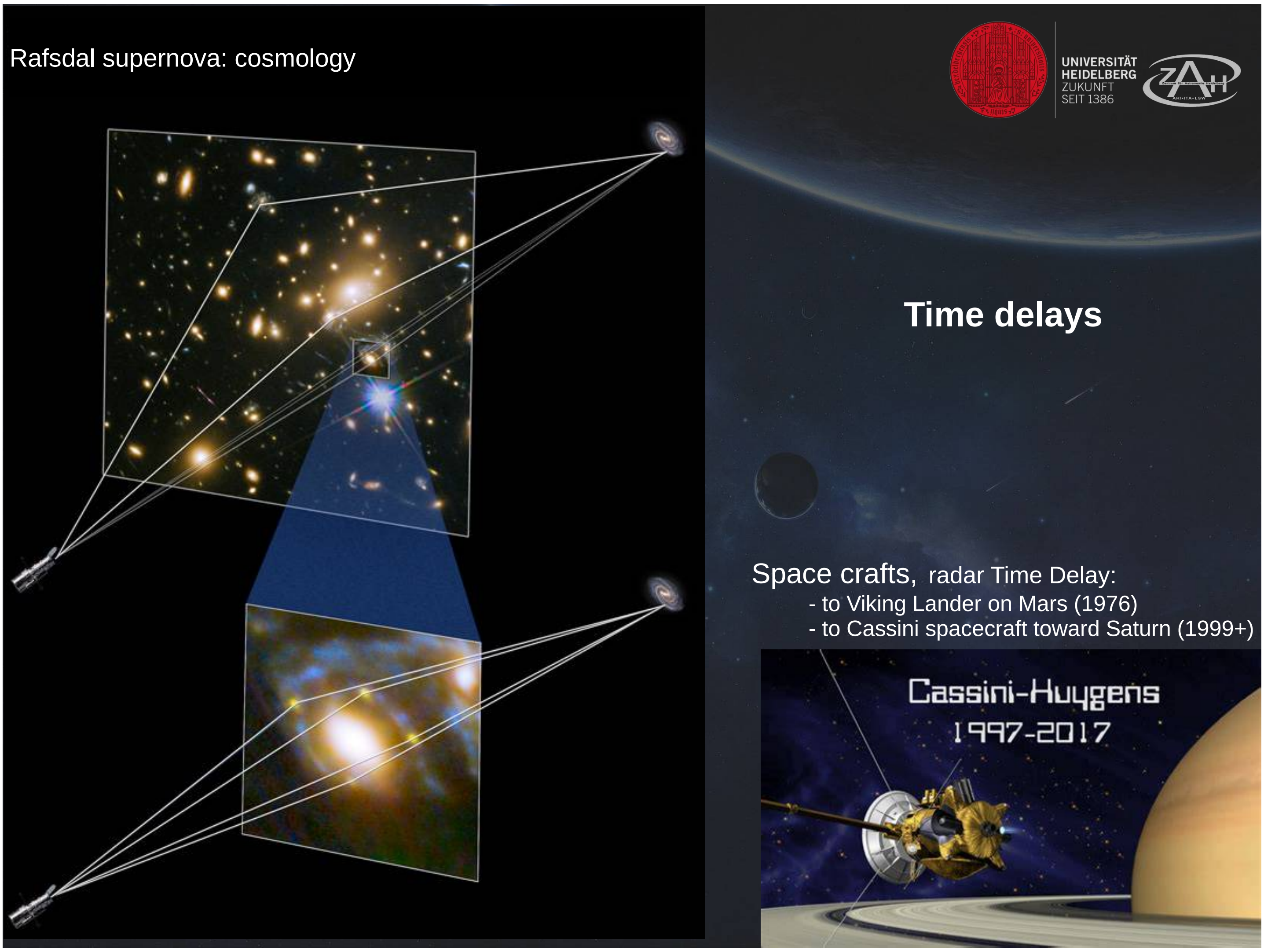
1919 Solar eclipse Eddington and Crommelin expedition

Credit:
ESO/Landessternwarte Heidelberg-Königstuhl/F. W. Dyson, A. S. Eddington, & C. Davidson



Strong gravitational lensing

- Measure distribution of dark matter
- Investigating modified gravity
- Cosmological constraints
- Magnification of the most distant galaxies



Rafsdal supernova: cosmology

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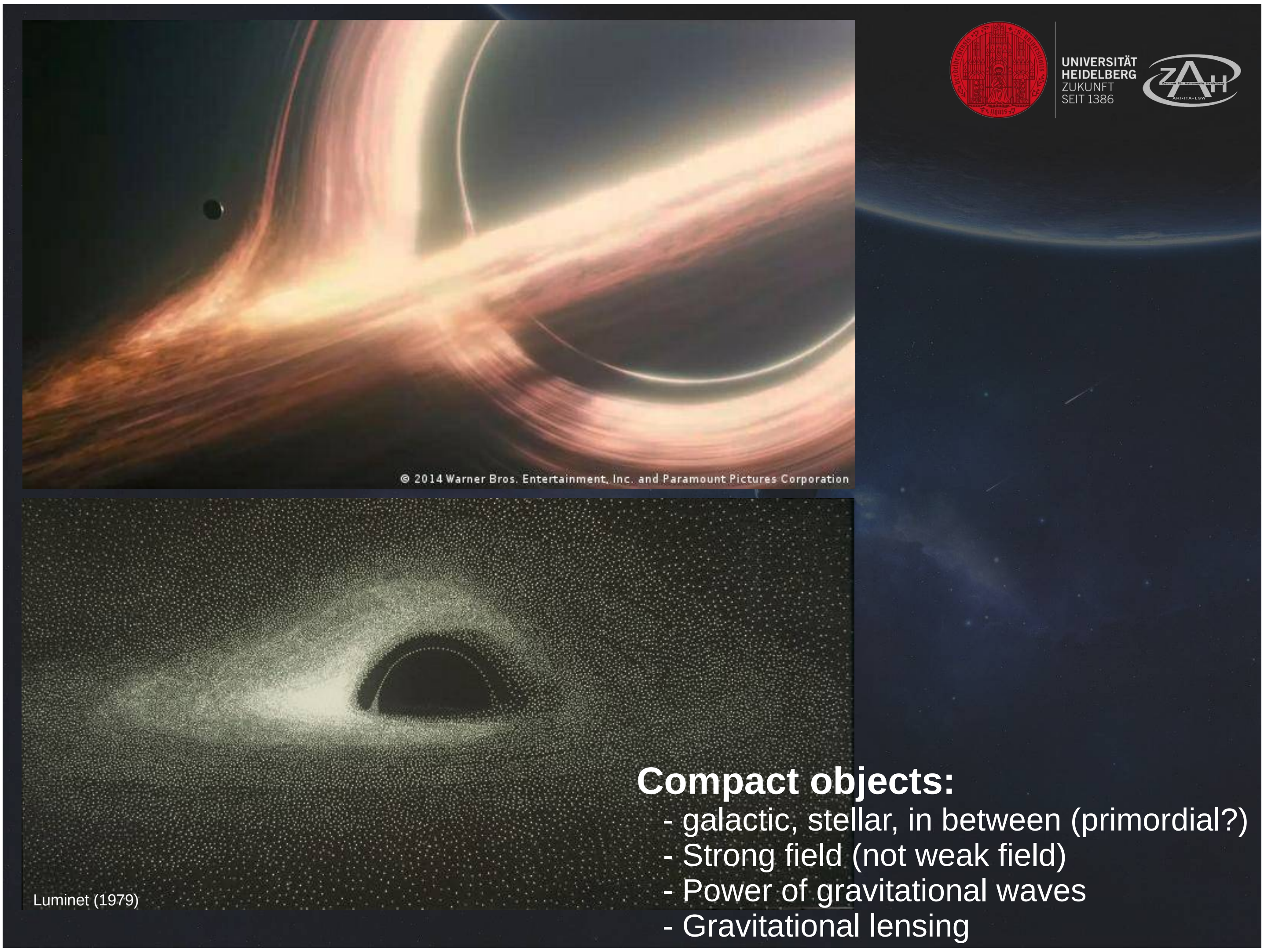
Time delays

Space crafts, radar Time Delay:
- to Viking Lander on Mars (1976)
- to Cassini spacecraft toward Saturn (1999+)



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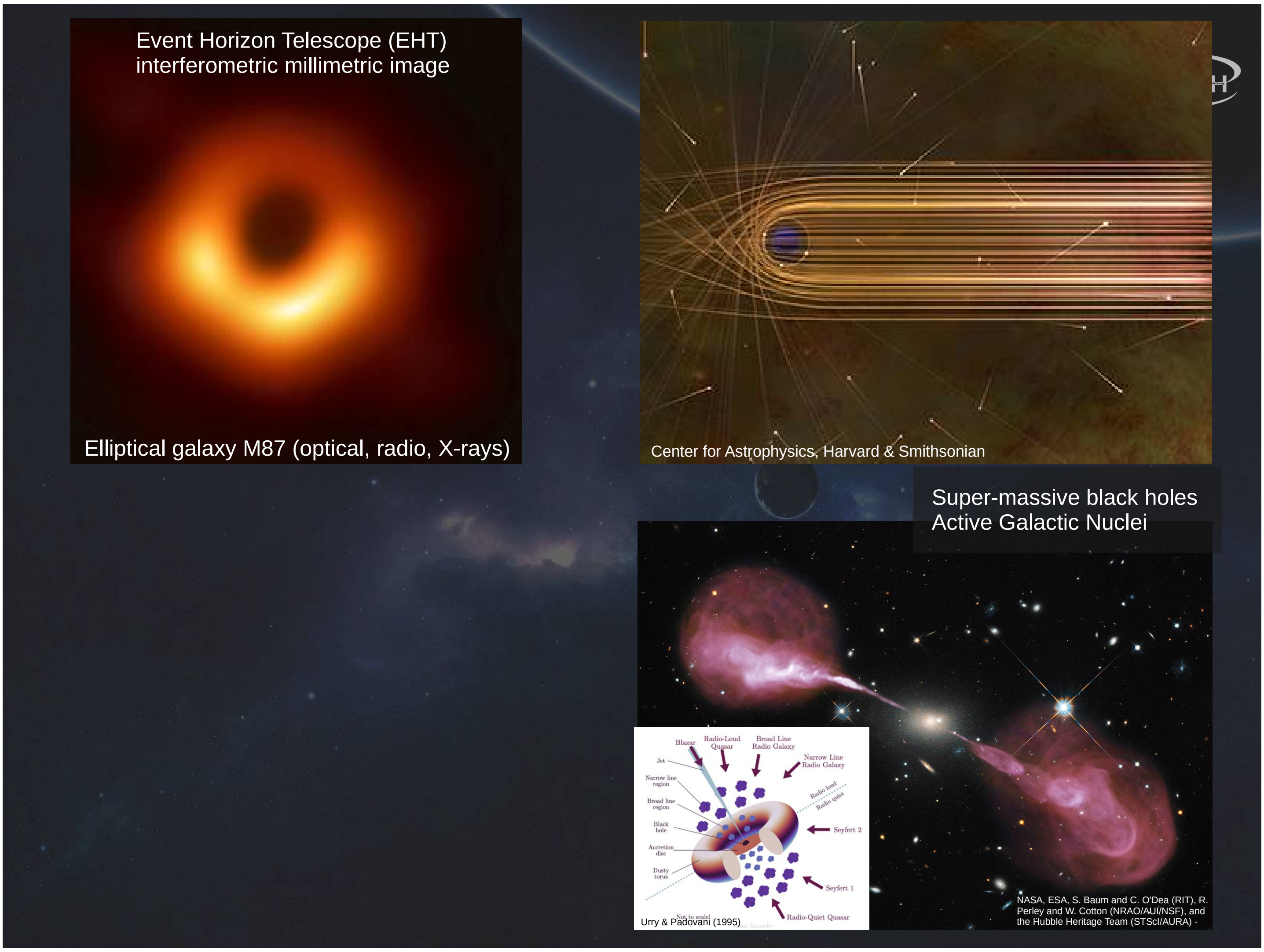
© 2014 Warner Bros. Entertainment, Inc. and Paramount Pictures Corporation

Luminet (1979)

Compact objects:

- galactic, stellar, in between (primordial?)
- Strong field (not weak field)
- Power of gravitational waves
- Gravitational lensing



Event Horizon Telescope (EHT)
interferometric millimetric image

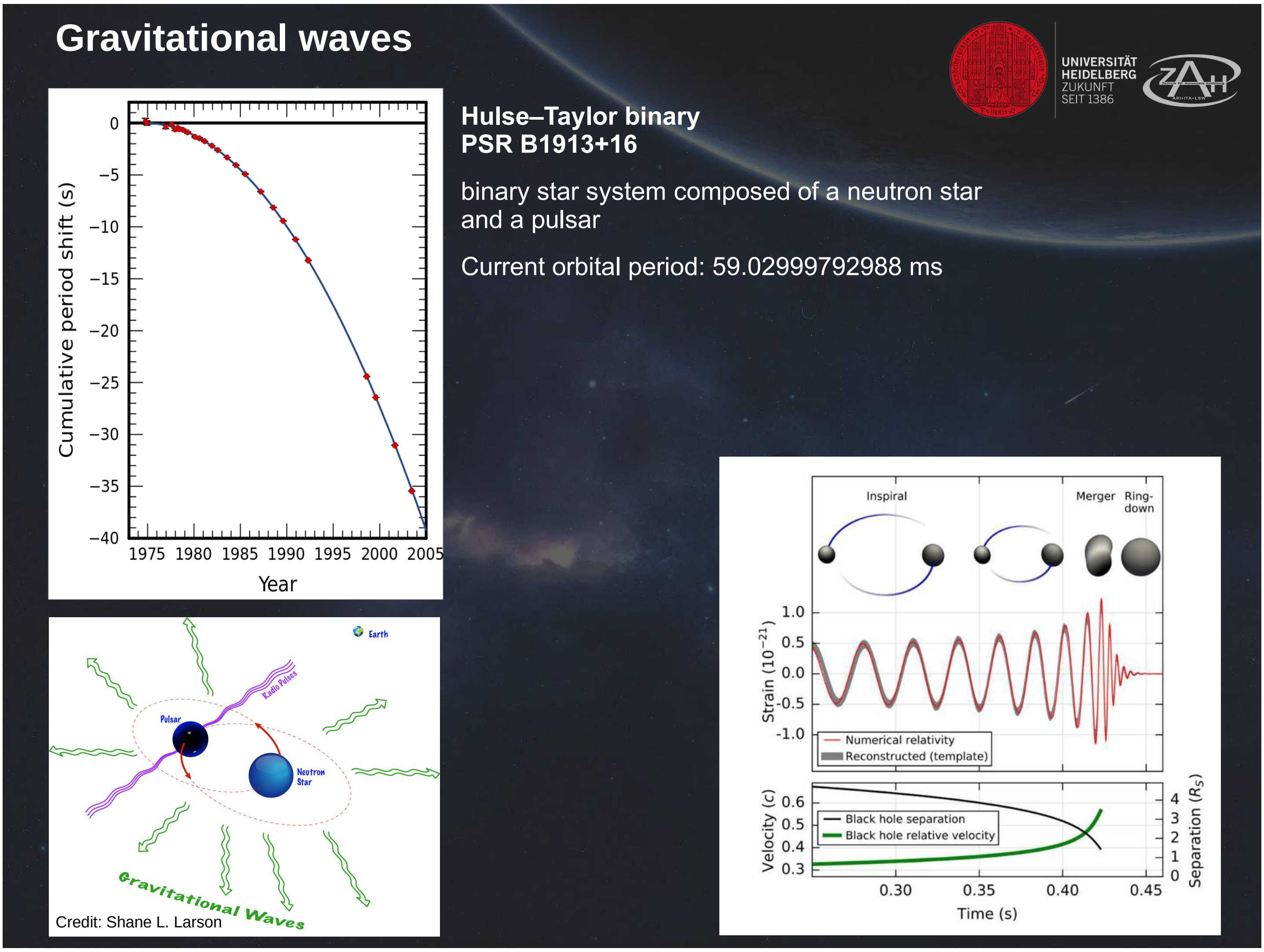
Elliptical galaxy M87 (optical, radio, X-rays)

Center for Astrophysics, Harvard & Smithsonian

Super-massive black holes
Active Galactic Nuclei

Urry & Padovani (1995)

NASA, ESA, S. Baum and C. O'Dea (RIT), R. Perley and W. Cotton (NRAO/AUI/NSF), and the Hubble Heritage Team (STScI/AURA) -



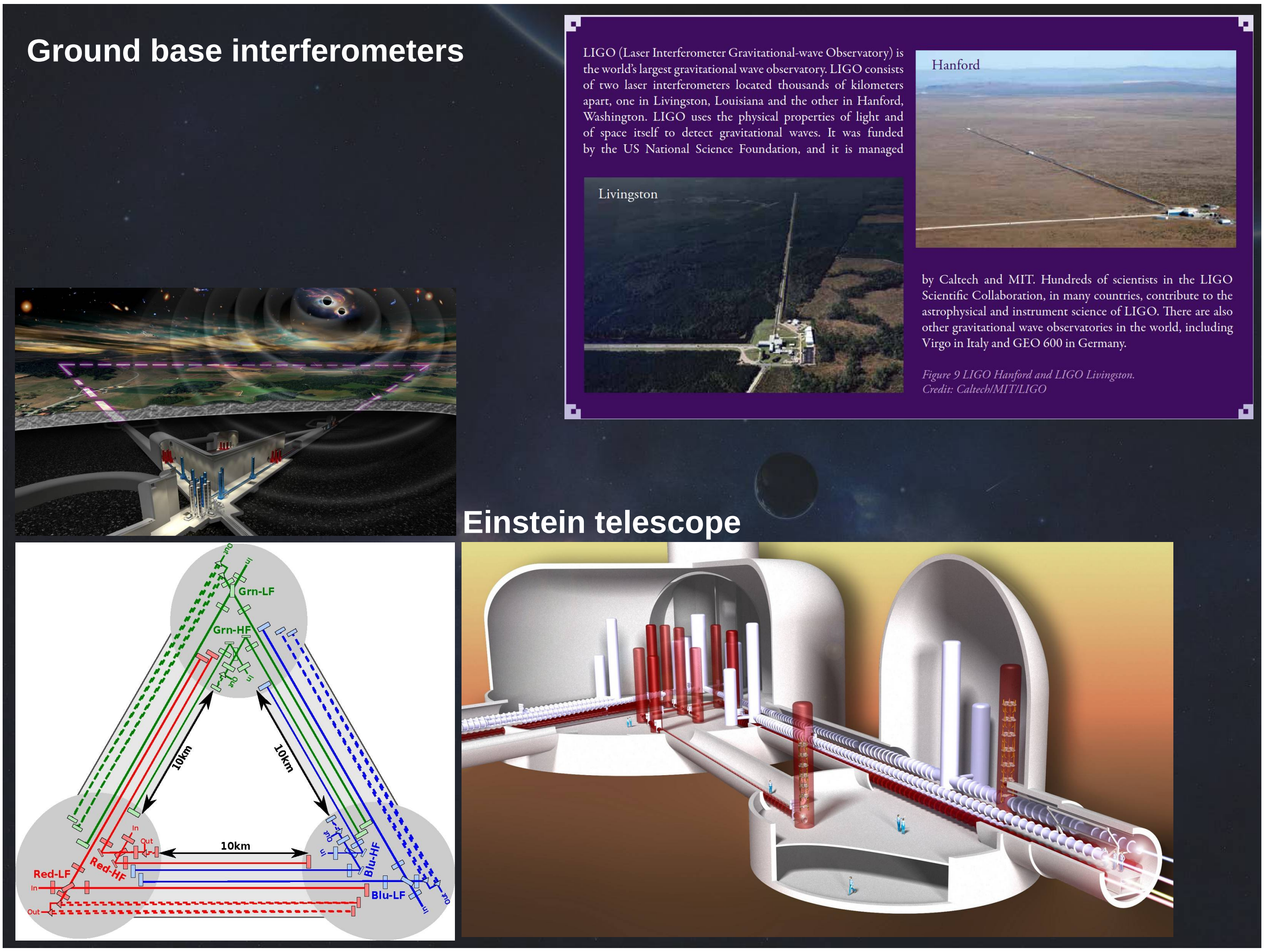
Gravitational waves

Hulse–Taylor binary
PSR B1913+16

binary star system composed of a neutron star and a pulsar

Current orbital period: 59.02999792988 ms

Credit: Shane L. Larson

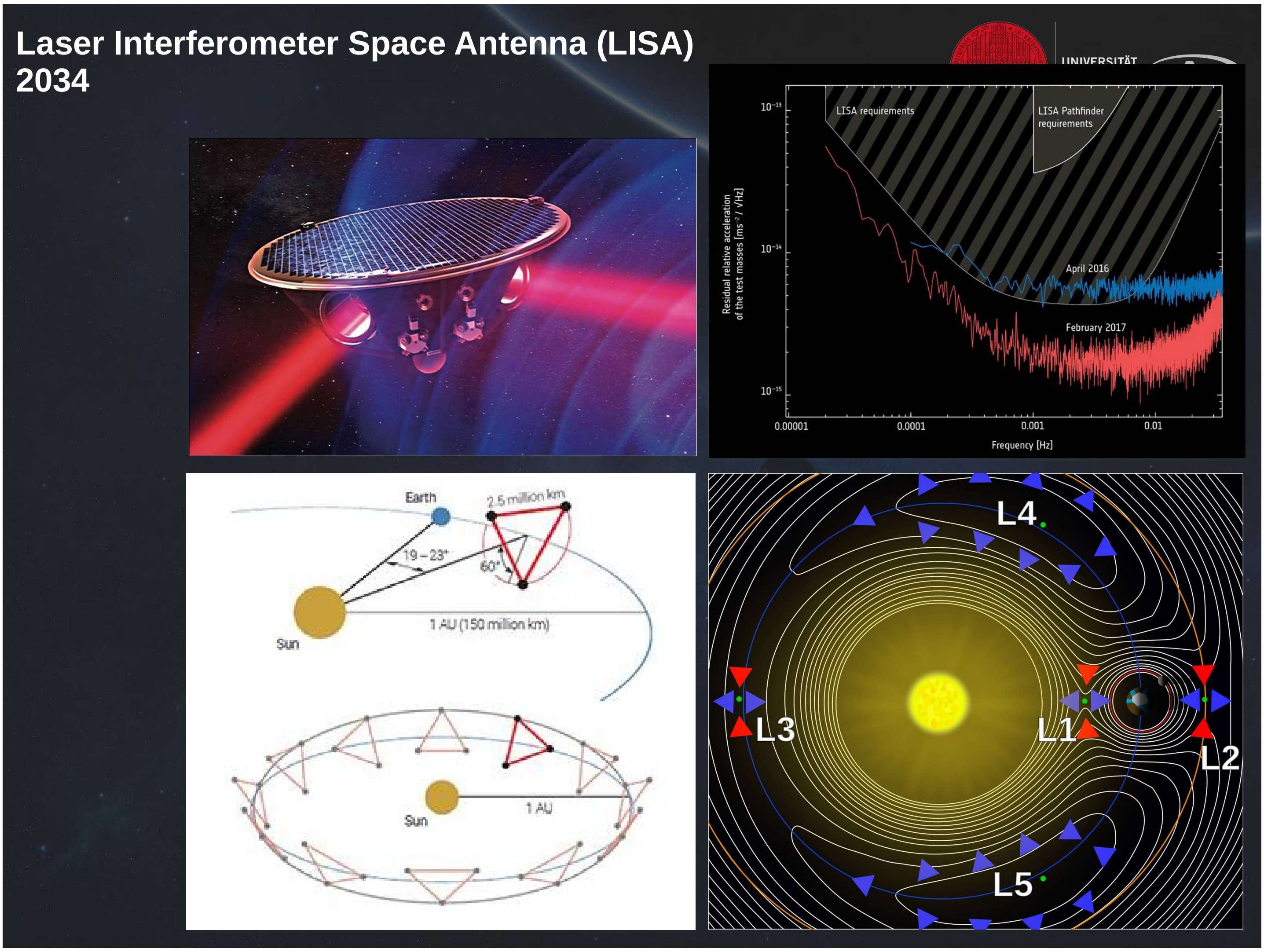


Ground base interferometers

LIGO (Laser Interferometer Gravitational-wave Observatory) is the world's largest gravitational wave observatory. LIGO consists of two laser interferometers located thousands of kilometers apart, one in Livingston, Louisiana and the other in Hanford, Washington. LIGO uses the physical properties of light and of space itself to detect gravitational waves. It was funded by the US National Science Foundation, and it is managed by Caltech and MIT. Hundreds of scientists in the LIGO Scientific Collaboration, in many countries, contribute to the astrophysical and instrument science of LIGO. There are also other gravitational wave observatories in the world, including Virgo in Italy and GEO 600 in Germany.

Figure 9 LIGO Hanford and LIGO Livingston.
Credit: Caltech/MIT/LIGO

Einstein telescope



Laser Interferometer Space Antenna (LISA) 2034

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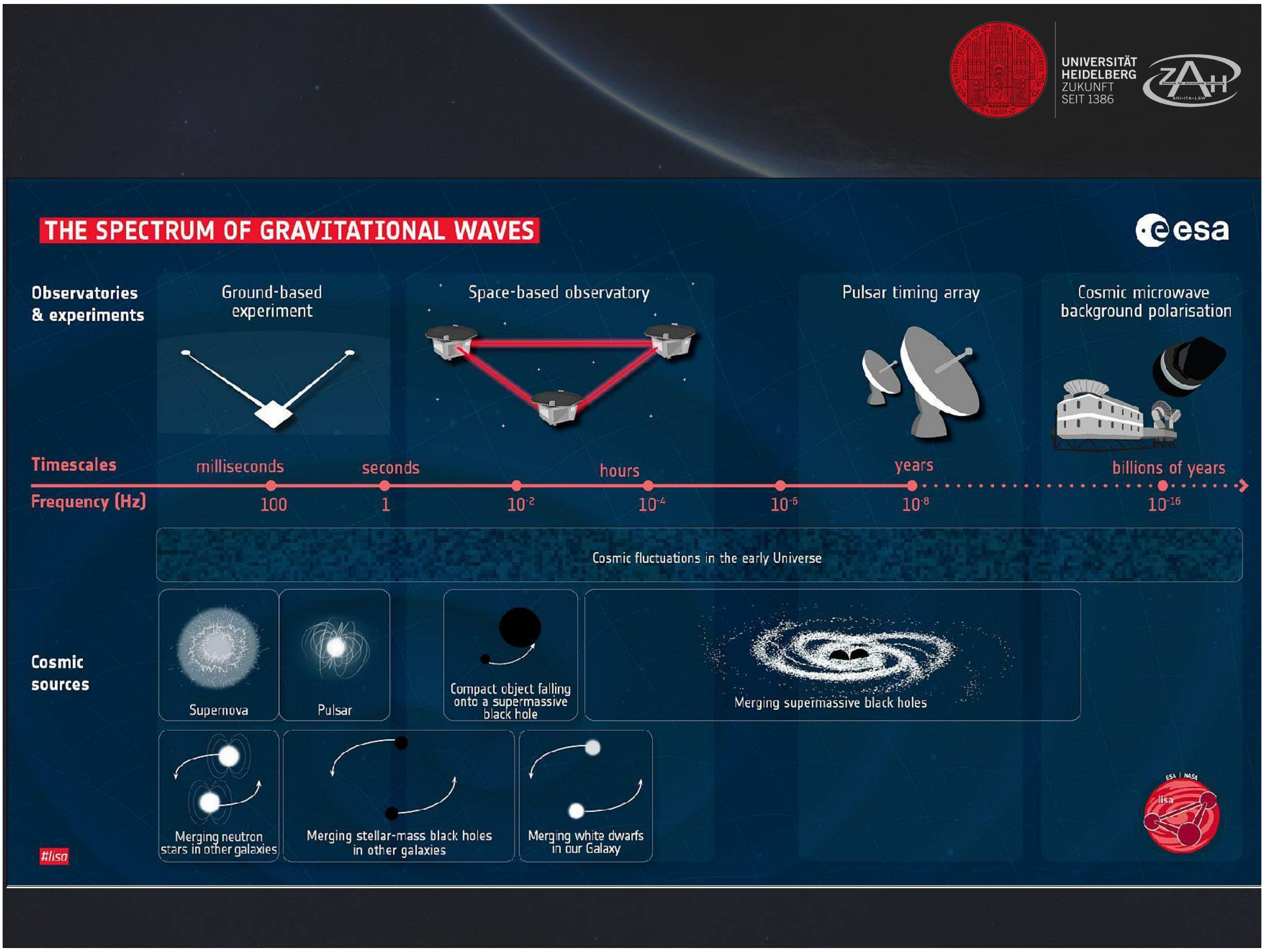
IPTA-International Pulsar Timing Array-Clockwise

- nHz gravitational waves
- binary super massive black holes

credit: David J. Champion



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THE SPECTRUM OF GRAVITATIONAL WAVES

esa

Observatories & experiments: Ground-based experiment; Space-based observatory; Pulsar timing array; Cosmic microwave background polarisation

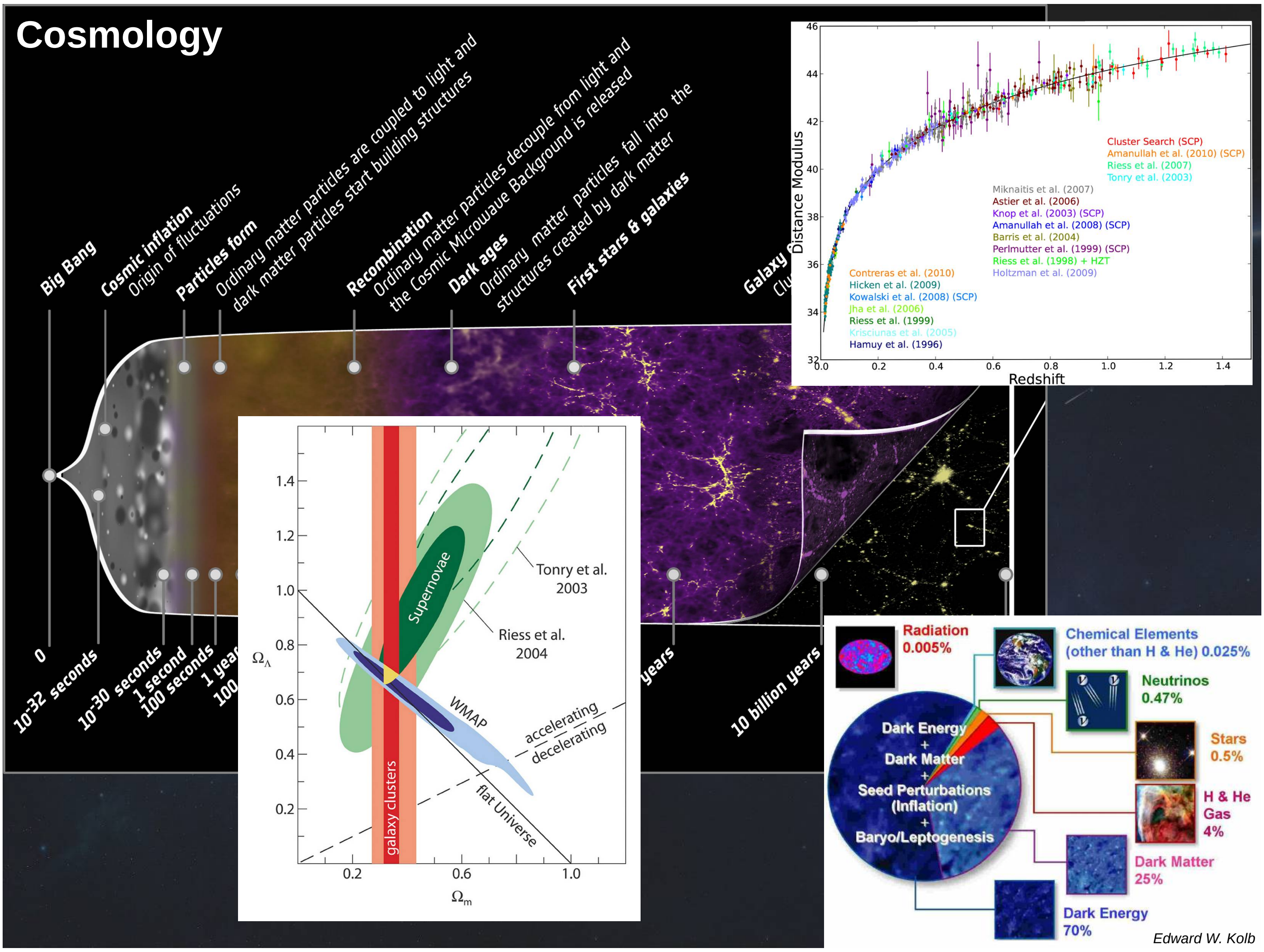
Timescales: milliseconds; seconds; hours; years; billions of years

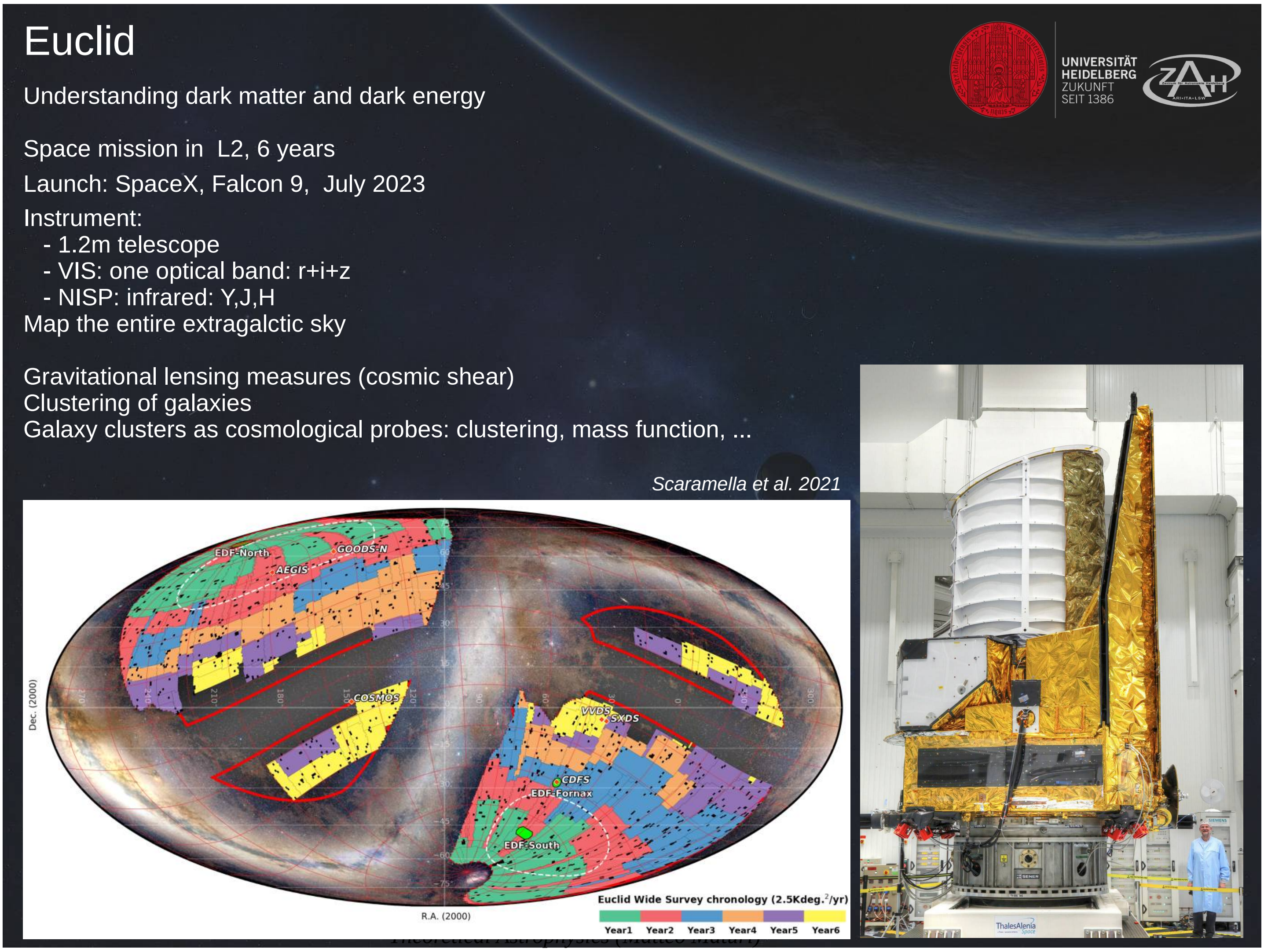
Frequency (Hz): 100; 1; 10^{-2}; 10^{-4}; 10^{-6}; 10^{-8}; 10^{-16}

Cosmic fluctuations in the early Universe

Cosmic sources: Supernova; Pulsar; Compact object falling onto a supermassive black hole; Merging supermassive black holes; Merging neutron stars in other galaxies; Merging stellar-mass black holes in other galaxies; Merging white dwarfs in our Galaxy

#lisa


Cosmology
Big Bang
Cosmic inflation
Origin of fluctuations
Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwaue Background is released
Dark ages
Ordinary matter particles fall into the structures created by dark matter
First stars & galaxies
Galaxy
Clu
0
10-32 seconds
10-30 seconds
1 second
100 seconds
1 year
100
years
10 billion years
Distance Modulus
Redshift
46
44
42
40
38
36
34
32
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Cluster Search (SCP)
Amanullah et al. (2010) (SCP)
Riess et al. (2007)
Tonry et al. (2003)
Miknaitis et al. (2007)
Astier et al. (2006)
Knop et al. (2003) (SCP)
Amanullah et al. (2008) (SCP)
Barris et al. (2004)
Perlmutter et al. (1999) (SCP)
Riess et al. (1998) + HZT
Holtzman et al. (2009)
Contreras et al. (2010)
Hicken et al. (2009)
Kowalski et al. (2008) (SCP)
Jha et al. (2006)
Riess et al. (1999)
Krisciunas et al. (2005)
Hamuy et al. (1996)
Ω_Λ
Ω_m
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.2
0.6
1.0
Supernovae
Tonry et al. 2003
Riess et al. 2004
WMAP
accelerating
decelerating
flat Universe
galaxy clusters
Radiation 0.005%
Chemical Elements (other than H & He) 0.025%
Neutrinos 0.47%
Stars 0.5%
H & He Gas 4%
Dark Matter 25%
Dark Energy 70%
Dark Energy + Dark Matter + Seed Perturbations (Inflation) + Baryo/Leptogenesis
Edward W. Kolb



Euclid

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Understanding dark matter and dark energy

Space mission in L2, 6 years

Launch: SpaceX, Falcon 9, July 2023

Instrument:
- 1.2m telescope
- VIS: one optical band: r+i+z
- NISP: infrared: Y,J,H

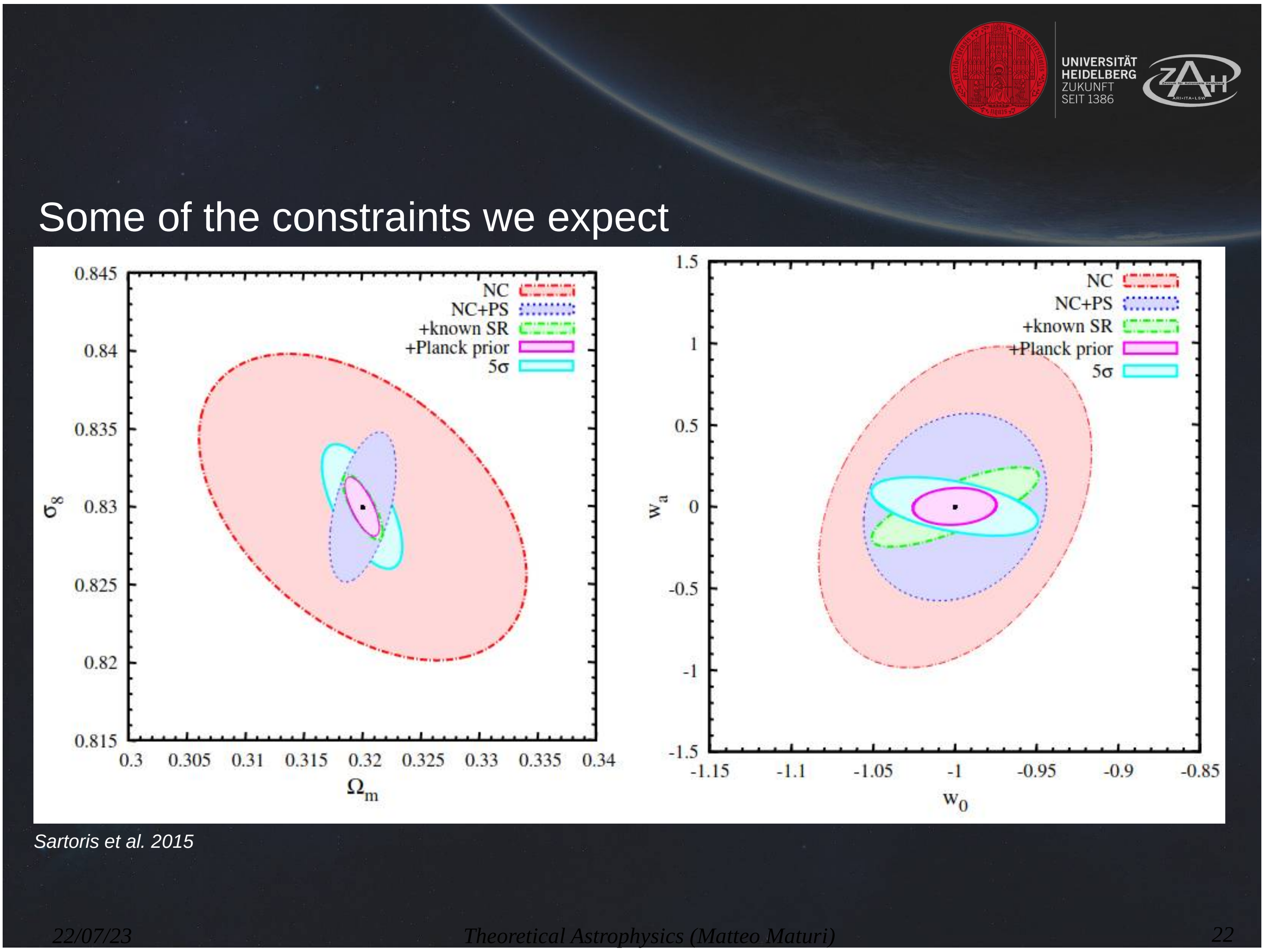
Map the entire extragalctic sky

Gravitational lensing measures (cosmic shear)
Clustering of galaxies
Galaxy clusters as cosmological probes: clustering, mass function, ...

Scaramella et al. 2021

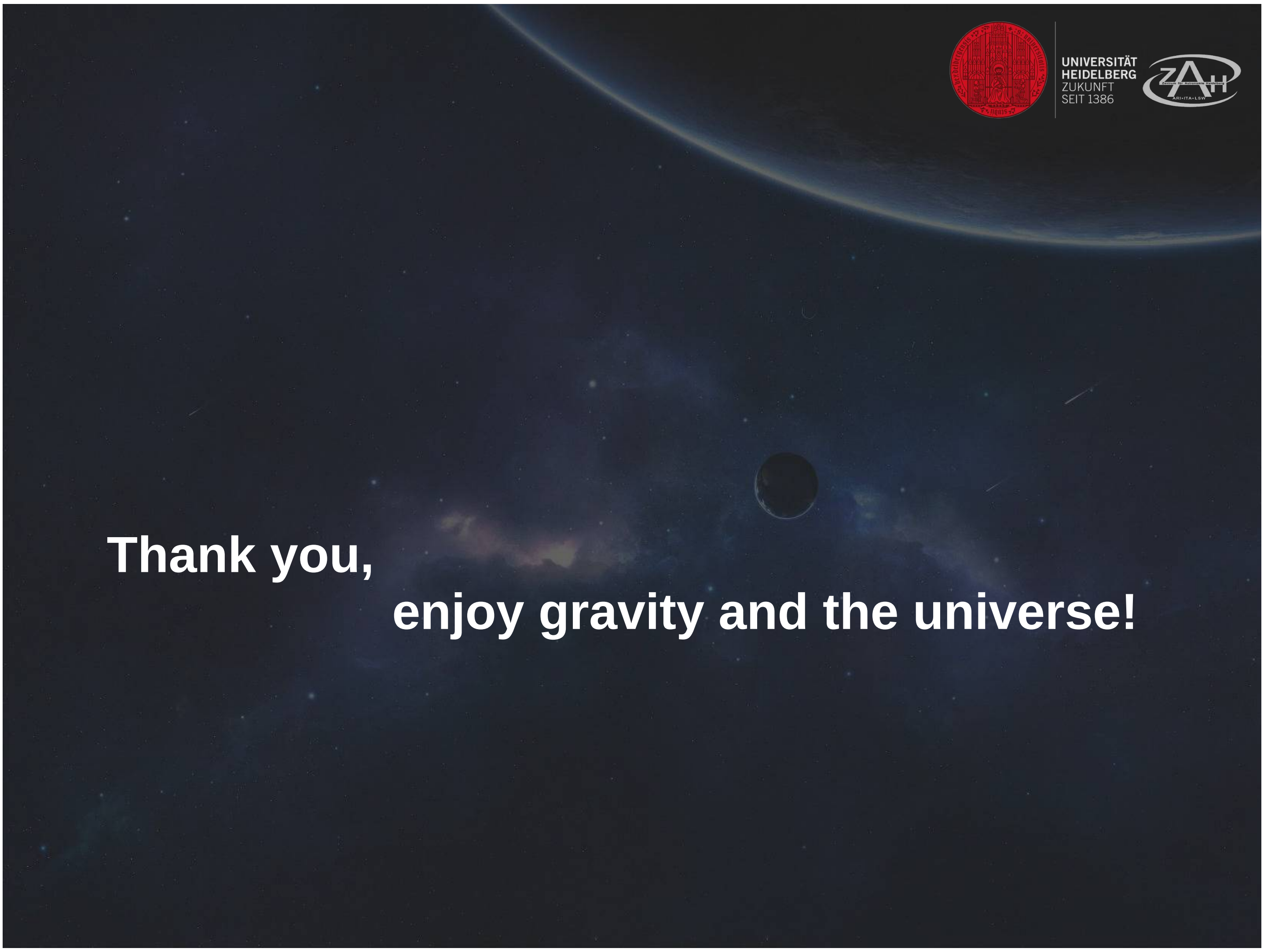
Euclid Wide Survey chronology (2.5Kdeg.^{2}/yr)

🟩	🟥	🟦	🟧	🟪	🟨
Year1	Year2	Year3	Year4	Year5	Year6



Some of the constraints we expect

Sartoris et al. 2015



Thank you,

enjoy gravity and the universe!

Conformal transformations

- conformal transformation: frame $x^{\mu'}$ such that $\boxed{g_{\mu'\nu'} = \Omega^2(x^{\alpha'})\, g_{\mu\nu}}$ $\quad \Omega: M \to \mathbb{R}$

$$g_{\mu'\nu'} = x^{\alpha}_{,\mu'}\, x^{\beta}_{,\nu'}\, g_{\alpha\beta} = \Omega^2(x')\, g_{\mu\nu}$$

- conformal flatness: $\boxed{g'_{\mu\nu} = \Omega^2(x')\, \eta_{\mu\nu}}$

- Light-cone: $ds^2 = g'_{\mu\nu} k^\mu k^\mu = \Omega^2(x) \underbrace{\eta_{\mu\nu} k^\mu k^\nu}_{=0} = 0 \quad \forall\, \Omega$ => Minkowskian light cones

- any 2D (pseudo-) Riemanian manifold is conformally flat
  i.e. it always exists a coord. transf. such that ds^2 takes that form

↳ Prove: x^i $i = 0,1$ signature $(-,+)$

$$g'^{ij} = \frac{\delta x^{i'}}{\delta x^\alpha} \frac{\delta x^{j'}}{\delta x^\beta} g^{\alpha\beta} \overset{!}{=} \Omega^2(x)\, \eta^{ij}$$

$\alpha \neq \beta$ $\quad g'^{01} = x^{0'}_{,i}\, x^{1'}_{,j}\, g^{ij} \overset{!}{=} \Omega^2(x)\, \eta^{01} = 0 \qquad (1)$

$\alpha = \beta$ $\quad g'^{00} = x^{0'}_{,i}\, x^{0'}_{j}\, g^{ij} \overset{!}{=} \Omega^2(x)\, \eta^{00} = -\Omega^2(x)$
$\qquad\quad g'^{11} = x^{1'}_{,i}\, x^{1'}_{,j}\, g^{ij} \overset{!}{=} \Omega^2(x)\, \eta^{11} = \Omega^2(x) \qquad (2)$

conditions defining the Jacobian $J^{i'}_{j} \equiv \frac{\delta x^{i'}}{\delta x^j}$

$$\boxed{\begin{array}{ll} J^{0'}_{i} J^{1'}_{j} g^{ij} = 0 & * \\ \left(J^{0'}_{i} J^{0'}_{j} + J^{1'}_{i} J^{1'}_{j}\right) g^{ij} = 0 & \end{array}} \begin{array}{l} (1) \\ (2) \end{array}$$

$*$ identically satisfied for $J^{0'}_{i} = k\, \epsilon_{ij}\, g^{jl} J^{1}_{l}$ (**) $\quad k$ arbitrary function, $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$

plugging (**) in (2) $\longrightarrow$ $\left(\frac{k^2}{g} - 1\right) J^{1}_{i} J^{1}_{j}\, g^{ij} = 0$ $\quad g = \det(g_{ij})$ $\quad$ => $k^2 = g$ ✓

- A non-conformally flat manifold, can have a conformally flat sub-manifold
  eg. Schwarzschild

- The Weyl tensor is conformal invariant $\quad C_{\alpha\beta\gamma\delta} \to C_{\alpha\beta\gamma\delta}$

- Conformal because angles between vectors keep the same value

Conformal flatness example

e.g. $$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$ Schwarzschild metric

$$= \left(1-\frac{r_s}{r}\right)\left(-c^2dt^2 + d\tilde{r}^2\right) + r^2 d\Omega^2$$

$$\cancel{\left(1-\frac{r_s}{r}\right)}^{1} d\tilde{r}^2 \overset{!}{=} \left(1-\frac{r_s}{r}\right)^{\cancel{-1}\,-2} dr^2 \Rightarrow d\tilde{r} = \left(1-\frac{r_s}{r}\right)^{-1} dr$$

2D space ($\vartheta = \text{const}$, $\phi = \text{const}$)

$$ds^2 = \left(1-\frac{r_s}{r}\right)\left(-c^2dt^2 + d\tilde{r}^2\right) = \Omega^2(x)\,\eta_{\mu\nu}dx^\mu dx^\nu$$

$\Omega(x)$: conformal scaling factor

$x^0 = ct \quad x^1 = \tilde{r}$

- 2D space is curved because Ω depends on position (Ω enters in the curvature tensor)

But line element is conformally flat : conformal flatness

Conformal invariance example

$$ds'^2 = \left(1-\frac{r_s}{r}\right)\underbrace{\Omega^2c^2dt^2} - \left(1-\frac{r_s}{r}\right)^{-1}\underbrace{\Omega^2dr^2} - \underbrace{r^2\Omega^2}\left(d\vartheta^2 + \sin^2\vartheta\, d\phi^2\right),$$ i.e. $ds^2 \to \Omega^2 ds^2$

$t \to \tau \equiv \Omega t$ * $\qquad r \to R \equiv \Omega r$ * $\qquad \swarrow\ r_s \to R_s \equiv \Omega r$ *

$$= \left(1-\frac{\Omega r_s}{R}\right)c^2d\tau^2 - \left(1-\frac{\Omega r_s}{R}\right)^{-1}dR^2 - R^2\left(d\vartheta^2 + \sin^2\vartheta\, d\phi^2\right)$$ i.e. rescaling of r_s (i.e. of mass m)

$\Rightarrow$ Schwarzschild metric is invariant under conformal transformation *

Another example of conformal flatness (FLRW)

- Pure Ricci curvature (Weyl-tensor $= 0$ $\quad C_{\alpha\beta\gamma\delta} = 0$)

$\Rightarrow$ Conformally flat $\quad g_{\mu\nu} = \Omega^2(t)\,\eta_{\mu\nu}$

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + r^2\left(d\vartheta + \sin^2\vartheta\, d\phi^2\right)\right] = a(t)\overbrace{\left[-c^2d\eta^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\phi^2\right)\right]}^{\text{Minkowski structure}}$$

conformal time $\quad d\eta \equiv \dfrac{dt}{a(t)} \qquad \eta = \displaystyle\int \frac{dt}{a(t)} \neq t$

$a(t)$ plays the role of the conformal factor Ω

$d\eta = dt/a(t)$ might be divergent $\Rightarrow$ horizon appears

$$r_{PH} = c\int_{t_i}^{t_0}\frac{dt}{a} = c\int_{-\infty}^{0}d\eta$$ particle horizon: max dist. reached by photon (finite because of finite age of universe

$$r_{EH} = c\int_{t_0}^{t_f}\frac{dt}{a} = c\int_{0}^{+\infty}d\eta$$ event horizon: max dist. of photon emitted today can possibly reach in the future

Quantities after having applied a conformal transformation

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\alpha\gamma}\left(\partial_\mu \overset{\downarrow}{\tilde{g}}_{\gamma\nu} + \partial_\nu \overset{\downarrow}{\tilde{g}}_{\gamma\mu} - \partial_\gamma \overset{\downarrow}{\tilde{g}}_{\mu\nu}\right) \qquad \tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}$$

just product rule of derivatives

$$= \Gamma^{\alpha}_{\mu\nu} + \partial^{\alpha}_{\nu}\partial_\mu \ln\Omega + \partial^{\alpha}_{\mu}\partial_\nu \ln\Omega - g_{\mu\nu}\partial^{\alpha}\ln\Omega$$

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\alpha}{}_{\mu\alpha\nu} = \partial_\alpha \overset{\downarrow}{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu} - \partial_\nu \overset{\downarrow}{\Gamma}{}^{\alpha}_{\mu\alpha} + \overset{\downarrow}{\tilde{\Gamma}}{}^{\alpha}_{\alpha\lambda}\overset{\downarrow}{\tilde{\Gamma}}{}^{\lambda}_{\nu\mu} - \overset{\downarrow}{\tilde{\Gamma}}{}^{\alpha}_{\nu\lambda}\overset{\downarrow}{\tilde{\Gamma}}{}^{\lambda}_{\alpha\mu}$$

$$= R_{\mu\nu} - 2\nabla_\mu\nabla_\nu \ln\Omega - g_{\mu\nu}\nabla_\lambda\nabla_\lambda \ln\Omega + 2\partial_\nu \ln\Omega\ \partial_\mu \ln\Omega - 2 g_{\mu\nu}\partial_\lambda \ln\Omega\, \partial^\lambda \ln\Omega$$

$$\tilde{R} = \tilde{R}^{\alpha}{}_{\alpha} = \frac{1}{\Omega^2}\left(R - \frac{6\,\Box\Omega}{\Omega}\right)$$